MATHEMATICAL METHODS IN ENGINEERING

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MATHEMATICAL METHODS IN ENGINEERING

An Introduction to the Mathematical Treatment of Engineering Problems

BY

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PREFACE

> Neither seeking nor avoiding mathematical exercitations we enter into problems solely with a view to possible usefulness for physical science.
>
> LORD KELVIN and PETER GUTHRIE TAIT,
> "Treatise on Natural Philosophy," Part II.

The primary objective of this book is to introduce the reader to the mathematical treatment of engineering problems. It is often said that the engineer nowadays needs more and more mathematics. However, it is common experience that most engineers apply only a small portion of the knowledge presented to them in their college mathematics courses. It seems that in many cases the amount of mathematics included in the curriculums is quite adequate, but the ability to find the proper mathematical setup for given physical or engineering problems is not developed in the student to a sufficient degree. In other words, the need is not so much for "more mathematics" as for a better understanding of the potentialities of its application.

There are two ways of teaching the art of applying mathematics to engineering problems. One consists of a systematic course comprising selected branches of mathematics including a choice of appropriate examples for applications. The other chooses certain representative groups of engineering problems and demonstrates the mathematical approach to their solution. There are excellent books available that follow the first method. This book might be considered as an experiment in the direction of the second method. Of course, it would not be advantageous to push this method to an extreme; hence, certain parts of the book are concerned primarily with mathematical subjects. However, also in these parts the mathematical concepts are presented from the viewpoint of their application rather than from that of their purely logical development.

To keep the book within a reasonable size the scope of topics was limited, although the authors realize that, for example, the study of partial differential equations, integral equations, functions of complex variables, and vector and tensor analysis is desirable to complete the mathematical equipment of a scientific engineer.

Mathematical prerequisites for an understanding of this book are ordinary calculus, the elements of analytical geometry, and the elements of algebra, which include linear equations, determinants, and some knowledge of algebraic equations. It is supposed that the reader can handle the elementary operations with complex numbers. Previous knowledge of differential equations is not necessary. To avoid ambiguity in the mathematical terminology, some of the fundamental definitions of algebra, calculus, and analytical geometry are compiled under the title Words and Phrases.

The first two chapters on differential equations and Bessel functions are fundamentally of a mathematical character. In the subsequent chapters we have tried to insert most of the information of mathematical nature between the problems that require their knowledge. The reader will notice that elliptic integrals, for example, are treated in connection with the classical problem of the pendulum, and that vector algebra appears in the chapter on the fundamental concepts of dynamics. The latter chapter was included in the belief that *mechanica rationalis* generally does not enjoy its due place between physics and applied mechanics in the engineering curriculum. The instructor who shuns Lagrange's equations may omit the last sections of Chapter III. The examples of Chapters V and VI, in which Lagrange's equations are used, can be treated without reference to the Lagrangian method. However, it is believed that familiarity with generalized coordinates, generalized forces, and other concepts of the mechanics of Lagrange is a positive advantage for the scientific minded engineer.

Most problems treated in detail in the text are drawn from the dynamics of particles, rigid bodies, and the theory of structures; a smaller number from the theory of electric circuits. Many of the problems of mechanical oscillations can easily be converted into problems of oscillations in electrical networks. The problems given at the end of each chapter contain, besides

exercises in mathematical operations, additional applications to various fields of engineering, *e.g.*, fluid mechanics, heat conduction, etc.

We ask the *mathematician* who uses this book for the instruction of engineering students not to forget that a book of this type cannot contain rigorous proofs for all mathematical statements included. We hope that most of the statements stand up even before the supreme court of pure mathematics. Possibly it cannot be avoided that at times an engineer will be seduced into applying some of the mathematical methods beyond their limit of validity and will run into difficulties. Perhaps the best advice he can be given is to consult a mathematician.

We ask the *engineering instructor* to take into consideration that we cannot always give that problem which he deems to be the most significant for the demonstration of certain mathematical methods. He may work out his own problems and substitute them for those given in this book.

We ask the *student* to remember that mere following of the detailed treatment of a certain problem does not assure a real understanding of the method unless he has solved analogous problems by himself.

We ask the *practicing engineer* to forgive us if we do not give precisely the particular problem that he has lately encountered or the particular method that he hoped to find in the book. He may also read those chapters that momentarily do not interest him so much, for perhaps he will face similar problems tomorrow.

The authors outlined each chapter in close cooperation. Then each author worked out independently a certain number of chapters. Finally the whole manuscript was worked over in view of continuity through mutual discussions. The close cooperation of the authors was made possible, especially during the initial period, by a fellowship granted to the junior author by the Belgian-American Educational Foundation. The authors wish to express their gratitude to the Foundation and especially to its vice-president, Mr. Perrin C. Galpin.

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CONTENTS

CHAPTER IX

COMPLEX REPRESENTATION OF PERIODIC PHENOMENA

CHAPTER X

TRANSIENT PHENOMENA. OPERATIONAL CALCULUS

CHAPTER XI

EQUATIONS OF FINITE DIFFERENCES APPLIED TO ENGINEERING

CHAPTER I

INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

> Mathematicians do not study objects, but the relations between objects. Matter does not engage their attention, they are interested in form alone.
>
> —H. POINCARÉ
> "Science et Hypothèse."

Introduction.—The first part of this chapter, including sections 1 to 7, gives a short review of the fundamental theorems on differential equations of one independent variable. We begin in section 1 with the simplest form of a differential equation: one which can be solved by pure quadrature. In sections 3 and 4 the elements of the theory of differential equations of the first order are given. Section 6 discusses numerical methods for the solution of differential equations. Section 7 contains some general remarks on differential equations of higher order. The last sections are devoted to linear differential equations, especially to equations with constant coefficients and their special methods of solution.

1. The Fundamental Problem of Integral Calculus.—The simplest example of a differential equation is a relation of the form:

$$\frac{dy}{dx} = f(x) \tag{1.1}$$

It requires the determination of a function $y(x)$ whose first derivative is equal to a given function $f(x)$ of the independent variable x. We assume that $f(x)$ is single valued and continuous.

The solution of Eq. (1.1) is furnished by the fundamental theorem of the integral calculus which states that the differential quotient of the *definite integral* $\int_a^x f(\xi)\, d\xi$ with respect to the upper limit x is equal to $f(x)$; the lower limit a is an arbitrary constant. In fact, if we plot $f(\xi)$ (Fig. 1.1) as function of ξ, the definite integral in question is equal to the area $ABCD$ enclosed by the

ξ-axis, the curve whose ordinates are $f(\xi)$ and the ordinates AD erected at $\xi = a$ and BC at $\xi = x$. The area $ABCD$ is defined as the limiting value of the sum of the small rectangles shown schematically in Fig. 1.1. If we increase the upper limit by Δx, the area increases by the amount $f(x)\,\Delta x$; therefore, the derivative of the integral as function of the upper limit is equal to $f(x)$, as required by Eq. (1.1). Hence,

$$y(x) = \int_a^x f(\xi)\, d\xi \tag{1.2}$$

is a solution of the differential equation (1.1). When the solution of a differential equation is obtained, as in this case, by

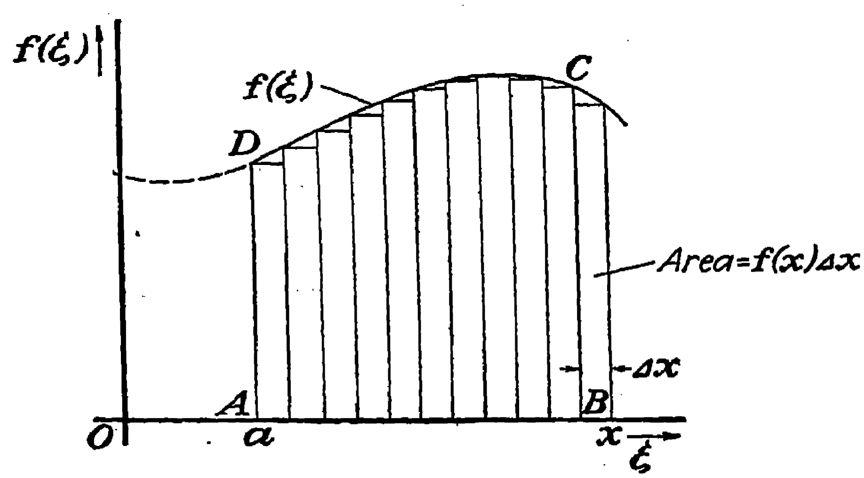

FIG. 1.1.—Geometrical interpretation of a definite integral.

integration of a given function, we say that the equation is solved by *quadrature* or *direct integration.*

If we plot the solution $y = \int_a^x f(\xi)\, d\xi$, as function of x, we obtain a so-called *integral curve* of the differential equation (1.1). This integral curve passes through the point $x = a$, $y = 0$. In order to obtain the integral curve passing through any point $x = x_0$, $y = y_0$, we write $y = y_0 + \int_{x_0}^x f(\xi)\, d\xi$. If we vary x_0, the integral is changed only by a constant. Let us choose the origin $\xi = 0$ within the domain for which $f(x)$ is defined. Then we can write:

$$y = \int_0^x f(\xi)\, d\xi + C \tag{1.3}$$

where C is an arbitrary constant.

For the sake of simplicity in notation, we shall write, in general, a definite integral that is a function of the upper limit x, in the form $\int_0^x f(x)\, dx$, except when such simplification may lead to confusion.

The expression (1.3), containing the arbitrary constant C, constitutes the *general solution* of Eq. (1.1); any function $y(x)$ corresponding to a particular value of C is a *particular solution* of (1.1). The general solution represents a *one-parameter family* of curves, which, in this special case, are identical equidistant curves.

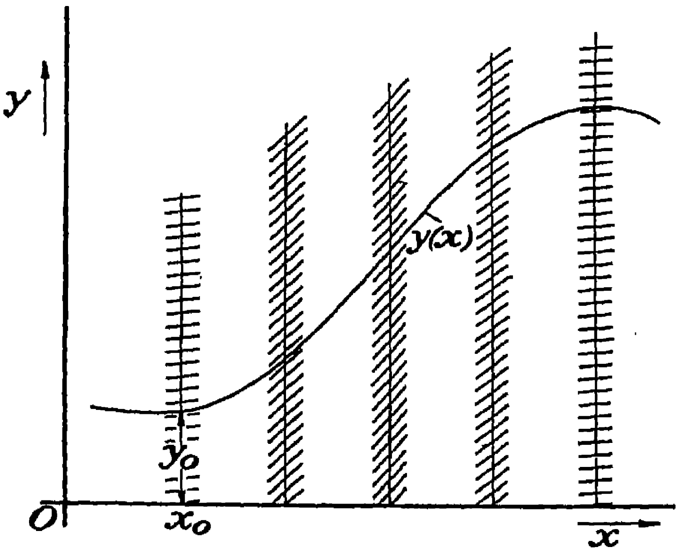

FIG. 1.2.—Construction of the integral curve passing through the point x_0, y_0.

Another method of finding the solution of Eq. (1.1) is connected with the fundamental geometrical interpretation of the derivative of a function. Equation (1.1) states that the slope of the integral curves is a given function of the abscissa x, *i.e.*, all integral curves crossing the same ordinate have the same slope (Fig. 1.2). Hence, if we start from an arbitrary point x_0,y_0 and draw a continuous curve through x_0,y_0, which crosses every ordinate with the slope prescribed by Eq. (1.1), we obtain the integral curve passing through the point x_0,y_0. The equation of this curve is, as shown above, $y = y_0 + \int_{x_0}^{x} f(x)\, dx$. Since $f(x)$ is assumed to be single valued, there is only one such curve. It is evident that by varying x_0,y_0 an infinite number of such curves can be drawn and that they constitute a one-parameter family of equidistant curves.

2. Numerical Evaluation of Integrals.—The solution of Eq. (1.1) involves the evaluation of definite integrals of the form:

$$S = \int_a^b f(x)\, dx \tag{2.1}$$

This integral can be evaluated approximately in the following way. We divide the range ab into n equal intervals h, such that $b - a = nh$ (Fig. 2.1). Two successive ordinates

$$y_r = f(a + rh)$$
$$y_{r+1} = f[a + (r + 1)h]$$

define a trapezoid $ABCD$ whose area is $\frac{1}{2}(y_r + y_{r+1})h$. The total

area representing the definite integral (2.1) is approximated by the sum of the areas of all the trapezoids defined by the ordinates $y_0, y_1, \ldots, y_n$:

$$S = \tfrac{1}{2}\left[(y_0 + y_1) + (y_1 + y_2) + \cdots + (y_{n-1} + y_n)\right] h$$

or

$$S = (\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \tfrac{1}{2}y_n)h \qquad (2.2)$$

The formula (2.2) is known as the *trapezoidal rule.* It is equiva-

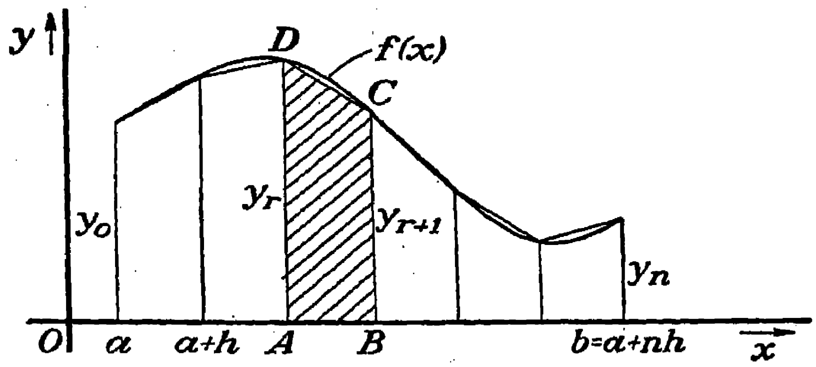

FIG. 2.1.—The evaluation of the integral $\int_a^b f(x)\,dx$ by the trapezoidal rule.

lent to approximating the curve $y = f(x)$ between equidistant ordinates by segments of straight lines.

A closer value of the definite integral may be obtained by approximating the curve $y = f(x)$ by a succession of parabolic arcs. This leads to *Simpson's rule.*

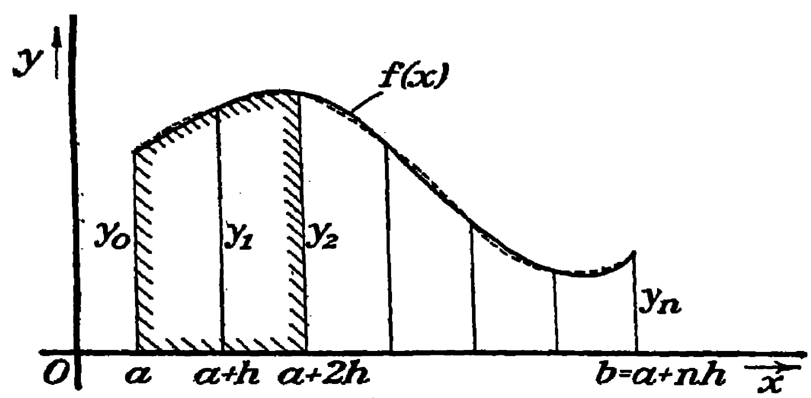

FIG. 2.2.—The evaluation of $\int_a^b f(x)\,dx$ by Simpson's rule.

Consider, for instance, the portion of the curve in the interval $a \leqq x \leqq a + 2h$. We denote the ordinates corresponding to a, $a + h$, and $a + 2h$ by y_0, y_1, and y_2, respectively (Fig. 2.2). The equation of any parabola passing through the point $(a + h, y_1)$ is of the form:

$$\eta = y_1 + \alpha\left[x - (a + h)\right] + \beta\left[x - (a + h)\right]^2 \qquad (2.3)$$

We determine the two constants α and β in such a way that the curve also passes through the points (a,y_0) and $(a + 2h,\ y_2)$. This gives

$$\alpha = \frac{y_2 - y_0}{2h}$$

$$\beta = \frac{y_2 + y_0 - 2y_1}{2h^2}$$

The area between this parabolic arc and the x-axis is

$$\int_a^{a+2h} \eta\, dx = \int_a^{a+2h} y_1\, dx + \alpha \int_a^{a+2h} [x - (a + h)]\, dx + \beta \int_a^{a+2h} [x - (a + h)]^2\, dx$$

$$= \tfrac{1}{3}(y_0 + 4y_1 + y_2)h$$

We proceed in the same way for the set of three points $(a + 2h,\ y_2)$, $(a + 3h,\ y_3)$, $(a + 4h,\ y_4)$ and evaluate the area between the parabolic arc passing through these three points and the x-axis: $\frac{1}{3}(y_2 + 4y_3 + y_4)h$, and so on for the other intervals $(a + 4h)$ to $(a + 6h)$, $(a + 6h)$ to $(a + 8h)$, etc. Each parabolic arc covers an interval $2h$, and if we have divided the range ab into an *even number* n of equal intervals h, we can approximate the entire curve $y = f(x)$ from a to b by a series of $n/2$ parabolic arcs. The area representing the integral (2.1) is then approximated by the sum of all the areas under these parabolic arcs.

$$S = \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)]$$

$$= \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots \quad 2y_{n-2} + 4y_{n-1} + y_n] \tag{2.4}$$

where the number n of intervals is *even*. This formula is known as *Simpson's rule*.

The degree of approximation of these formulas depends, of course, on the magnitude of the interval h. It is possible to establish formulas which are more accurate than Simpson's rule by using polynomials of higher degree to approximate the curve $y = f(x)$ in the successive intervals h.* In most practical cases, however, Simpson's rule will be satisfactory for engineering use.

* See, for instance, J. B. Scarborough, "Numerical Mathematical Analysis," pp. 117–152.

Example.—The value of π is given by the integral

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1 + x^2} = [\tan^{-1} x]_0^1$$

To evaluate this integral, we compute the values of the function $y = \dfrac{1}{1 + x^2}$ between $x = 0$ and $x = 1$ using as interval $h = 0.25$. We obtain $y_0 = 1.00000000$, $y_1 = 0.94117647$, $y_2 = 0.80000000$, $y_3 = 0.64000000$, $y_4 = 0.50000000$.

The trapezoidal rule gives

$$\frac{\pi}{4} = 0.25 \left[\frac{1}{2} y_0 + y_1 + y_2 + y_3 + \frac{1}{2} y_4 \right]$$

and

$$\pi = 3.13117647$$

Simpson's rule yields the value

$$\frac{\pi}{4} = \frac{0.25}{3} [y_0 + 4y_1 + 2y_2 + 4y_3 + y_4]$$

and

$$\pi = 3.14156896$$

The correct value of π to nine digits is

$$\pi = 3.14159265$$

Comparison with the values above shows the superiority of Simpson's rule which gives a result correct to five digits, while the trapezoidal rule gives only the first two digits correctly. By using a smaller interval $h = 0.1$, the trapezoidal rule yields $\pi = 3.13992597$, and Simpson's rule yields $\pi = 3.14159260$. The latter value is now correct to eight digits, whereas that given by the trapezoidal rule has not been greatly improved.

3. Differential Equations of the First Order.—We define a differential equation of the first order as a relation between the independent variable x, the unknown function $y(x)$, and its first derivative dy/dx:

$$\frac{dy}{dx} = f(x,y) \tag{3.1}$$

Obviously, Eq. (1.1) is the simplest case of such a relation.

We first assume that $f(x,y)$ is a single-valued function of x and y, that it can be differentiated with respect to x and y, and that its derivatives are finite.

Although we were able to find the solution of (1.1) by quadrature (direct integration), in the case of an equation of the form

(3.1) this will be possible in exceptional cases only. However, the second method of solution applied to Eq. (1.1) can be generalized for the case of (3.1). We draw a family of curves defined by the equation

$$f(x,y) = \text{const.} \tag{3.2}$$

Obviously, each of these curves connects the points in which the integral curves of (3.1) have the same slope. Varying the value of the constant in Eq. (3.2), we obtain a family of such curves, called the *isoclines* (Fig. 3.1). The problem of finding solutions of (2.1) consists in plotting curves which cross each of the iso-

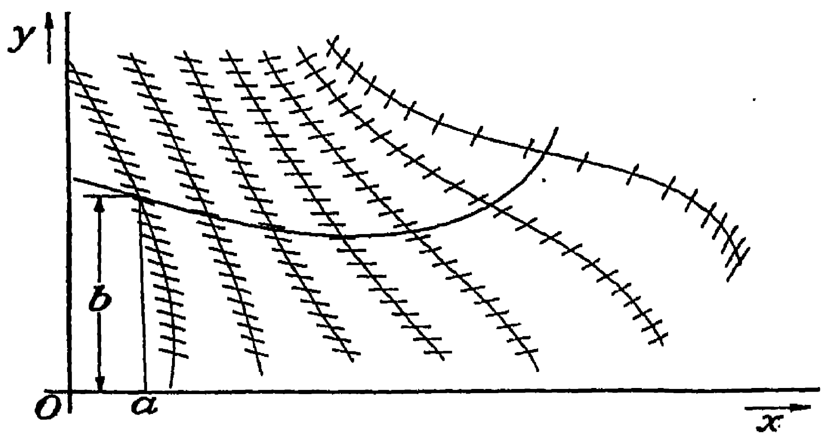

FIG. 3.1.—To illustrate the isocline method. The parallel line segments along a line $f(x, y) = \text{const.}$ indicate the slope dy/dx of the integral curves crossing that line.

clines with the slope associated with that isocline. It is seen that an infinite number of such integral curves can be obtained by starting from different points of a certain isocline. Figure 3.1 shows the integral curve passing through the point $x = a$, $y = b$. In the special case treated in section 1 the isoclines were straight lines $x = \text{const.}$

This method of solving a differential equation of the first order is called the *isocline method.* It can always be used as a step-by-step method provided that $f(x,y)$ is a single-valued continuous function of x and y. However, we shall later encounter cases in which for some particular points these conditions are not fulfilled. Let us assume, for instance, that $f(x,y)$ is given as the quotient of two functions φ and ψ, $f(x,y) = \varphi(x,y)/\psi(x,y)$. Then if φ and ψ vanish at the same point, $f(x,y)$ has the form $0/0$ and the isocline method cannot be applied without further discussion. The behavior of the integral curves in the neighborhood of such *singular points* will be discussed later in connection with

the problems in which they occur (Chapter IV, sections 10 and 11).

Separation of Variables.—It was said above that, in general, the solution of (3.1) cannot be found by quadratures. However, there are some important cases in which this is possible. One of these particular cases occurs if the variables x and y can be *separated.*

Assume that $f(x,y)$ has the form $f(x,y) = \varphi(x)/\psi(y)$; then Eq. (3.1) becomes

$$\frac{dy}{dx} = f(x,y) = \frac{\varphi(x)}{\psi(y)} \tag{3.3}$$

and, therefore,

$$\psi(y)\frac{dy}{dx} = \varphi(x) \tag{3.4}$$

The left side of (3.4) is obviously the derivative of a function $\Psi(y)$ with respect to x, where $\Psi(y)$ is defined by

$$\Psi(y) = \int_b^y \psi(y)\,dy \tag{3.5}$$

and b is an arbitrary constant. Hence, (3.4) reads

$$\frac{d\Psi}{dx} = \varphi(x)$$

This equation is of the form (1.1); hence, we have by integration

$$\Psi = \int_b^y \psi(y)\,dy = \int_a^x \varphi(x)\,dx \tag{3.6}$$

The values of x and y which satisfy Eq. (3.6) determine an integral curve of the differential equation (3.3). Writing (3.6) in the form:

$$G(x,y) = \int_b^y \psi(y)\,dy - \int_a^x \varphi(x)\,dx = 0$$

we verify by differentiation that

$$\frac{dy}{dx} = -\frac{\partial G/\partial x}{\partial G/\partial y} = \frac{\varphi(x)}{\psi(x)}$$

For $x = a$ and $y = b$ both integrals vanish; hence, $x = a$, $y = b$ is one point of the integral curve; in other words, (3.6) is

the equation of the integral curve passing through an arbitrary point $x = a$, $y = b$. The integral curves constitute a one-parameter family of curves. This is seen by choosing for a and b fixed values, for example, $a = 0$, $b = 0$. Then we can write Eq. (3.6) in the form:

$$\int_0^y \psi(y)\,dy - \int_0^x \varphi(x)\,dx = C \tag{3.7}$$

The relation (3.7) containing the arbitrary constant C is the *general solution* or *general integral* of the differential equation (3.3), since C can be determined in such a way that the integral curve passes through an arbitrary point.

Example: The Catenary.—We shall derive the equation of the so-called *catenary, i.e.*, the equilibrium configuration of a cable loaded by its own weight (Fig. 3.2). It follows from the equilibrium condition that the horizontal component H of the cable tension must be a constant. Denoting the inclination of the tangent by θ, the vertical component of the cable tension will be equal to $H \tan \theta$. The weight of the cable per unit length will be denoted by w; then the weight of an element ds, the horizontal projection of which is equal to dx, will be given by $w\,dx/\cos\theta$. The difference of the vertical components of the cable tension at x and $x + dx$ is equal to the weight of this element; hence,

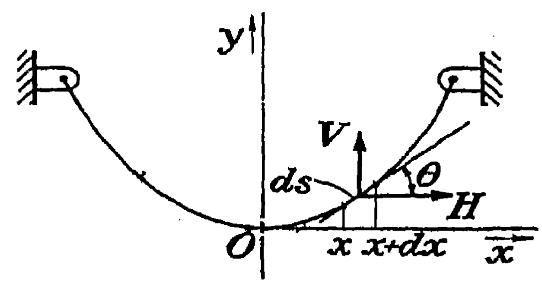

FIG. 3.2.—The catenary.

$$H \frac{d}{dx}(\tan\theta) = \frac{w}{\cos\theta}$$

or

$$\frac{d\theta}{dx} = \frac{w}{H}\cos\theta \tag{3.8}$$

Separating the variables, we obtain

$$\frac{d\theta}{\cos\theta} = \frac{w}{H}\,dx \tag{3.9}$$

and

$$\int_0^\theta \frac{d\theta}{\cos\theta} = \frac{w}{H}\int_0^x dx \tag{3.10}$$

where we assume that $\theta = 0$ when $x = 0$, *i.e.*, $x = 0$ corresponds to the lowest point of the catenary. Integrating both sides

$$\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{w}{H}x$$

or

$$\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = e^{\frac{wx}{H}}$$

Now

$$\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) - \cot\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = 2\tan\theta$$

Consequently,

$$\tan\theta = \frac{1}{2}\left(e^{\frac{wx}{H}} - e^{-\frac{wx}{H}}\right) \tag{3.11}$$

Introducing $y = y(x)$ as the equation of the catenary, we obtain from (3.11)

$$\frac{dy}{dx} = \tan\theta = \frac{1}{2}\left(e^{\frac{wx}{H}} - e^{-\frac{wx}{H}}\right)$$

and integrating both sides

$$y = \frac{H}{2w}\left(e^{\frac{wx}{H}} + e^{-\frac{wx}{H}}\right) + \text{const.} \tag{3.12}$$

Homogeneous Equation.—A homogeneous equation of the first order has the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \tag{3.13}$$

Equation (3.13) is reduced to a form in which the variables can be separated by the substitution $u = y/x$ or $y = xu$. We have by differentiation

$$\frac{dy}{dx} = x\frac{du}{dx} + u$$

and, therefore, Eq. (3.13) becomes

$$x\frac{du}{dx} = f(u) - u \tag{3.14}$$

The variables in this equation are separable; we have

$$\frac{du}{f(u) - u} = \frac{dx}{x}$$

Hence, putting

$$F(u) = \int_a^u \frac{du}{f(u) - u}$$

the general solution of (3.13) is given by

$$F\left(\frac{y}{x}\right) = \log x + C$$

4. Linear Differential Equations of the First Order.—The first-order linear equation represents a further fundamental case in which the solution can be obtained by quadratures. The general form of a first-order linear equation is

$$\frac{dy}{dx} = \varphi(x)y + \psi(x) \tag{4.1}$$

We show two methods of obtaining the solution of (4.1). The first method proceeds in the following way: We substitute in (4.1)

$$y = uv \tag{4.2}$$

where both u and v are unknown functions of x. Then from (4.1), we have

$$\frac{du}{dx} v + u \frac{dv}{dx} = \varphi(x)uv + \psi(x) \tag{4.3}$$

Obviously, we can choose arbitrarily one of the two functions u and v. Let us choose v such that

$$u \frac{dv}{dx} = \varphi(x)uv$$

i.e.,

$$\frac{1}{v} \frac{dv}{dx} = \varphi(x) \tag{4.4}$$

Then the rest of Eq. (4.3) can be solved by quadratures. We integrate (4.4) and write $\log v = \int_0^x \varphi(x)\, dx$ or $v = e^{\int_0^x \varphi(x)\, dx}$. Substituting this expression in Eq. (4.3), we obtain

$$\frac{du}{dx} = \frac{\psi(x)}{v} = e^{-\int_0^x \varphi(x)\, dx} \psi(x) \tag{4.5}$$

and by integration

$$u = \int_0^x e^{-\int_0^x \varphi(x)\, dx} \psi(x)\, dx + \text{const.} \tag{4.6}$$

Substituting (4.6) in (4.2), we obtain the general solution of Eq. (4.1)

$$y = e^{\int_0^x \varphi(x)\,dx}\left(\int_0^x e^{-\int_0^x \varphi(x)\,dx}\psi(x)\,dx + C\right) \tag{4.7}$$

Obviously, we could use any other lower limits in the integrals occurring in (4.7). However, it is easily shown that Eq. (4.7) represents the complete family of integral curves; *viz.*, if we vary the lower limits in the integrals occurring in (4.7), we obtain the same functions that we had before; they merely belong to other values of the arbitrary constant C.

The second method of solving Eq. (4.1) consists in finding the so-called *integrating factor* for this differential equation.

We write Eq. (4.1) in the form:

$$\frac{dy}{dx} - \varphi(x)y = \psi(x) \tag{4.8}$$

Then we try to determine a function $M(x)$ of the independent variable x such that the left side of (4.8) when multiplied by $M(x)$ represents the derivative of a function of x. We call such a function $M(x)$ an integrating factor of the differential equation.

Writing

$$M\frac{dy}{dx} - M\varphi(x)y = M\psi(x) \tag{4.9}$$

it is seen that the left side will be the differential quotient of My if

$$M\varphi(x) = -\frac{dM}{dx}, \qquad i.e., \qquad \frac{dM}{M} = -\varphi(x)\,dx$$

or by integration

$$\log M = -\int_a^x \varphi(x)\,dx + \text{const.}$$

Hence, for instance, $M = e^{-\int_0^x \varphi(x)\,dx}$ is an integrating factor of Eq. (4.8). Substituting this expression for M in Eq. (4.9), we obtain

$$\frac{d}{dx}\left(ye^{-\int_0^x \varphi(x)\,dx}\right) = \psi(x)e^{-\int_0^x \varphi(x)\,dx} \tag{4.10}$$

and

$$y = e^{\int_0^x \varphi(x)\,dx}\left(\int_0^x \psi(x)e^{-\int_0^x \varphi(x)\,dx} + C\right) \tag{4.11}$$

where C is a constant. It is seen that the expressions (4.7) and

(4.11) are identical. They represent the *general solution* or *general integral* of the differential equation (4.1).

As an example let us assume that $\varphi(x)$ is equal to a constant m. Then $\int_0^x \varphi(x)\,dx = mx$, and we obtain

$$y = e^{mx}\left(\int_0^x e^{-mx}\psi(x)\,dx + C\right) \tag{4.12}$$

5. Singular Solutions of First-order Equations.—If the function $f(x,y)$ in Eq. (3.1) is multivalued, in general more than one integral curve will go through an arbitrary point x,y. Let us consider, for example, the differential equation

$$\frac{dy}{dx} = \sqrt{1 - y} \tag{5.1}$$

Obviously dy/dx is real only if $y \leqq 1$; hence, the real integral curves cover only the portion of the plane for which $y \leqq 1$. If $y < 1$, we obtain two real values for the slope of the integral curves passing through the point x,y-corresponding to the positive and negative signs of the radical on the right side of (5.1). Separating the variables in Eq. (5.1), we obtain

$$\frac{dy}{\sqrt{1 - y}} = dx \tag{5.2}$$

and by integration

$$-2\sqrt{1 - y} = x - a$$

where a is an arbitrary constant. Thus we obtain

$$y = 1 - \tfrac{1}{4}(x - a)^2 \tag{5.3}$$

This equation represents a one-parameter family of parabolas (Fig. 5.1) which are all tangent to the straight line $y = 1$. The parabola corresponding to the particular value a of the arbitrary constant has its vertex at the point $x = a$, $y = 1$. Now it is seen that the straight line $y = 1$, which is the *envelope* of the family of the parabolas, also satisfies the differential equation (5.1), since in this case $dy/dx = 0$. However, this solution is not contained in the expression (5.3), *i.e.*, it cannot be obtained by a particular value of the parameter a. The expression (5.3) is called the *general solution* of (5.1); any solution corresponding

to a particular value of a is called a *particular solution.* The solution $y = 1$, because it is not included in the general solution, is called a *singular solution* or a *singular integral.* It was said that it represents the envelope of the family of the parabolas; it touches at every point one of the particular solutions.

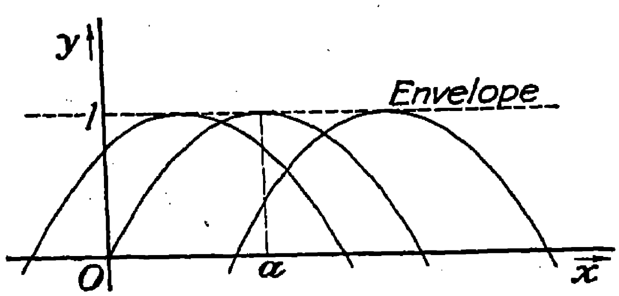

FIG. 5.1.—The envelope of the family of parabolas is a singular solution of the differential equation (5.1).

Equations of first order can have singular solutions only if the function $f(x,y)$ determining dy/dx is multivalued.*

6. Numerical Solution of Differential Equations of the First Order.—The analytical methods shown in the previous sections are restricted to certain classes of differential equations of the first order. Hence, there is a need for numerical methods which enable one to calculate particular solutions of any differential equation of the first order with a higher accuracy than can be done, for example, by the isocline method shown in section 3. Several such methods have been suggested by different mathematicians. They are in general *step-by-step* methods; those developed by I. C. Adams and by C. Runge jointly with R. Kutta are perhaps the most practical ones.

Let us consider the equation

$$\frac{dy}{dx} = f(x,y) \tag{6.1}$$

with the initial condition that $y = y_0$ at $x = x_0$. Then we try to find an approximate method for the calculation of the values of y for equidistant values of x, say

$$x_1 = x_0 + h, \qquad x_2 = x_0 + 2h,$$

and so on. Let us assume for the moment that we have determined the solution up to the point $x_n = x_0 + nh$. Then the crudest approximation for y_{n+1} corresponding to

$$x_{n+1} = x_0 + (n + 1)h$$

* More information on singular solutions may be found in any treatise on differential equations, such as, for instance, H. T. H. Piaggio, "An Elementary Treatise on Differential Equations and Their Applications."

will be

$$y_{n+1} = y_n + q_n h \tag{6.2}$$

where q_n is the value of dy/dx for $x = x_n$ and $y = y_n$, *i.e.*, $q_n = f(x_n, y_n)$. We propose to refine this approximation by assuming that portions of the function $y(x)$ can be approximated fairly well by portions of polynomials. Let us use, for instance, polynomials of the second degree. This means for the graphical representation of $y = y(x)$ that portions of the curve $y = y(x)$ are replaced by parabolic arcs. A parabola being determined by three points, let us assume that the points x_{n-1}, y_{n-1}; x_n, y_n; and x_{n+1}, y_{n+1} belong to one parabola and that the equation of the parabola is

$$y = y_n + a(x - x_n) + b(x - x_n)^2 \tag{6.3}$$

In order to determine the coefficients a and b in Eq. (6.3), let us express the values q_n, q_{n-1} of dy/dx at $x = x_n$ and $x = x_{n-1}$ in terms of a and b. Obviously,

$$\frac{dy}{dx} = a + 2b(x - x_n) \tag{6.4}$$

and

$$q_n = a, \qquad q_{n-1} = a - 2bh \tag{6.5}$$

Hence,

$$a = q_n \qquad \text{and} \qquad b = \frac{q_n - q_{n-1}}{2h}$$

and Eq. (6.3) becomes

$$y = y_n + q_n(x - x_n) + \frac{q_n - q_{n-1}}{2h}(x - x_n)^2 \tag{6.6}$$

We now use Eq. (6.6) to compute y_{n+1}:

$$y_{n+1} - y_n = \left(q_n + \frac{q_n - q_{n-1}}{2}\right) h \tag{6.7}$$

Equation (6.7) evidently represents a refinement of the first approximation (6.2).

The first ordinate that can be computed by means of Eq. (6.7) is y_2, provided we know, besides the initial value y_0 for $x = x_0$, the value y_1 for $x_1 = x_0 + h$ and the first derivatives of y for $x = x_0$ and $x_1 = x_0 + h$. We calculate these data approximately by means of the Taylor series

$$y = y_0 + \left(\frac{dy}{dx}\right)_0 (x - x_0) + \frac{1}{2}\left(\frac{d^2y}{dx^2}\right)_0 (x - x_0)^2 + \cdots \tag{6.8}$$

and

$$\frac{dy}{dx} = \left(\frac{dy}{dx}\right)_0 + \left(\frac{d^2y}{dx^2}\right)_0 (x - x_0) + \cdots \tag{6.9}$$

by substituting successively $x - x_0 = 0$ and then $x - x_0 = h$. The coefficients in these series can be determined by repeated differentiation of the function $f(x,y)$. Evidently

$$\begin{aligned} \left(\frac{dy}{dx}\right)_0 &= f(x_0,y_0) \\ \left(\frac{d^2y}{dx^2}\right)_0 &= \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right)_0 = \left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right)_0 \end{aligned} \tag{6.10}$$

and so on. The subscript means that the values of the function f and of its derivatives are to be taken for $x = x_0$ and $y = y_0$. Calculating in this way $y_1 = y(x_0 + h)$, $q_0 = (dy/dx)_{x=x_0}$, $q_1 = (dy/dx)_{x=x_0+h}$, we obtain from Eq. (6.7)

$$y_2 - y_1 = \left(q_1 + \frac{q_1 - q_0}{2}\right)h \tag{6.11}$$

Now y_2 being known, we compute q_2 from

$$q_2 = f(x_0 + 2h, y_2)$$

Hence, we can continue the calculation by writing, according to (6.7),

$$y_3 - y_2 = \left(q_2 + \frac{q_2 - q_1}{2}\right)h \tag{6.12}$$

and this procedure can be continued until the desired range of integration is covered.

In order to obtain better approximations, polynomials of higher (4th or 5th) degree will be used. Using, for instance, a polynomial of 5th degree, the reader will verify by similar elementary calculation that the equation which gives the difference $y_{n+1} - y_n$ expressed by the values of the slope at five equidistant points x_n, $x_n - h$, $x_n - 2h$, $x_n - 3h$, $x_n - 4h$ is of the form:

$$\begin{aligned} y_{n+1} - y_n = h(q_n &+ \tfrac{1}{2}\,\Delta q_{n-1} + \tfrac{5}{12}\,\Delta^2 q_{n-2} + \tfrac{3}{8}\,\Delta^3 q_{n-3} \\ &+ \tfrac{251}{720}\,\Delta^4 q_{n-4}) \end{aligned} \tag{6.13}$$

where the differences Δq, $\Delta^2 q$, $\Delta^3 q$, and $\Delta^4 q$ are defined by

$$\begin{aligned} \Delta q_r &= q_{r+1} - q_r \\ \Delta^2 q_r &= \Delta q_{r+1} - \Delta q_r \\ \Delta^3 q_r &= \Delta^2 q_{r+1} - \Delta^2 q_r \\ \Delta^4 q_r &= \Delta^3 q_{r+1} - \Delta^3 q_r \end{aligned} \tag{6.14}$$

This generalizes Eq. (6.7).

Assuming that we know y_0, y_1, y_2, y_3, y_4 at five equidistant points, x_0, $x_0 + h$, $x_0 + 2h$, $x_0 + 3h$, $x_0 + 4h$, we calculate the differences (6.14), and applying (6.13), we obtain the value of y_5 at the point $x + 5h$. Repeating now the same procedure for the five points $x_0 + h$, $x_0 + 2h$, $x_0 + 3h$, $x_0 + 4h$, $x_0 + 5h$, we find y_6. By successive application of the procedure we determine all other ordinates y_7, y_8, . . . step by step. To start the procedure five initial ordinates at equidistant points must be calculated. This is done by a Taylor series expansion as illustrated in the example below.

The formula (6.13) has the advantage that the sequence of coefficients is independent of the degree of the polynomials used. It is seen that we obtain Eq. (6.7) by retaining the first two terms in parentheses on the right side of Eq. (6.13). If we want to use polynomials of the 4th order, for instance, we can cut off the expression on the right side of (6.13) after the fourth term.

Example.—We shall apply Adams' method to the numerical integration of the differential equation

$$\frac{dy}{dx} = x + y$$

with the initial condition $y = 0$ at $x = 0$. We first calculate Taylor's expansion of the solution in the vicinity of $x = 0$. The first derivative is

$$\frac{dy}{dx} = x + y$$

The second derivative is found by differentiating this equation with respect to x:

$$\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx} = 1 + x + y$$

Differentiating again,

$$\frac{d^3y}{dx^3} = 1 + \frac{dy}{dx} = 1 + x + y$$

$$\frac{d^4y}{dx^4} = 1 + \frac{dy}{dx} = 1 + x + y$$

Using the values of these derivatives at $x = 0$, $y = 0$, we find Taylor's expansion

$$y = \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4$$

This gives a good approximation of the solution up to $x = 0.4$. We use as increment $h = 0.1$ and compute from Taylor's expansion the values of the function

$$\begin{aligned} y_1 &= 0.0051 &\quad \text{for} \quad x &= 0.1 \\ y_2 &= 0.0213 &\quad \text{for} \quad x &= 0.2 \\ y_3 &= 0.0498 &\quad \text{for} \quad x &= 0.3 \\ y_4 &= 0.0916 &\quad \text{for} \quad x &= 0.4 \end{aligned}$$

We also compute the corresponding values of $q = f(x,y) = x + y$ and its four first differences. We find in particular

$$\begin{aligned} q_4 &= 0.4916 \\ \Delta q_3 &= 0.1418 \\ \Delta^2 q_2 &= 0.0133 \\ \Delta^3 q_1 &= 0.0010 \\ \Delta^4 q_0 &= -0.0002 \end{aligned}$$

Substituting these values in expression (6.13), we obtain

$$y_5 - y_4 = 0.0568$$

and then proceed similarly for the following steps. The calculation up to $x = 1$ is recorded in the following table. The result is compared with the exact solution: $y = e^x - x - 1$, which can be easily obtained from Eq. (4.12). (Cf. section 10, Example *a*.)

By comparison with the exact value $e^x - x - 1$, it can be seen that the error on the numerically calculated value of y is of the order of 0.0001. The irregular character of the $\Delta^4 q$ column shows that these values are inexact; however, their influence on the value of y is very small.

The procedure sketched above corresponds to I. C. Adams' method. The procedure of Runge and Kutta is somewhat different. They express the difference $y_n - y_{n-1}$ by linear combination

of values of $q = dy/dx$ at certain suitably chosen points within the intervals instead of using the values of q corresponding to the end points of the intervals. For practical use of either of these methods the references on numerical methods at the end of this chapter should be consulted.

x	Exact $e^x - x - 1$	Numerical y	$q = x + y$	Δq	$\Delta^2 q$	$\Delta^3 q$	$\Delta^4 q$
0	0	0	0				
				0.1051			
0.1	0.0052	0.0051	0.1051		0.0111		
				0.1162		0.0012	
0.2	0.0214	0.0213	0.2213		0.0123		−0.0002
				0.1285		0.0010	
0.3	0.0499	0.0498	0.3498		0.0133		0.0007
				0.1418		0.0017	
0.4	0.0918	0.0916	0.4916		0.0150		0.0003
				0.1568		0.0020	
0.5	0.1487	0.1484	0.6484		0.0170		−0.0011
				0.1738		0.0009	
0.6	0.2221	0.2222	0.8222		0.0179		0.0012
				0.1917		0.0021	
0.7	0.3138	0.3139	1.0139		0.0200		0.0003
				0.2117		0.0024	
0.8	0.4255	0.4256	1.2256		0.0224		
				0.2341			
0.9	0.5596	0.5597	1.4597				
1.0	0.7183	0.7184					

The methods of Adams or Runge and Kutta can be extended to a system of simultaneous equations of the first order. This is of great practical importance, since—as will be shown in the next section—a differential equation containing higher order derivatives of the unknown function can be replaced by a system of simultaneous equations of the first order.

In addition to numerical methods various mechanical devices are available for the solution of differential equations. For example, the so-called *differential analyzer*, developed by V. Bush,* has been shown to be an extremely useful tool for solving

* V. Bush, "The Differential Analyzer. A New Machine for Solving Differential Equations," *Journal of the Franklin Institute*, Vol. 212, 1931.

engineering problems involving differential equations or systems of differential equations.

7. Differential Equations of Higher Order. General Remarks. The order of a differential equation means the order of the highest derivative occurring in the equation. The equation

$$\frac{d^3y}{dx^3} = f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) \tag{7.1}$$

is, for instance, a differential equation of third order. It is equivalent to a system of three linear differential equations of the first order. We can introduce the derivatives dy/dx and d^2y/dx^2 as new *auxiliary* variables z and u. Then the following system of equations:

$$\begin{aligned} \frac{dy}{dx} &= z \\ \frac{dz}{dx} &= u \\ \frac{du}{dx} &= f(x, y, z, u) \end{aligned} \tag{7.2}$$

is equivalent to Eq. (7.1). To solve Eqs. (7.2) by a step-by-step method, we plot the functions $y(x)$, $z(x)$, $u(x)$ as curves in three planes, using x as abscissa and y, z, u as ordinates, respectively. Starting from the arbitrary initial values $x = x_0$, $y = y_0$, $z = z_0$, $u = u_0$, the slopes of the integral curves in all three planes are given by (7.2). Proceeding from x_0 to $x_0 + \Delta x$ with the slopes given by (7.2), we obtain new values of y, z, u, and, consequently, we can calculate by means of Eqs. (7.2) the new values for the three respective slopes at the new points. The solution obtained in this way will depend on three arbitrary parameters, *viz.*, y_0, u_0, z_0, or y_0, $(dy/dx)_0$, $(d^2y/dx^2)_0$. In other words, the solution of the equation considered is given if the initial values of the ordinate, the initial slope, and the initial curvature are known. In the case of an equation of the order n the general solution will depend on n arbitrary constants; for instance, on the initial values of the function $y(x)$ and its first $n - 1$ derivatives.

We shall not discuss the case where the initial values of the unknown function and its first $n - 1$ derivatives do not determine a unique solution. This occurs, for example, if the highest

derivative is not a single-valued finite function of x, y and of the lower derivatives.

The method applied above to substitute a system of equations of the first order for one differential equation of higher order can be readily transferred to the case of simultaneous differential equations of higher order. The reader will verify that if we have m equations of the order $n_1, n_2, \ldots, n_m$, the total number of the equations of first order will be $n_1 + n_2 + \cdots + n_m$. Hence, the general solution will depend on the same number of arbitrary constants. The solution of a system of the form:

$$\begin{aligned} \frac{d^2y}{dx^2} &= f\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right) \\ \frac{d^2z}{dx^2} &= g\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right) \end{aligned} \tag{7.3}$$

is determined by four constants; for instance, by the values of y, z, dy/dx, dz/dx corresponding to a certain abscissa x_0.

If the second derivative d^2y/dx^2 is given as function of dy/dx and y only, for example,

$$\frac{d^2y}{dx^2} = f\left(\frac{dy}{dx}, y\right) \tag{7.4}$$

such an equation can be integrated in two steps. Introducing $q = \frac{1}{2}(dy/dx)^2$ as the new unknown variable and y as the independent variable, we have

$$\frac{dq}{dx} = \frac{dy}{dx}\frac{d^2y}{dx^2}$$

or

$$\frac{d^2y}{dx^2} = \frac{dq}{dx}\frac{dx}{dy} = \frac{dq}{dy} \tag{7.5}$$

Hence, Eq. (7.4) can be written in the form:

$$\frac{dq}{dy} = f(\sqrt{2q}, y) \tag{7.6}$$

The general solution of this equation of first order,

$$q = \varphi(y, C) = \frac{1}{2}($$

where C is a constant of integratio

of the differential equation of s

For example, if we have the equation

$$\frac{d^2y}{dx^2} + \beta\left(\frac{dy}{dx}\right)^2 + ky = 0 \tag{7.8}$$

we obtain by the substitution $q = \frac{1}{2}(dy/dx)^2$

$$\frac{dq}{dy} = -2\beta q - ky \tag{7.9}$$

The general solution of (7.9) is, according to Eq. (4.12),

$$q = e^{-2\beta y}\left(-k\int_0^y e^{2\beta y}y\,dy + C\right)$$

or

$$q = Ce^{-2\beta y} + \frac{k}{4\beta^2} - \frac{k}{2\beta}y$$

Thus we have

$$\frac{dy}{dx} = \sqrt{\frac{k}{2\beta^2} - \frac{k}{\beta}y + 2Ce^{-2\beta y}} \tag{7.10}$$

which is a first integral of the differential equation (7.8). Another integration gives y as function of x.

Initial Conditions and Boundary Conditions.—When all the constants of the general solution of a differential equation are determined by conditions at one point these conditions are called *initial conditions.* For instance, in the case of a second-order equation the integral curve is entirely determined by the value of the function and the slope for a given abscissa. In many problems the constants are determined by other types of conditions. For instance, it may be imposed upon the integral curve of a second-order equation to pass through two given points. Such conditions are called *boundary conditions.* From the standpoint of numerical and approximate methods of integration these two types of problems must be treated quite differently. The difference will appear more clearly from the study of Chapter IV, which is chiefly concerned with *initial-value problems*, and of Chapter VII, which deals with problems involving *boundary conditions.*

8. Linear Differential Equations of the First and Second Order with Constant Coefficients.—In the following sections of this chapter we are concerned with *linear differential equations.*

The general form of a linear differential equation of the order n is

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)y = \psi(x) \tag{8.1}$$

where $a_1(x), \ldots, a_n(x)$ and $\psi(x)$ are given functions of the independent variable x. If $\psi(x) = 0$, the equation is called *homogeneous*. The functions $a_1(x), \ldots, a_n(x)$ are called the *coefficients* of the linear differential equation. If they are constants, we say that Eq. (8.1) is a linear differential equation of the order n with constant coefficients.

In section 11 we shall discuss some general properties of linear differential equations of any order, and in section 12 we shall give the methods for solving such equations with constant coefficients. As an introduction to the special methods involved in solving such equations, we treat in this section the case of linear equations of the first and second order with constant coefficients. Such equations occur very frequently in engineering problems, so that a discussion at some length is justified.

*The Homogeneous First-order Equation.**—This is a particular case of Eq. (4.1) for which $\varphi(x) = \text{const.}$ and $\psi(x) = 0$. Thus we write

$$\frac{dy}{dx} + ay = 0 \tag{8.2}$$

Obviously, by separation of the variables,

$$\frac{dy}{y} + a\,dx = 0$$

or

$$y = \text{const. } e^{-ax}$$

In order to develop a method that can be applied to equations of higher order we shall deduce the same result in another way. Remembering that the derivative of an exponential function is again an exponential function, we try a solution of the form:

$$y = Ce^{\lambda x}$$

* The reader will notice that the expression *homogeneous equation* was used before in section 3 with a different meaning. In that case the expression referred to homogeneity with respect to both variables x and y, while here it refers only to y.

where C and λ are undetermined constants. Substituting this expression in Eq. (8.2), we obtain

$$Ce^{\lambda x}(\lambda + a) = 0$$

It is seen that the equation is satisfied for an arbitrary value of the constant C, provided that

$$\lambda + a = 0 \tag{8.3}$$

We call Eq. (8.3) the *characteristic equation* of the linear differential equation (8.2).

The expression

$$y = Ce^{-ax} \tag{8.4}$$

represents the general solution of Eq. (8.2) since it contains an arbitrary constant C, *i.e.*, it is a general expression for the one-parameter family of integrals of Eq. (8.2).

The equation $\dfrac{d^2y}{dx^2} + by = 0$.—A further step will be to consider the equation of the second order:

$$\frac{d^2y}{dx^2} + by = 0 \tag{8.5}$$

By trying a solution of the form const. $e^{\lambda x}$ as before, we find the characteristic equation

$$\lambda^2 + b = 0 \tag{8.6}$$

Obviously, two cases must be considered. If $b < 0$, the roots $\sqrt{-b}$ and $-\sqrt{-b}$ are real; if $b > 0$, imaginary.

In the first case the solution corresponding to $\sqrt{-b}$ is $y = C_1e^{\sqrt{-b}x}$, where C_1 is an arbitrary constant. We obtain a second distinct solution using the root $-\sqrt{-b}$; *viz.*,

$$y = C_2e^{-\sqrt{-b}x}$$

where C_2 is a second arbitrary constant. Substituting the sum

$$y = C_1e^{\sqrt{-b}x} + C_2e^{-\sqrt{-b}x} \tag{8.7}$$

in Eq. (8.5), we see that (8.7) also satisfies the differential equation. The expression (8.7) is the general solution of Eq. (8.5) in the sense explained in section 7, because the constants can be

adjusted so as to give the function y and its first derivative any initial values. If $b > 0$, λ is imaginary, *i.e.*, Eq. (8.5) cannot be solved by exponential functions. However, remembering the property of the trigonometric function, *viz.*, that their derivatives are again trigonometric functions, we tentatively substitute $y = C_1 \cos \mu x$ or $y = C_2 \sin \mu x$. Then we obtain from (8.5) $-\mu^2 + b = 0$ or $\mu = \sqrt{b}$; C_1 and C_2 remain arbitrary. Hence, when b is positive, the general solution of Eq. (8.5) is

$$y = C_1 \cos \sqrt{b}x + C_2 \sin \sqrt{b}x \tag{8.8}$$

These two distinct cases where $b < 0$ and $b > 0$ can be formally merged into one, if we consider the complex expression

$$f(\varphi) = \cos \varphi + i \sin \varphi \tag{8.9}$$

where $i = \sqrt{-1}$. We readily verify that this function satisfies the following relations:

$$(\cos \varphi_1 + i \sin \varphi_1)(\cos \varphi_2 + i \sin \varphi_2) = \cos (\varphi_1 + \varphi_2) + i \sin (\varphi_1 + \varphi_2)$$

and

$$\frac{d}{d\varphi} (\cos \varphi + i \sin \varphi) = i(\cos \varphi + i \sin \varphi)$$

or

$$f(\varphi_1)f(\varphi_2) = f(\varphi_1 + \varphi_2) \qquad \text{and} \qquad \frac{d}{d\varphi} [f(\varphi)] = if(\varphi)$$

Now the exponential function $e^{\beta x}$ satisfies the same relations

$$e^{\beta x_1} e^{\beta x_2} = e^{\beta x_1 + \beta x_2}$$

$$\frac{d}{dx} (e^{\beta x}) = \beta e^{\beta x}$$

Hence, if we use the notation $e^{i\varphi}$ for $f(\varphi) = \cos \varphi + i \sin \varphi$, we can carry out all calculations for $f(\varphi) = e^{i\varphi}$ using the known rules for exponential functions with real exponents.

A deeper reason for these properties appears by considering the power series expansions of the exponential and the trigonometric functions. We have

$$\sin \varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \cdots$$

$$\cos \varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \cdots$$

Hence,

$$(\cos\varphi + i\sin\varphi) = \left(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \cdots\right) + i\left(\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \cdots\right)$$

or, after rearrangement of the terms,

$$e^{i\varphi} = (\cos\varphi + i\sin\varphi) = 1 + (i\varphi) + \frac{(i\varphi)^2}{2!} + \frac{(i\varphi)^3}{3!} + \frac{(i\varphi)^4}{4!} + \frac{(i\varphi)^5}{5!} + \cdots \tag{8.10}$$

It is seen that the infinite series on the right side can be formally obtained by expanding e^x in a power series and substituting

$$x = i\varphi.$$

Let us now consider again Eq. (8.5) when b is positive. We find that the characteristic equation

$$\lambda^2 + b = 0$$

has two purely imaginary solutions, *viz.*,

$$\lambda_1 = i\sqrt{b}$$
$$\lambda_2 = -i\sqrt{b}$$

Hence, $e^{i\sqrt{b}x}$ and $e^{-i\sqrt{b}x}$ are solutions of the equation (8.5).

The meaning of this statement is that if we substitute any one of these functions into the differential equation, we obtain identically zero, *i.e.*, the real part and the imaginary part of any of the functions satisfy the differential equation. Since $e^{i\sqrt{b}x}$ and $e^{-i\sqrt{b}x}$ are conjugate complex, we obtain the real and the imaginary parts of $e^{i\sqrt{b}x}$ by the formulas

$$\begin{aligned} \cos\sqrt{b}x &= \frac{e^{i\sqrt{b}x} + e^{-i\sqrt{b}x}}{2} \\ \sin\sqrt{b}x &= \frac{e^{i\sqrt{b}x} - e^{-i\sqrt{b}x}}{2i} \end{aligned} \tag{8.11}$$

Hence, it is verified that $\cos\sqrt{b}x$ and $\sin\sqrt{b}x$ are solutions of (8.5), and we may write the general solution in the real form (8.8) given above.

9. Hyperbolic Functions.—By the definition

$$e^{ix} = \cos x + i \sin x$$

we established a relation between the trigonometric functions $\cos x$ and $\sin x$ and the *exponential function with imaginary exponent* e^{ix}. The trigonometric functions can be expressed in the form:

$$\begin{aligned} \cos x &= \frac{e^{ix} + e^{-ix}}{2} \\ \sin x &= \frac{e^{ix} - e^{-ix}}{2i} \end{aligned} \tag{9.1}$$

By analogy with (9.1) we shall derive two new functions from the exponential function with a real exponent e^x, *viz.*,

$$\begin{aligned} \cosh x &= \frac{e^x + e^{-x}}{2} \\ \sinh x &= \frac{e^x - e^{-x}}{2} \end{aligned} \tag{9.2}$$

We shall show that there is a remarkable analogy between the properties of these functions and of the trigonometric functions.

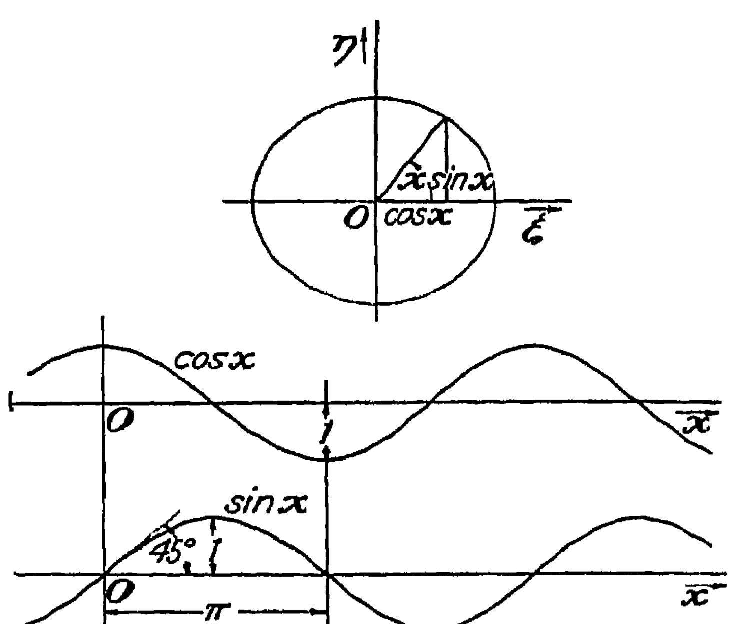

FIG. 9.1.—The circular functions $\cos x$ and $\sin x$.

If ξ and η are rectangular coordinates in a plane and we put

$$\begin{aligned} \xi &= \cos x \\ \eta &= \sin x \end{aligned} \tag{9.3}$$

from

$$\cos^2 x + \sin^2 x = 1 \tag{9.4}$$

it follows that

$$\xi^2 + \eta^2 = 1 \tag{9.5}$$

Equation (9.5) is the equation of a circle of unit radius, and, therefore, Eqs. (9.3) are the parametric equations of this circle with the angle x as parameter (Fig. 9.1). For this reason the trigonometric functions are sometimes called *circular functions.*

Now from Eqs. (9.2) we have

$$\cosh^2 x - \sinh^2 x = 1 \tag{9.6}$$

Hence, if we put

$$\begin{aligned} \xi &= \cosh x \\ \eta &= \sinh x \end{aligned} \tag{9.7}$$

we obtain

$$\xi^2 - \eta^2 = 1 \tag{9.8}$$

This is the equation of a hyperbola whose semiaxis is equal to 1 and whose asymptotes are straight lines at 45° with the axes ξ and η. Therefore, Eqs. (9.7) are the parametric equations of this hyperbola (Fig. 9.2). For this reason the functions defined by (9.2) are called *hyperbolic functions;* $\cosh x$, the *hyperbolic cosine;* $\sinh x$, the *hyperbolic sine* of the variable x.

We can easily verify by means of the definitions (9.2) the following identities for the hyperbolic functions:

$$\begin{aligned} \cosh (x_1 + x_2) &= \cosh x_1 \cosh x_2 + \sinh x_1 \sinh x_2 \\ \sinh (x_1 + x_2) &= \sinh x_1 \cosh x_2 + \cosh x_1 \sinh x_2 \end{aligned} \tag{9.9}$$

and

$$\begin{aligned} \frac{d}{dx} \cosh x &= \sinh x \\ \frac{d}{dx} \sinh x &= \cosh x \end{aligned} \tag{9.10}$$

These rules become formally identical with the corresponding rules valid for trigonometric functions when we write the hyperbolic functions as trigonometric functions with an imaginary variable. Substituting $x = iy$ in Eqs. (9.1), we obtain

$$\begin{aligned} \cos (iy) &= \frac{e^{-y} + e^{y}}{2} = \cosh y \\ \sin (iy) &= \frac{e^{-y} - e^{y}}{2i} = i \sinh y \end{aligned} \tag{9.11}$$

On the other hand, the circular functions can be expressed as hyperbolic functions with an imaginary argument. With $x = iy$ we obtain from Eqs. (9.1)

$$\begin{aligned} \cosh (iy) &= \frac{e^{iy} + e^{-iy}}{2} = \cos y \\ \sinh (iy) &= \frac{e^{iy} - e^{-iy}}{2} = i \sin y \end{aligned} \tag{9.12}$$

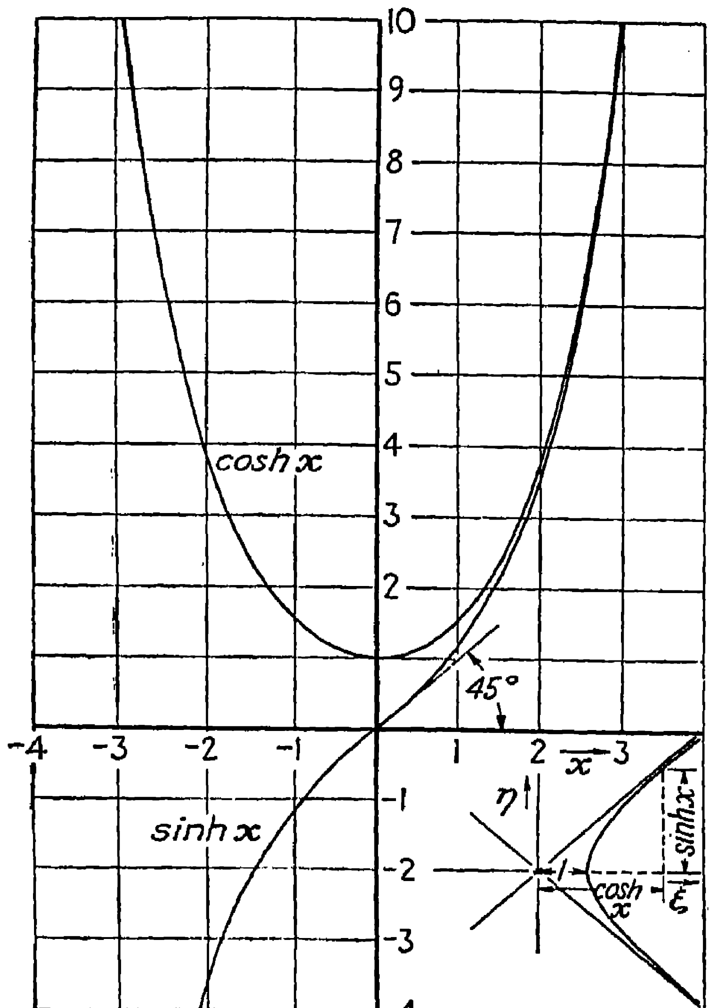

FIG. 9.2.—The hyperbolic functions cosh x and sinh x.

Using these relations, for example, the first identity (9.9) can be written

$$\cos (ix_1 + ix_2) = \cos (ix_1) \cos (ix_2) - \sin (ix_1) \sin (ix_2)$$

which formally coincides with the well-known formula for trigonometric functions. The first differentiation rule (9.10) becomes

$$\frac{d}{dx} \cos (ix) = -i \sin (ix)$$

which again is in accordance with the formal rules for differentiation of the cosine function. The hyperbolic cosine is an even function equal to unity at $x = 0$. The hyperbolic sine is an odd function equal to zero at $x = 0$ and starting with a slope equal to unity. These functions are plotted in Fig. 9.2.*

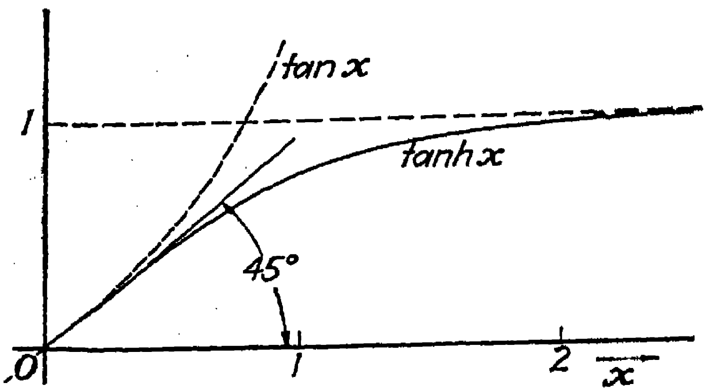

FIG. 9.3.—The hyperbolic function tanh x.

Both cosh x and sinh x tend asymptotically toward $\frac{1}{2}e^x$ for large values of x, so that

$$\lim_{x \to \infty} \frac{\sinh x}{\cosh x} = 1$$

The *hyperbolic tangent* is defined in a way analogous to the circular tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{9.13}$$

It is plotted in Fig. 9.3.

The hyperbolic functions may be used to express the general solution of the differential equation (8.5). When $b < 0$, we write, with $b = -k^2$,

$$\frac{d^2y}{dx^2} - k^2y = 0 \tag{9.14}$$

The general solution of (9.14) is a linear combination of e^{kx} and e^{-kx} which may also be written as

$$y = C_1 \cosh kx + C_2 \sinh kx \tag{9.15}$$

Accurate values of the hyperbolic functions may be found in Hayashi's tables, "Kreis und Hyperbelfunktionen."

Also, "Tables of Hyperbolic Functions," are published by the Smithsonian Institution, Washington.

This is now in formal analogy with the solution of Eq. (8.5) for the case $b > 0$. Writing $b = k^2$, we have

$$\frac{d^2y}{dx^2} + k^2y = 0 \tag{9.16}$$

The general solution of this equation is

$$y = C_1 \cos kx + C_2 \sin kx$$

10. Linear Differential Equations of the Second Order with Constant Coefficients—(*Continued*).—Let us consider now the second-order equation

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \tag{10.1}$$

Trying again a solution of the form $y = \text{const. } e^{\lambda x}$, we find the characteristic equation of Eq. (10.1)

$$\lambda^2 + a\lambda + b = 0 \tag{10.2}$$

Three cases must be examined separately:

a. The roots are real and distinct.

b. The roots are real and equal.

c. The roots are complex.

a. First we shall assume that the roots are real and distinct, and write

$$\begin{aligned} \lambda_1 &= -\frac{a}{2} + \sqrt{\frac{a^2}{4} - b} \\ \lambda_2 &= -\frac{a}{2} - \sqrt{\frac{a^2}{4} - b} \end{aligned} \tag{10.3}$$

In this case we find two distinct solutions $C_1e^{\lambda_1 x}$ and $C_2e^{\lambda_2 x}$. Hence, the general solution is given by

$$y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x} \tag{10.4}$$

b. If the two roots are equal, $a^2/4 = b$ and $\lambda_1 = \lambda_2 = -a/2$. In this case we get only one solution of the form $Ce^{\lambda x}$, *viz.*,

$$y = Ce^{-\frac{ax}{2}} \tag{10.5}$$

However, we know that the general solution must contain two arbitrary constants, hence, there must be particular solutions

which are not included in Eq. (10.5). To obtain the general solution, we write tentatively

$$y = ue^{-\frac{ax}{2}} \tag{10.6}$$

where u is an undetermined function of x. Substituting expression (10.6) in Eq. (10.1), which in this case has the form

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + \frac{a^2}{4}y = 0 \tag{10.7}$$

we obtain

$$e^{-\frac{ax}{2}}\left(\frac{d^2u}{dx^2} - a\frac{du}{dx} + \frac{a^2}{4}u + a\frac{du}{dx} - \frac{a^2}{2}u + \frac{a^2}{4}u\right) = 0$$

or

$$\frac{d^2u}{dx^2} = 0$$

Hence, $u = C_1 + C_2x$ where C_1 and C_2 are arbitrary constants. The expression

$$y = (C_1 + C_2x)e^{-\frac{ax}{2}} \tag{10.8}$$

represents the general solution of Eq. (10.7), since it contains two arbitrary constants. We found the solution $C_1e^{-\frac{a}{2}x}$ before by means of the substitution $y = Ce^{\lambda x}$ [cf. Eq. (10.5)]. The function $y = xe^{-\frac{a}{2}x}$ is a particular solution that is not contained in (10.5).

c. If the roots are imaginary, we again try a solution of the form $y = ue^{-\frac{a}{2}x}$, and by putting this value into Eq. (10.1), we find

$$\frac{d^2u}{dx^2} + \left(b - \frac{a^2}{4}\right)u = 0$$

Since $b - \frac{a^2}{4} > 0$, we obtain $u = C_1 \cos \beta x + C_2 \sin \beta x$ where $\beta = \sqrt{b - \frac{a^2}{4}}$ (cf. section 8). Hence, putting $\frac{a}{2} = -\alpha$ the general solution of (10.1) has the form:

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) \tag{10.9}$$

The cases *a* and *c* can be formally merged into one by using exponential functions with complex exponents. We define the

exponential function with the complex exponent by the relation

$$e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}(\cos \beta x + i \sin \beta x) \tag{10.10}$$

In the case where the roots of the characteristic equations (10.2) are complex, we have

$$\lambda_1 = -\frac{a}{2} + i\sqrt{b - \frac{a^2}{4}} = \alpha + i\beta$$

$$\lambda_2 = -\frac{a}{2} - i\sqrt{b - \frac{a^2}{4}} = \alpha - i\beta$$

We have, therefore, two solutions of Eq. (10.1): $e^{(\alpha+i\beta)x}$ and $e^{(\alpha-i\beta)x}$. Their linear combinations,

$$\frac{1}{2}(e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x}) = e^{\alpha x} \cos \beta x$$

$$\frac{1}{2i}(e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}) = e^{\alpha x} \sin \beta x$$

which give the real and the imaginary parts of the two conjugate complex functions, are also solutions. Hence, the general solution of Eq. (10.1) can be written in the form of Eq. (10.9).

The Nonhomogeneous Equation $A\,\frac{d^2y}{dx^2} + B\,\frac{dy}{dx} + Cy = \psi(x)$.—Let us consider now the case of the nonhomogeneous equation

$$\frac{d^2y}{dx^2} + a\,\frac{dy}{dx} + by = \psi(x) \tag{10.11}$$

where $\psi(x)$ is a given function of x. We shall show that if we know a particular solution of this equation, in other words, a function $\eta(x)$, such that

$$\frac{d^2\eta(x)}{dx^2} + a\,\frac{d\eta(x)}{dx} + b\eta(x) = \psi(x) \tag{10.12}$$

then the general solution of (10.11) is obtained by adding $\eta(x)$ to the general solution of the homogeneous equation (10.1) which we call the *homogeneous equation associated with Eq.* (10.11). In other words, if $C_1y_1(x) + C_2y_2(x)$ is the general solution of this homogeneous equation, then

$$y(x) = C_1y_1(x) + C_2y_2(x) + \eta(x) \tag{10.13}$$

is the general solution of the nonhomogeneous equation (10.11). That the expression (10.13) satisfies the differential equa-

tion (10.11) can be immediately verified by substituting it in (10.11). The left side of Eq. (10.11) then becomes $\frac{d^2\eta}{dx^2} + a\frac{d\eta}{dx} + b\eta$, and this expression according to (10.12) is equal to $\psi(x)$. Moreover, (10.13) is the general solution of the nonhomogeneous equation because the two arbitrary constants appearing in (10.13) can be determined in such a way that any arbitrary initial value at a given point can be chosen for the function and its first derivative.

Example.—*a.* Consider the equation

$$\frac{dy}{dx} = x + y \tag{10.14}$$

which we have integrated numerically by Adams' method in section 6.

The associated homogeneous equation is

$$\frac{dy}{dx} - y = 0$$

of which the general solution is Ce^x. To find a particular solution of Eq. (10.14) we might apply Eq. (4.12), but it is easier to try a linear function of the form $ax + b$. We find by substitution in Eq. (10.14)

$$a = b = -1$$

Therefore, $\eta = -(x + 1)$ is a particular solution of Eq. (10.14), and the general solution is

$$y = Ce^x - x - 1$$

b. The nonhomogeneous equation

$$\frac{d^2y}{dx^2} - y = x + 3x^2 \tag{10.15}$$

will have a particular solution of the form

$$\eta = \alpha + \beta x + \gamma x^2$$

By substitution of this expression in the equation, we find

$$2\gamma - \alpha - \beta x - \gamma x^2 = x + 3x^2$$

Therefore,

$$\begin{aligned} 2\gamma - \alpha &= 0 \\ \beta &= -1 \\ \gamma &= -3 \end{aligned}$$

Hence,

$$\eta = -(6 + x + 3x^2)$$

is a particular solution, and the general solution is

$$y = C_1e^x + C_2e^{-x} - (6 + x + 3x^2) \tag{10.16}$$

Now we can verify by substitution that, for example,

$$\eta = -(6 + x + 3x^2) + \cosh x$$

is also a particular solution of the nonhomogeneous equation. Hence, the general solution can also be written in the form:

$$y = B_1e^x + B_2e^{-x} - (6 + x + 3x^2) + \cosh x \tag{10.17}$$

However, it is seen that the expressions (10.16) and (10.17) are identical, *i.e.*, they contain the same two-parameter family of particular solutions. For example, the solution obtained from (10.16) by choosing the values $C_1 = C_2 = \frac{1}{2}$ for the arbitrary constants results from (10.17) by substituting $B_1 = B_2 = 0$.

11. General Remarks on Linear Differential Equations.—In this section we shall prove certain general properties of linear differential equations. As a matter of fact we discovered and used these properties in the case of the second-order equation with constant coefficients (sections 8 and 10). The reader will see easily that they are general properties of linear equations independent of the order of the equation, and they hold if the coefficients are constants or not.

The general form of a linear differential equation of the order n is the following (cf. Eq. 8.1):

$$\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = \psi(x) \tag{11.1}$$

We remember that if $\psi(x) = 0$, the equation is called a *homogeneous linear differential equation*, and if $\psi(x)$ is a given function of x, the equation is called a *nonhomogeneous linear differential equation*.

a. If a function $y(x)$ is a solution of the homogeneous linear equation, we might multiply $y(x)$ with an arbitrary constant C, and the product $Cy(x)$ also satisfies the same equation. The left side of Eq. (11.1) being a homogeneous linear expression in y and its derivatives, it is seen that substituting $Cy(x)$ instead of $y(x)$, every term on the left side is multiplied by the same constant C. Therefore, if upon substituting a certain function $y(x)$ in the equation, the left side of the equation becomes equal to zero, it will be zero also if we substitute $Cy(x)$.

b. If two functions $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation, then any linear combination of $y_1(x)$ and

$y_2(x)$, say $C_1y_1(x) + C_2y_2(x)$, where C_1 and C_2 are arbitrary constants, also satisfies the same equation. Substituting $C_1y_1 + C_2y_2$ in the equation and taking into account that $\frac{d^k(y_1 + y_2)}{dx^k} = \frac{d^ky_1}{dx^k} + \frac{d^ky_2}{dx^k}$, we obtain for the left side of (11.1) the expression

$$C_1\left(\frac{d^ny_1}{dx^n} + a_1\frac{d^{n-1}y_1}{dx^{n-1}} + \cdots + a_ny_1\right) + C_2\left(\frac{d^ny_2}{dx^n} + a_1\frac{d^{n-1}y_2}{dx^{n-1}} + \cdots + a_ny_2\right)$$

Now, since both y_1 and y_2 satisfy the equation. both expressions in the parentheses vanish. Hence, the sum of all the terms on the left side is zero, *i.e.*, $C_1y_1 + C_2y_2$ satisfies the equation.

c. If $\eta(x)$ is a solution of the nonhomogeneous equation, and $y_k(x)$ is a solution of the associated homogeneous equation, *i.e.*, of the homogeneous equation obtained from Eq. (11.1) by putting $\psi(x) = 0$, then $\eta(x) + Cy_k(x)$, where C is an arbitrary constant, also satisfies the nonhomogeneous equation. Substituting $\eta(x) + Cy_k(x)$ for $y(x)$ into the homogeneous equation, the left side of (11.1) becomes

$$\frac{d^n\eta}{dx^n} + \cdots + a_n\eta + C\left(\frac{d^ny_k}{dx^n} + \cdots + a_ny_k\right) \tag{11.2}$$

Now η being a solution of the nonhomogeneous equation, the sum of the terms containing η is equal to $\psi(x)$, while the expression in the parentheses vanishes since y_k satisfies the homogeneous equation. Consequently, the whole expression (11.2) is equal to $\psi(x)$, and $\eta + Cy_k$ is a solution of the nonhomogeneous equation.

These fundamental properties of linear differential equations help us to find the general solution if particular solutions are known. We have seen in section 3 that the general solution of a differential equation of the order n contains n arbitrary constants. To every set of these constants corresponds a particular solution. On the other hand, if n initial conditions are given, for example, the function y and its $n - 1$ first derivatives, these initial conditions must determine the particular values of the n arbitrary constants. Let us first consider the homogeneous equation. If $y_1(x)$, $y_2(x)$, . . . , $y_n(x)$ are solutions of the

homogeneous equation, the expression

$$y = C_1y_1 + C_2y_2 + \cdot \cdot \cdot + C_ny_n \tag{11.3}$$

might satisfy the requirements for the general solution. First, it follows from the theorems proved under a and b that y is a solution of the equation for arbitrary values of the constants. Second, if we put $(y)_0$, $(dy/dx)_0$, . . . , $(d^{n-1}y/dx^{n-1})_0$ equal to n given values, we obtain n linear equations for the constants C_1, C_2, . . . , C_n. If these linear equations are independent of each other, they give us definite values for C_1, C_2, . . . , C_n and, therefore, in this case (11.3) is the general solution of our differential equation.

It can be shown that if the linear equations obtained for the constants C_1, C_2, . . . , C_n are not independent, at least one of the solutions y_1, y_2, . . . , y_n can be expressed as a linear combination of some of the others. We do not give the proof for this statement; however, it is evident that if some y can be expressed as a linear combination of others, expression (11.3) does not represent the complete general solution, because it can be reduced to a sum containing less than n arbitrary constants. In order to obtain the general solution, we need n independent solutions, none of which can be expressed as a linear function of the others. Such n particular solutions constitute a so-called *fundamental system of solutions*.

In the following investigations of linear differential equations our first aim will be to find such a fundamental system of solutions, *i.e.*, n independent particular solutions of the differential equations in question.

Now, as far as the nonhomogeneous equation is concerned, its general solution is given by the sum of the general solution of the associated homogeneous equation and an arbitrary particular solution of the nonhomogeneous equation. According to c, this sum satisfies the nonhomogeneous differential equation. On the other hand, it contains n arbitrary constants, which can be determined to meet n arbitrary initial conditions.

In the following section we return to linear differential equations with constant coefficients, *i.e.*, to the special case that all coefficients $a_1(x)$, $a_2(x)$, . . . , $a_n(x)$ in Eq. (11.1) are constants.

12. Linear Differential Equations of Higher Order with Constant Coefficients. System of Linear Equations.—The method

developed in the case of the equations of second order with constant coefficients will now be extended to the case of higher order equations.

Let us consider a homogeneous equation of order n:

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_n y = 0 \qquad (12.1)$$

where $a_1, a_2, \ldots, a_n$ are given constants. We try to satisfy the equation by a solution of the type $e^{\lambda x}$ where λ is either real or complex. By substituting this expression in the equation, we find an algebraic equation for λ of degree n

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0 \qquad (12.2)$$

This is our characteristic equation in this case. Let us first assume that Eq. (12.2) has n distinct real roots, $\lambda_1, \lambda_2, \ldots \lambda_n$, then the general solution of the differential equation above is

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots + C_n e^{\lambda_n x} \qquad (12.3)$$

where $C_1, C_2, \ldots C_n$ are n arbitrary constants. The functions $e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_n x}$ constitute a fundamental system in the sense of the last section. If there are multiple roots, the above expression is not the general solution because it would contain less than n arbitrary constants. If, for instance, k roots $\lambda_1, \lambda_2, \ldots,$ λ_k are equal, the solution (12.3) is reduced to

$$y = C_1 e^{\lambda_1 x} + C_{k+1} e^{\lambda_{k+1} x} + \cdots + C_n e^{\lambda_n x}$$

We find that in this expression $k - 1$ arbitrary constants are missing because there are only $n - k + 1$ independent solutions available.

Therefore, we try the same method which has proved to be successful in the case of the second-order equation. Introducing an undetermined function u, substituting $y = ue^{\lambda_1 x}$ in the differential equation, and taking into account that λ_1 is a root of the characteristic equation repeated k times, the differential equation (12.1) yields

$$\frac{d^k u}{dx^k} = 0$$

Therefore,

$$u = C_1 + C_2 x + C_3 x^2 + \cdots + C_k x^{k-1}$$

The substitution may be conveniently carried through by noticing that we may write

$$\frac{d}{dx}(ue^{\lambda_1 x}) = e^{\lambda_1 x}\left(\frac{du}{dx} + \lambda_1 u\right)$$

or symbolically replacing d/dx by the *operator* D

$$\frac{d}{dx}(ue^{\lambda_1 x}) = e^{\lambda_1 x}(D + \lambda_1)u$$

Similarly we write

$$\frac{d^2}{dx^2}(ue^{\lambda_1 x}) = e^{\lambda_1 x}(D^2 + 2\lambda_1 D + \lambda_1^2)u = e^{\lambda_1 x}(D + \lambda_1)^2 u$$

and $\dfrac{d^n}{dx^n}(ue^{\lambda_1 x}) = e^{\lambda_1 x}(D + \lambda_1)^n u.$ The result of the substitution of $y = ue^{\lambda_1 x}$ in the differential equation (12.1) can, therefore, be written symbolically

$$[(D + \lambda_1)^n + a_1(D + \lambda_1)^{n-1} + \cdots + a_n]\, u = 0 \tag{12.4}$$

Now, because λ_1 is a multiple root of (12.2) repeated k times, the polynomial on the left side of (12.2) may be written in the form:

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = f(\lambda)(\lambda - \lambda_1)^k$$

where $f(\lambda)$ is a polynomial of degree $n - k$. Hence, substituting $D + \lambda_1$ for λ in this identity, the differential equation (12.4) may be written symbolically

$$f(D + \lambda_1)\, D^k u = 0$$

This equation is satisfied if $D^k u = d^k u/dx^k = 0$, *i.e.*, if

$$= C_1 + C_2 x + C_3 x^2 + \cdots + C_k x^{k-1}$$

Hence, in addition to $e^{\lambda_1 x}$, the following functions, $xe^{\lambda_1 x}$, $x^2 e^{\lambda_1 x}$, $\ldots$, $x^{k-1}e^{\lambda_1 x}$, are also particular solutions of the differential equation (12.1), and the general solution is equal to

$$y(x) = (C_1 + C_2 x + C_3 x^2 + \cdots + C_k x^{k-1})e^{\lambda_1 x} + C_{k+1}e^{\lambda_{k+1} x} + \cdots + C_n e^{\lambda_n x} \tag{12.5}$$

It contains n arbitrary constants, and, therefore, any arbitrary initial values at a given point can be chosen for the function and its first $n - 1$ derivatives.

Let us now consider the case when there are complex roots among $\lambda_1, \lambda_2, \ldots, \lambda_n$. The complex roots of an algebraic equation with real coefficients always appear as pairs of conjugate

complex quantities. Hence, the roots of the characteristic equation (12.2) are either real or conjugate complex. Let us assume, for example, that λ_1 and λ_2 are complex and conjugate. Then we can write $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, and according to Eq. (10.10), $e^{\lambda_1 x} = e^{\alpha x}(\cos \beta x + i \sin \beta x)$, $e^{\lambda_2 x} = e^{\alpha x}(\cos \beta x - i \sin \beta x)$. Instead of the particular solutions $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$, we use

$$\begin{aligned} \frac{1}{2}(e^{\lambda_1 x} + e^{\lambda_2 x}) &= e^{\alpha x} \cos \beta x \\ \frac{1}{2i}(e^{\lambda_1 x} - e^{\lambda_2 x}) &= e^{\alpha x} \sin \beta x \end{aligned} \tag{12.6}$$

These functions are real and represent independent particular solutions of the differential equation (12.1), whose general solution will be in this case

$$y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x + C_3 e^{\lambda_3 x} + \cdots + C_n e^{\lambda_n x} \tag{12.7}$$

Obviously, the same procedure can be used if the characteristic equation has more than one pair of conjugate complex roots.

The case of multiple complex roots can also be treated without difficulty. Assume, for example, that $\lambda_1 = \lambda_3$, $\lambda_2 = \lambda_4$ where λ_1, λ_2 and λ_3, λ_4 are conjugate complex. Then the terms contributed by these four roots to the general solution will be given by

$$(C_1 + C_2 x)e^{\alpha x} \cos \beta x + (C_3 + C_4 x)e^{\alpha x} \sin \beta x$$

where $\lambda_1 = \lambda_3 = \alpha + i\beta$.

The general solution of a nonhomogeneous equation can be obtained using the general properties of linear differential equations proved in the last section. The general solution of

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = \psi(x)$$

is given by

$$y = Y(x) + \eta(x)$$

where $Y(x)$ is the general solution of the homogeneous equation and $\eta(x)$ is an arbitrary particular solution of the nonhomogeneous equation. For example, if the roots of the characteristic equation are all real and different, the general solution takes the form:

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots + C_n e^{\lambda_n x} + \eta(x)$$

The same procedure which was used for the single linear equation with constant coefficients applies to a system of such equa-

tions. Consider, for instance, the system of two second-order homogeneous equations

$$
\begin{aligned}
\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + b_1 y + c_1 \frac{dz}{dx} &= 0 \\
\frac{d^2z}{dx^2} + a_2 \frac{dz}{dx} + b_2 z + c_2 \frac{dy}{dx} &= 0
\end{aligned}
\tag{12.8}
$$

Let us try a solution

$$
\begin{aligned}
y &= Ae^{\lambda x} \\
z &= Be^{\lambda x}
\end{aligned}
$$

The system of differential equations is satisfied if

$$
\begin{aligned}
A(\lambda^2 + a_1\lambda + b_1) + Bc_1\lambda &= 0 \\
B(\lambda^2 + a_2\lambda + b_2) + Ac_2\lambda &= 0
\end{aligned}
\tag{12.9}
$$

These equations are compatible if

$$
\begin{vmatrix} \lambda^2 + a_1\lambda + b_1 & c_1\lambda \\ c_2\lambda & \lambda^2 + a_2\lambda + b_2 \end{vmatrix} = 0 \tag{12.10}
$$

This 4th-degree equation is called the characteristic equation of the system (12.8). Let us assume that it has four distinct roots λ_1, λ_2, λ_3, and λ_4. Then the general solution of the system is

$$
y = \sum_{k=1}^{4} A_k e^{\lambda_k x}, \qquad z = \sum_{k=1}^{4} B_k e^{\lambda_k x} \tag{12.11}
$$

where A_k and B_k are constants satisfying the system of Eqs. (12.9) in which the root $\lambda = \lambda_k$ is substituted for λ.

If the four roots are *real*, the constants A_k, B_k are real. It appears at first sight as though there were eight arbitrary constants. However, Eq. (12.9) must hold for all four values of λ_k. Hence,

$$
\frac{B_k}{A_k} = -\left(\frac{\lambda_k^2 + a_1\lambda_k + b_1}{c_1\lambda_k}\right) = -\left(\frac{c_2\lambda_k}{\lambda_k^2 + a_2\lambda_k + b_2}\right) \tag{12.12}
$$

and eliminating the B_k's,

$$
y = \sum_{k=1}^{4} A_k e^{\lambda_k x}, \qquad z = -\sum_{k=1}^{4} A_k \left(\frac{\lambda_k^2 + a_1\lambda_k + b_1}{c_1\lambda_k}\right) e^{\lambda_k x}
$$

This solution contains four arbitrary constants, which is the

required number for the general solution of our system. If the characteristic equation has multiple roots, the required number of independent solutions can be found by a procedure similar to that used for the single equation.

Let us now assume that some roots of (12.10) are *conjugate complex*. In this case, we obtain in general complex values for the A_k's and B_k's. However, the expressions (12.11) must be real, and, therefore, if, for example, the roots λ_1 and λ_2 are complex conjugates, the corresponding factors A_1 and A_2, B_1 and B_2 must also be conjugate complex quantities.

Furthermore, B_1/A_1 and B_2/A_2 have to satisfy two complex equations (12.12). Since B_1/A_1 and B_2/A_2 are conjugates, these equations are reduced to two equations with real coefficients. Hence, from the four real quantities which appear in the complex coefficients A_1, A_2, B_1, and B_2, two remain arbitrary. It is seen that also in this case the number of arbitrary real constants is equal to the number of roots λ_k.

In general practice, to write the solution (12.11) in real form, one will carry out the entire calculation in complex form, and then choose either the real or the imaginary part of the expression (12.11) as the real general solution of the system (12.8). The coefficients of this solution can be determined so that the given initial conditions are met.

In the case of the nonhomogeneous system,

$$\begin{aligned} \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + b_1 y + c_1 \frac{dz}{dx} &= \varphi(x) \\ \frac{d^2z}{dx^2} + a_2 \frac{dz}{dx} + b_2 z + c_2 \frac{dy}{dx} &= \psi(x) \end{aligned} \qquad (12.13)$$

the general solution will be found by adding to the expressions (12.11), which represent the general solution of the homogeneous system (12.8), a particular solution of the system of Eqs. (12.13).

The methods of this section will be made clearer to the reader through applications in later chapters.

Problems

1. Evaluate numerically the integral

$$I = \int_0^{\pi/2} \log_e (\sin x)\, dx$$

and compare the result with the exact value $I = -\frac{\pi}{2} \log_e 2$.

Hint: Since the integrand is infinite for $x = 0$, write

$$I = \int_0^{\pi/2} \log_e x \, dx + \int_0^{\pi/2} \log_e \frac{\sin x}{x} \, dx$$

The first integral can be integrated by parts; the second integral can be evaluated numerically, *e.g.*, by Simpson's rule, since the integrand is now finite everywhere.

2. Evaluate numerically the integral

$$\int_1^2 \frac{dx}{x} = \log_e 2$$

Use Simpson's rule with an interval $h = 0.1$.

3. Integrate numerically the differential equation

$$\frac{dy}{dx} = y$$

with the initial condition $y = 1$ for $x = 0$. Use Adams' method with intervals $h = 0.1$ in the range from $x = 0$ to $x = 1$. Compare the result with a table of e^x from $x = 0$ to $x = 1$ with intervals 0.1.

4. Integrate numerically the differential equation

$$\frac{d^2y}{dx^2} - xy = 0$$

in the interval between $x = 0$ and $x = 1$ with the initial conditions $y = 1$ and $dy/dx = 0$ for $x = 0$.

Hint: Introduce a new unknown variable u by the transformation $u = \frac{1}{y}\frac{dy}{dx}$ or $u = \frac{d}{dx}(\log_e y)$. Calculate and substitute $\frac{d^2y}{dx^2}$; show that the above differential equation becomes $\frac{du}{dx} + u^2 - x = 0$. Integrate this nonlinear first-order equation by Adams' method with the initial condition $u = 0$ for $x = 0$, using as intervals $h = 0.1$. After determining $u(x)$, find $y(x)$ by quadrature from $u = d/dx(\log_e y)$.

5. Find the general solution of

$$\frac{dy}{dx} = \frac{(1 + x^2)y^3}{(y^2 - 1)x^3}$$

6. Find the general solution of

$$\frac{dy}{dx} = \frac{a^2 + y^2}{2x\sqrt{ax - a^2}}$$

7. Find the family of curves which cut at right angles the parabolas given by

$$y^2 = 2p(x - \alpha)$$

where p is a given constant and α is a variable parameter.

8. Find the general solution of the differential equation

$$x^2 \frac{dy}{dx} = x^2 + y^2$$

9. Find the general solution of the differential equations:

$$\frac{d^4y}{dx^4} + y = 0$$

$$\frac{d^4y}{dx^4} - y = 0$$

10. Find the general solution of the following equations:

$$\frac{dy}{dx} + y \cos x = \sin x \cos x$$

$$\frac{dy}{dx} = \frac{ny}{x+1} + e^x(x+1)^n$$

11. *From the Theory of Heat Transfer.*—A constant voltage E is applied to a resistance at the time $t = 0$. The heat capacity of the resistance is Q, its heat-transfer coefficient per unit time and unit surface is α, and its surface area is S. The resistance as a function of the temperature is given by

$$R = R_0(1 + \beta\theta)$$

where θ is the temperature of the resistance above room temperature. Find a relation between the temperature θ and the time.

Hint: We find a differential equation for the temperature by expressing the fact that the heat produced per second by the current is partly stored in the resistance and partly lost by transfer through the surface. The equation of heat balance is $\dfrac{E^2}{R_0(1+\beta\theta)} - \alpha S\theta = Q\dfrac{d\theta}{dt}$. The final temperature θ_f must correspond to $d\theta/dt = 0$, hence it is given by $E^2/R_0(1 + \beta\theta_f) = \alpha S\theta_f$. Introducing the dimensionless variable $\eta = \theta/\theta_f$, the differential equation may be written $\dfrac{Q}{\alpha S}\dfrac{d\eta}{dt} = \dfrac{1+\beta\theta_f}{1+\beta\theta_f\eta} - \eta$. Plot η as function of $\dfrac{\alpha S}{Q}t$ for $\beta\theta_f = \dfrac{1}{10}$, 1, 5. We have assumed that α and Q are expressed in units of electric energy.

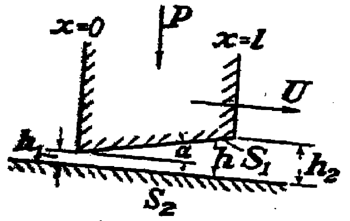

FIG. P.12.

12. *From the Theory of Lubrication.*—The pressure distribution $p(x)$ in the oil film between the inclined surface S_1 which moves with the velocity U and the fixed base S_2 (Fig. P.12) is given by the equation

$$\frac{dp}{dx} = \frac{12\mu}{h^3}\left(Q - \frac{Uh}{2}\right)$$

where $h = h_1 + x \tan \alpha$ is the thickness of the oil film at an arbitrary cross

section, Q is the volume of the oil flowing through between the surfaces per unit width and per unit time, and μ is the coefficient of viscosity of the oil. Determine the value of Q and the solution of the above equation so that the pressure $p = 0$ for $x = 0$ and $x = l$. Find the location of the highest pressure, and plot the total load carried by a strip of unit width of the surface S_1 as function of h_1.

Hint: From $\int^l \frac{dp}{dx}\,dx = 0$, it follows that $Q\left(\frac{1}{h_1^2} - \frac{1}{h_2^2}\right) = U\left(\frac{1}{h_1} - \frac{1}{h_2}\right)$ where $h_2 = h_1 + l \tan \alpha$. Hence $Q = \frac{h_1 h_2}{h_1 + h_2} U$. The total load is

$$P = \int_0^l p\,dx = \cot\alpha \int_{h_1}^{h_2} p\,dh = [ph]_{h_1}^{h_2} \cot\alpha - \cot\alpha \int_{h_1}^{h_2} \frac{dp}{dh}\,h\,dh$$
$$= -\cot^2\alpha \int_{h_1}^{h_2} \frac{dp}{dx}\,h\,dh$$

13. *From the Theory of Plasticity.*—A steel slab—kept at the temperature of white heat—is pulled through the slightly inclined surfaces AB and CD shown in Fig. P.13 by a force F per unit width. According to the laws of plastic deformation the longitudinal stress σ is equal to the yield-point stress of the material k minus the transverse pressure p. A friction of the amount fp (f = coefficient of friction) per unit area acts along the surfaces AB and CD in the opposite direction to F. Establish the equation of equilibrium for an element of the slab of unit width, thickness $h = h_1 - 2\alpha x$ and length dx. (α is the small angle of inclination between the fixed surfaces and the force F.) Determine the distribution of σ and p from the condition that $\sigma = 0$ for $x = 0$. Find the value of F. Discuss the influence of the magnitude of the coefficient of friction f and the inclination of the surfaces AB, CD.

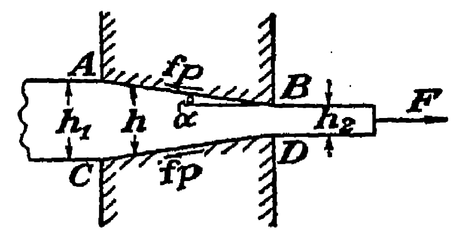

FIG. P.13.

Solve the same problem under the assumption that the slab is pushed through by a force F; in this case $\sigma = 0$ for $x = l$.

Hint: The equilibrium condition for an element of the slab of length dx and thickness h is $d(\sigma h) = 2(f + \alpha)p\,dx$. Since $h = h_1 - 2\alpha x$, we may use the variable h instead of x. The equation becomes

$$-\alpha \frac{d}{dh}(\sigma h) = (f + \alpha)p$$

Using $\sigma = k - p$, we find a differential equation for σ as function of h. The process is physically possible only if $k \geqslant \sigma$. Find the smallest ratio h_2/h_1 as function of f/α that is compatible with this condition. Show that in the limiting case where $f = 0$ the smallest value is $h_2/h_1 = 1/e = 1/2.71$.

14. Find the general solution of the following differential equations:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

15. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \cos 2x$$

Hint: Try a particular solution of the form: $y = \alpha \cos 2x + \beta \sin 2x$ for the nonhomogeneous equation.

16. Separate the real and imaginary parts of the following expression

$$y = \cos (2 + 5i)x + \sin (3 - 4i)x$$

17. Expand the function $y = \cosh^5 x$ in a series of hyperbolic cosines.

References

Differential equations:

1. FORSYTH, A. R.: "A Treatise on Differential Equations," Macmillan & Company, Ltd., London, 1929.
2. FRY, T. C.: "Elementary Differential Equations," D. Van Nostrand Company, Inc., New York, 1929.
3. PIAGGIO, H. T. H.: "An Elementary Treatise on Differential Equations and their Applications," George Bell & Sons, Ltd., London, 1929.
4. COHEN, A.: "An Elementary Treatise on Differential Equations," D. C. Heath & Company, Boston, 2d ed., 1933.
5. INCE, E. L.: "Ordinary Differential Equations," Longmans, Green & Company, New York, 1927.

Numerical methods:

6. SCARBOROUGH, J. B.: "Numerical Mathematical Analysis," Johns Hopkins Press, Baltimore, 1930.
7. WHITTAKER, E. J., and G. ROBINSON: "The Calculus of Observations," Blackie & Sons, Ltd., Glasgow, 1929.
8. RUNGE, C., and H. KÖNIG: "Numerisches Rechnen," Verlag Julius Springer, Berlin, 1924.

CHAPTER II

SOME INFORMATION ON BESSEL FUNCTIONS

> Friedrich Wilhelm Bessel (1784–1846) was a native of Minden in Westphalia. Fondness for figures and a distaste for Latin grammar led him to the choice of a mercantile career. Hoping some day to become a supercargo on trading expeditions, he became interested in observations at sea. . . . His success in this inspired him to astronomical study. . . . Encouraged by Olbers, Bessel turned his back to the prospect of affluence, chose poverty and the stars.
>
> —CAJORI,
> "A History of Mathematics," p. 448.

Introduction.—The solutions of linear equations of second or higher order can be expressed by elementary functions only in exceptional cases. As a matter of fact, certain types of differential equations define certain classes of new functions, and to solve such differential equations means to disclose the behavior and properties of the corresponding special functions. Many such special functions play an important role in physical and in engineering problems. The so-called *Bessel functions* or solutions of *Bessel's differential equation* are perhaps most frequently encountered. Besides that, they show many analogies to the elementary exponential and trigonometric functions which constitute the solutions of the linear differential equations with constant coefficients discussed in the previous chapter. Therefore, a somewhat detailed discussion of Bessel's differential equation and Bessel functions appears desirable.

1. Bessel's Differential Equation and Bessel Functions of Zero Order.—The second-order linear differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)y = 0 \tag{1.1}$$

is known as *Bessel's differential equation.* Every value of the parameter ν is associated with a pair of fundamental solutions called *Bessel functions of order* ν. One of them, which is finite at $x = 0$, is called the *Bessel function of the first kind,* and the other, the *Bessel function of the second kind.*

We shall first consider the particular case $\nu = 0$ (Bessel's equation of zero order):

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0 \qquad (1.2)$$

The general solution of this equation is of the form:

$$y = C_1y_1 + C_2y_2$$

where y_1 and y_2 are particular solutions of Eq. (1.2) and C_1 and C_2 are arbitrary constants. We try to find approximations for such particular solutions which are valid for very small or very large values of the independent variable x.

Let us try to determine a particular solution for the initial values $y = a_0$ and $(dy/dx)_0 = a_1$. If we substitute these initial values in Eq. (1.2), we see that the second term becomes infinite for $x = 0$. Hence, the first derivative of a particular solution of Eq. (1.2) that goes through the point $x = 0$, $y = a_0$, must be equal to zero for $x = 0$. We try, therefore, a series of the form:

$$y = a_0 + a_2x^2 + a_3x^3 + a_4x^4 + \cdots \qquad (1.3)$$

We obtain by differentiation

$$\begin{aligned} \frac{dy}{dx} &= 2a_2x + 3a_3x^2 + \cdots \\ \frac{d^2y}{dx^2} &= 2a_2 + 6a_3x + \cdots \end{aligned} \qquad (1.4)$$

Substituting Eqs. (1.3) and (1.4) in Eq. (1.2), we have

$$(2^2a_2 + a_0) + 3^2a_3x + (4^2a_4 + a_2)x^2 + \cdots = 0$$

Hence, the differential equation is satisfied if we have

$$a_2 = -\frac{a_0}{2^2}, \qquad a_4 = -\frac{a_2}{4^2}, \quad \cdots$$

and

$$a_3 = a_5 = \cdots = 0$$

We obtain for y the series

$$y = a_0\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots\right)$$

or

$$y = a_0 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{2k}(k!)^2} \tag{1.5}$$

We remember that $k! = 1 \cdot 2 \cdot 3 \cdots k$ and $0! = 1$. The above series *converges for all values of* x, and not only in the vicinity of $x = 0$.* It defines a solution of the differential equation (1.2). The function obtained from Eq. (1.5) by putting $a_0 = 1$,

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \tag{1.6}$$

is known as the *Bessel function of the first kind of zero order*.

The process of series development leads to a solution $CJ_0(x)$ of the differential equation with only one arbitrary constant C. In order to find the general solution, we must obtain another solution independent of $J_0(x)$.

There exists a method by which the general solution of a linear second-order differential equation can be found when a particular solution is known. In fact, we used this very method for differential equations with constant coefficients (Chapter I, section 10) when the characteristic equation had multiple roots. We introduce an undetermined function $u(x)$ and put

$$y = uJ_0(x) \tag{1.7}$$

Introducing this expression for y into the differential equation (1.2), we find

$$\frac{d^2u}{dx^2} + \left(2\frac{J_0'(x)}{J_0(x)} + \frac{1}{x}\right)\frac{du}{dx} = 0$$

where

$$J_0'(x) = \frac{dJ_0(x)}{dx}.$$

This is a first-order differential equation for $q = du/dx$. We obtain by separation of the variables

$$\frac{dq}{q} = \left[-\frac{1}{x} - 2\frac{J_0'(x)}{J_0(x)}\right] dx$$

* For a treatment of the properties of convergence of series, see for example, W. A. Granville, "Elements of the Differential and Integral Calculus," p. 214.

or

$$q = \frac{du}{dx} = \frac{B}{x[J_0(x)]^2}$$

where B is a constant. A second integration yields

$$u = A + B\int^x \frac{dx}{x[J_0(x)]^2}$$

where A and B are constants of integration. The lower limit a of the integral is arbitrary; however, it cannot be equal to zero. Varying the limit a amounts to varying the arbitrary constant A. The general solution of Bessel's equation (1.2) is, therefore,

$$y = AJ_0(x) + BJ_0(x)\int^x \frac{dx}{x[J_0(x)]^2} \tag{1.8}$$

It is seen that the second term of this expression yields the second solution which we did not obtain by expansion in power series. It is not easy to evaluate (1.8) by direct integration. However, it is not difficult to determine the nature of the singularity of the solution at $x = 0$. When x approaches zero, $J_0(x)$ approaches unity, and therefore,

$$\int^x \frac{dx}{x[J_0(x)]^2} \longrightarrow \int_a^x \frac{dx}{x} = \log x - \log a$$

Knowing the character of the singularity, we find directly the second solution of Bessel's equation by putting

$$y = J_0(x)\log x + b_0 + b_1x + b_2x^2 + \cdots \tag{1.9}$$

The coefficients of the series are determined by substitution into the differential equation (1.2). All but the coefficient b_0 can be determined in this way. However, as we standardized the first solution by putting $a_0 = 1$, it is also convenient to standardize the second independent solution. On Weber's suggestion the function

$$\begin{aligned} Y_0(x) = \frac{2}{\pi}\Bigg[J_0(x)\left(\log\frac{x}{2} + \gamma\right) + \left(\frac{x}{2}\right)^2 - \frac{(x/2)^4}{(2!)^2}\left(1 + \frac{1}{2}\right) \\ + \frac{(x/2)^6}{(3!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) - \cdots\Bigg] \end{aligned} \tag{1.10}$$

is chosen as the standard form. $Y_0(x)$ is known as the *Bessel*

function of zero order of the second kind. The constant γ (denoted in the German literature sometimes by C) is called *Euler's constant.* The numerical value of γ is 0.577216. . . . The choice of the constant γ and of the factor $2/\pi$ in the expression of Y_0 is made with the purpose of obtaining certain simple expressions of $J_0(x)$ and $Y_0(x)$ for large values of x. These expressions will be given later in this section. In German textbooks the notation $N_0(x)$ is generally used instead of $Y_0(x)$.

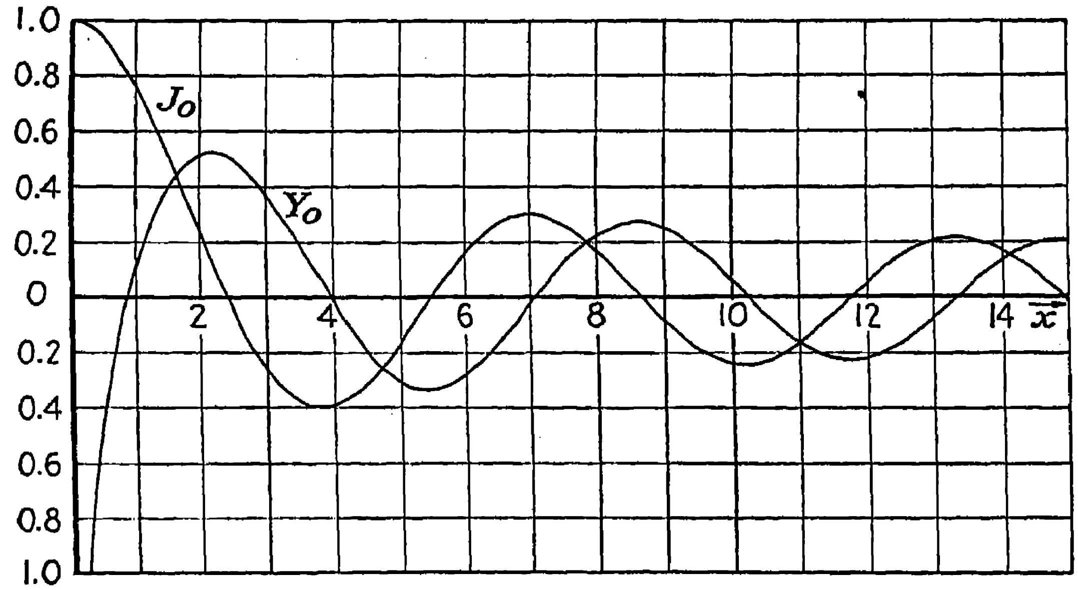

FIG. 1.1.—The Bessel functions of zero order.

We thus write the general solution of Eq. (1.2) in the form:

$$y = C_1J_0(x) + C_2Y_0(x) \tag{1.11}$$

Values of $J_0(x)$ and $Y_0(x)$ are plotted in Fig. (1.1). Tables for these functions will be found in many textbooks on Bessel functions (see References at end of chapter). The tables in Watson's "Theory of Bessel Functions" or in Jahnke-Emde's "Tables of Functions" will be found especially useful. The function $Y_0(x) \to -\infty$ when $x \to 0$. Its approximate value for small x is

$$Y_0(x) \cong \frac{2}{\pi}\left[\log \frac{x}{2} + \gamma\right] \tag{1.12}$$

The infinite series (1.6) and (1.10) converge very slowly for large values of x; therefore, we try to obtain approximate expressions for the particular solutions of Bessel's equation of zero order which make it possible to calculate their approximate values for large x by using a few terms only. We say that

$\varphi(x)$ is an *asymptotic approximation* of $y(x)$, if the ratio $y(x)/\varphi(x)$ converges to unity when $x \to \infty$.

We found in Chapter I that the particular solutions of an equation of the second order with constant coefficients have the form const. $e^{\lambda x}$. For Bessel's equation with variable coefficients we tentatively try the asymptotic approximation

$$\varphi(x) = \text{const. } x^\alpha e^{\lambda x} \tag{1.13}$$

Substitute $y = \varphi(x)$ in Eq. (1.2) and determine α and λ so that the terms of the highest order for $x \to \infty$ vanish identically. We obtain by differentiation

$$\begin{aligned} \frac{dy}{dx} &= \text{const. } x^\alpha \left(\frac{\alpha}{x} + \lambda\right) e^{\lambda x} \\ \frac{d^2y}{dx^2} &= \text{const. } x^\alpha \left[\frac{\alpha(\alpha-1)}{x^2} + \frac{2\alpha\lambda}{x} + \lambda^2\right] e^{\lambda x} \end{aligned} \tag{1.14}$$

Substituting the expressions (1.14) in Eq. (1.2), we obtain

$$x^\alpha e^{\lambda x}\left[(\lambda^2 + 1) + \frac{(2\alpha+1)\lambda}{x} + \frac{\alpha^2}{x^2}\right] = 0$$

It is seen that for $x \to \infty$ the term $\dfrac{(2\alpha+1)\lambda}{x}$ becomes small compared with the term $(\lambda^2 + 1)$, and α^2/x^2 becomes small compared with $\dfrac{(2\alpha+1)\lambda}{x}$. Hence, if we put

$$\lambda^2 + 1 = 0 \qquad \text{and} \qquad 2\alpha + 1 = 0 \tag{1.15}$$

Eq. (1.2) is satisfied, if terms of the order of $1/x^2$ are neglected.

The conditions (1.15) yield $\lambda = \pm i$ and $\alpha = -\frac{1}{2}$. Hence, we obtain for large values of x the following asymptotic approximations for the particular solutions of Eq. (1.2):

$$\begin{aligned} y(x) &\cong \text{const. } \frac{e^{ix}}{\sqrt{x}} \\ \text{and} \qquad & \\ y(x) &\cong \text{const. } \frac{e^{-ix}}{\sqrt{x}} \end{aligned} \tag{1.16}$$

Instead of the complex expressions (1.16), we can use their real linear combinations

$$y(x) \cong \text{const.}\ \frac{\cos x}{\sqrt{x}}$$

and (1.17)

$$y(x) \cong \text{const.}\ \frac{\sin x}{\sqrt{x}}$$

It follows that for large values of x both $J_0(x)$ and $Y_0(x)$ have the form:

$$y(x) \cong C_1 \frac{\cos x}{\sqrt{x}} + C_2 \frac{\sin x}{\sqrt{x}} \tag{1.18}$$

The determination of the constants C_1 and C_2 for the asymptotic approximations of the standard forms of $J_0(x)$ and $Y_0(x)$ involves analytical methods that are beyond the scope of this book. They can be found in advanced textbooks on Bessel functions. Here we give only the results:

$$\begin{aligned} J_0(x) \underset{x\to\infty}{\cong} \frac{\cos\left(x - \frac{\pi}{4}\right)}{\sqrt{\frac{1}{2}\pi x}} \\ Y_0(x) \underset{x\to\infty}{\cong} \frac{\sin\left(x - \frac{\pi}{4}\right)}{\sqrt{\frac{1}{2}\pi x}} \end{aligned} \tag{1.19}$$

The asymptotic approximations (1.16) can be improved by using a series of the form:

$$y = \text{const.}\ \frac{e^{ix}}{\sqrt{x}}\left(1 + \frac{b_1}{x} + \frac{b_2}{x^2} + \cdots\right) \tag{1.20}$$

By substituting this expression into the differential equation (1.2), we can determine the coefficient b_1 so that not only the terms with $e^{ix}/\sqrt{x}$, $e^{ix}/x\sqrt{x}$, but also those with $e^{ix}/x^2\sqrt{x}$, $e^{ix}/x^3\sqrt{x}$, . . . vanish, and continuing this procedure, we determine $b_2, b_3, \ldots$. However, the series (1.20) is not convergent in the sense of the convergence of a power series, for we shall find that after a certain number of terms the quantities b_k/x^k begin to increase, even in the case of an arbitrarily large x. Poincaré has shown that for a wide class of such expansions, if we cut off the series at a certain term, the error is not larger than the last

term retained. Series with this property are called *semiconvergent series.*

The semiconvergent series for $J_0(x)$ and $Y_0(x)$ have the form:

$$J_0(x) = \frac{\cos\left(x - \frac{\pi}{4}\right)}{\sqrt{\frac{1}{2}\pi x}}\left[1 - \frac{3^2}{2}\frac{1}{(8x)^2} + \frac{(3\cdot 5\cdot 7)^2}{4!}\frac{1}{(8x)^4}\cdots\right] + \frac{\sin\left(x - \frac{\pi}{4}\right)}{\sqrt{\frac{1}{2}\pi x}}\left[\frac{1}{8x} - \frac{(3\cdot 5)^2}{3!}\frac{1}{(8x)^3} + \cdots\right] \tag{1.21}$$

$$Y_0(x) = \frac{\sin\left(x - \frac{\pi}{4}\right)}{\sqrt{\frac{1}{2}\pi x}}\left[1 - \frac{3^2}{2}\frac{1}{(8x)^2} + \frac{(3\cdot 5\cdot 7)^2}{4!}\frac{1}{(8x)^4}\cdots\right] - \frac{\cos\left(x - \frac{\pi}{4}\right)}{\sqrt{\frac{1}{2}\pi x}}\left[\frac{1}{8x} - \frac{(3\cdot 5)^2}{3!}\frac{1}{(8x)^3} + \cdots\right]$$

Sometimes it is convenient to use linear combinations of $J_0(x)$ and $Y_0(x)$ whose asymptotic expansions are of the complex form (1.20). The functions

$$\begin{aligned} H_0^{(1)}(x) &= J_0(x) + iY_0(x) \\ H_0^{(2)}(x) &= J_0(x) - iY_0(x) \end{aligned} \tag{1.22}$$

are known as *Bessel functions of zero order of the third kind,* or as *Hankel functions of zero order.* Their asymptotic approximations for large values of x are

$$\begin{aligned} H_0^{(1)}(x) &\cong \frac{e^{i\left(x - \frac{\pi}{4}\right)}}{\sqrt{\frac{1}{2}\pi x}} \\ H_0^{(2)}(x) &\cong \frac{e^{-i\left(x - \frac{\pi}{4}\right)}}{\sqrt{\frac{1}{2}\pi x}} \end{aligned} \tag{1.23}$$

In practical problems we often encounter Bessel's equation in the form:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + k^2y = 0 \tag{1.24}$$

where k is a constant parameter. To reduce this equation to the standard form (1.2), we put $kx = \xi$ and obtain

$$\frac{d^2y}{d\xi^2} + \frac{1}{\xi}\frac{dy}{d\xi} + y = 0$$

Then the general solution of Eq. (1.24) becomes

$$y = C_1J_0(kx) + C_2Y_0(kx) \tag{1.25}$$

2. Bessel Functions of Higher Order.—To find particular integrals of Bessel's equation of higher order

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)y = 0 \tag{2.1}$$

we must use a somewhat different method from that used for the equation of zero order. If $\nu \neq 0$, no solution can be found for the initial values $x = 0$, $y = a_0$. Hence, we write

$$y = x^m(a_0 + a_1x + a_2x^2 + \cdots) \tag{2.2}$$

where a_0 shall be different from zero. Then we have

$$y = x^m(a_0 + a_1x + a_2x^2 + \cdots)$$

$$\frac{1}{x}\frac{dy}{dx} = x^m[a_0mx^{-2} + a_1(m+1)x^{-1} + a_2(m+2) + a_3(m+3)x + a_4(m+4)x^2 + \cdots]$$

$$\frac{d^2y}{dx^2} = x^m[a_0(m-1)mx^{-2} + a_1m(m+1)x^{-1} + a_2(m+1)(m+2) + a_3(m+2)(m+3)x + a_4(m+3)(m+4)x^2 + \cdots]$$

Substituting these expressions in Eq. (2.1), we obtain

$$x^m\left\{a_0(m^2 - \nu^2)\frac{1}{x^2} + a_1[(m+1)^2 - \nu^2]\frac{1}{x} + a_0 + a_2[(m+2)^2 - \nu^2] + \cdots\right\} = 0 \tag{2.3}$$

This equation must be satisfied identically, hence the coefficients of $1/x^2$, $1/x$, 1, x, . . . must vanish. The condition that the coefficient of a_0/x^2 is equal to zero gives $m^2 - \nu^2 = 0$, *i.e.*, $m = \nu$ or $m = -\nu$.

Let us assume first $m = \nu$, *i.e.*, m is equal to the positive root of the coefficient ν^2 in Eq. (2.1). In this case we find that in (2.2) the coefficients with odd subscripts must vanish, and we obtain a solution of the form:

$$y_1 = x^\nu(a_0 + a_2x^2 + a_4x^4 + \cdots) \tag{2.4}$$

where a_0 is arbitrary and the coefficients a_2, a_4, . . . are determined by the condition that (2.3) is identically satisfied.

If we put $m = -\nu$, we have to distinguish between the cases that ν is an integer or ν is an arbitrary nonintegral positive number.

a. If *ν is an integer*, say $\nu = n$, we find that the procedure of calculating $a_2, a_4, \ldots$ breaks down at a certain term. For example, take $\nu = 1$, $m = -1$. Then we have to satisfy the condition $a_0 + a_2(1 - 1) = 0$. Obviously, this would require $a_2 = \infty$.* Hence, the method that we are using does not give us a second independent solution of the differential equation (2.1). However, employing a method similar to that used in the last section for the equation of zero order, a second independent particular solution of the following form is found:

$$y_2 = y_1 \log x + x^{-n}(b_0 + b_2x^2 + \cdots) \tag{2.5}$$

where y_1 is the particular solution (2.4).

b. If *ν is not an integer*, we find without difficulty a second independent solution, which has the form:

$$y_2 = x^{-\nu}(a_0 + b_2x^2 + b_4x^4 + \cdots) \tag{2.6}$$

Hence, if ν is an integer, the two solutions are of the form (2.4) and (2.5); if ν is not an integer, they are of the form (2.4) and (2.6).

3. Standard Forms of Bessel Functions.—For practical calculations and for the use of tables, standard forms for Bessel functions have been introduced. Unfortunately, different forms and notations have been used by various authors. In this section we shall make the reader familiar with some of the most frequently used standard forms and notations. On page 64 a table of equivalence of symbols for Bessel functions is given for convenient use of the most important textbooks and tables.

We found in the last section that Bessel's differential equation has for arbitrary values of ν^2 a solution of the form:

$$y_1 = x^{\nu}(a_0 + a_2x^2 + \cdots) \tag{3.1}$$

where ν is the positive root of ν^2. We standardize this particular

* If we admit that a_0 can be equal to zero we could have $a_0 = a_1 = 0$ and start the series with an arbitrary constant a_2; the series would then have the form $x(a_2 + a_4x^2 + \cdots)$. However, the reader will easily verify that the series which we obtain in this way is identical with y_1. The same is true in the general case in which ν is an arbitrary integer.

solution by a certain choice of the arbitrary constant a_0, which depends on ν.

If ν is an integer, say $\nu = n$, we choose $a_0 = 1/2^\nu n!$. If ν is not an integer, $n!$ is to be replaced by the so-called *generalized factorial function* $\Gamma(\nu + 1)$, whose numerical values can be found in the references given at the end of the chapter. The function $\Gamma(\nu + 1)$ obeys the recurrence law of the factorial, *viz.*,

$$\Gamma(\nu + 1) = (\nu + 1)\Gamma(\nu).$$

If ν is equal to an integer n, $\Gamma(n + 1) = n!$. Substituting (3.1) into the differential equation, we obtain the values of a_2, a_4, etc. The function defined by this infinite series is denoted by $J_\nu(x)$ and is called the *Bessel function of the order* ν *of the first kind:*

$$J_\nu(x) = \frac{1}{2^\nu\Gamma(\nu + 1)}\, x^\nu\left[1 - \frac{x^2}{1!(\nu + 1)2^2}\right.$$
$$\left. + \frac{}{2!(\nu + 1)(\nu + 2)2^4} - \right.$$

The function $J_\nu(x)$ may be considered as a function of two variables x and ν and, therefore, can be represented by a surface,

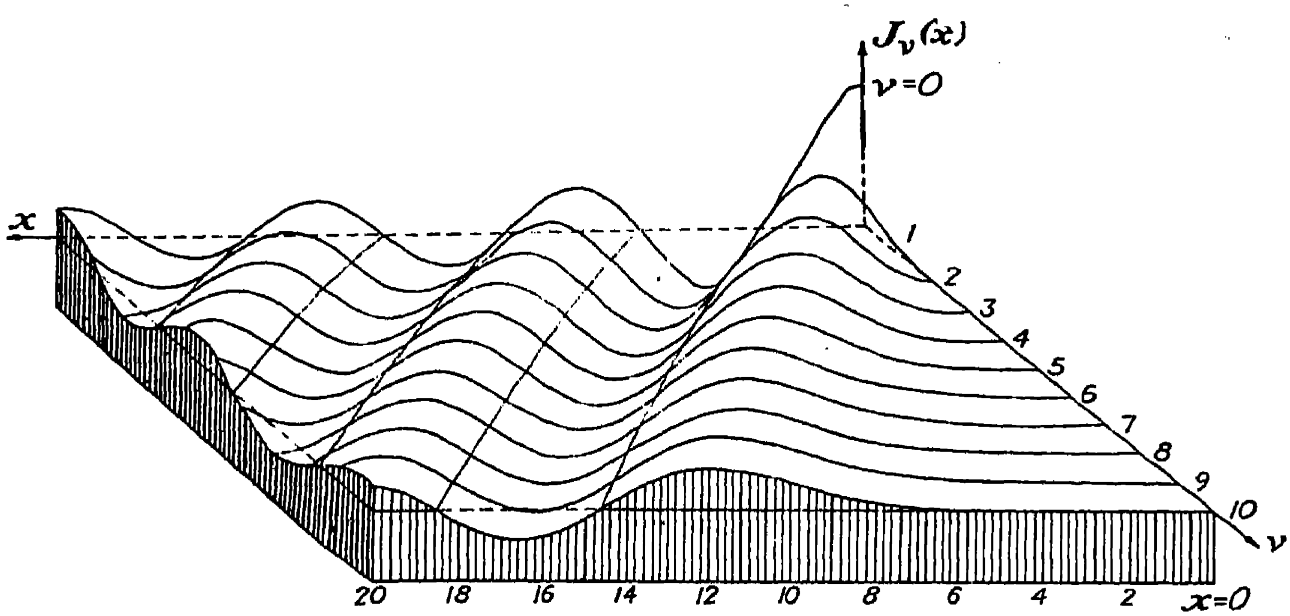

FIG. 3.1.—The Bessel function of the first kind $J_\nu(x)$ as function of the variables x and ν.

as shown in Fig. 3.1. In this figure the ordinates of the surface are the values $J_\nu(x)$ where x runs from 0 to 20 and ν varies from 0 to 10. The wavelike curves shown are intersections of the surface with the vertical planes $\nu = 0, 1, \cdots, 10$; they represent,

therefore, the Bessel functions whose orders are positive integers. Intersections made by intermediary planes give Bessel functions of nonintegral orders. For $x = 0$, $J_0(x) = 1$; all others start with zero. At $x = 0$, they have a vertical tangent if $0 < \nu < 1$, and a horizontal tangent if $\nu > 1$, whereas $J_1(x)$ has a finite slope. The transverse curves are the intersections of the surface with the horizontal plane $J_\nu(x) = 0$; hence, they show the values of the zeros of the Bessel functions.

The function obtained from $J_\nu(x)$ by change of ν to $-\nu$ in the exponent of x and in the expressions for $a_0, a_2, \ldots$ is denoted by $J_{-\nu}(x)$ and is sometimes called a *Bessel function of negative order*. If ν is not an integer, $J_{-\nu}(x)$ has the form y_2 given by Eq. (2.6); if ν is an integer, say $\nu = n$,

$$J_{-n}(x) = (-1)^n J_n(x) \tag{3.2}$$

The Bessel function of the order ν of the second kind is a particular solution of Bessel's differential equation (2.1) independent of $J_\nu(x)$ and is defined by the formula

$$Y_\nu(x) = \frac{1}{\sin \nu\pi}\,[\cos \nu\pi\, J_\nu(x) - J_{-\nu}(x)] \tag{3.3}$$

If ν *is not an integer*, the function $Y_\nu(x)$ is a linear combination of the solutions $y_1(x)$ [Eq. (2.4)] and $y_2(x)$ [Eq. (2.6)]. If ν *is an integer*, the expression (3.3) becomes $0/0$. However, if we take the limit of (3.3) when ν approaches an integer n, we can show that $Y_\nu(x)$ takes the form $y_2(x)$ given by Eq. (2.5). For $\nu \to 0$ the limiting value of the right side of Eq. (3.3) coincides with the definition of $Y_0(x)$ given by Eq. (1.10).

The standard form for the general solution of (2.1) is

$$y = AJ_\nu(x) + BY_\nu(x) \tag{3.4}$$

For large values of x the asymptotic approximation for $J_\nu(x)$ is

$$J_\nu(x) \cong \frac{\cos\left(x - \frac{\pi}{4} - \frac{\nu\pi}{2}\right)}{\sqrt{\frac{1}{2}\pi x}} \tag{3.5}$$

We have seen that for small values of x, $J_\nu(x)$ is of the order x^ν. The function $Y_\nu(x)$ is plotted in Fig. 3.2, for various integral values of ν. The asymptotic value of $Y_\nu(x)$ for large values of x is

$$Y_\nu(x) \cong \frac{\sin\left(x - \frac{\pi}{4} - \frac{\nu\pi}{2}\right)}{\sqrt{\frac{1}{2}\pi x}} \cdot \tag{3.6}$$

The value of $Y_\nu(x)$ is always infinite at $x = 0$. For small values of x this function is of the order $1/x^\nu$, if $\nu \neq 0$, and of the order of $\log x$ if $\nu = 0$.

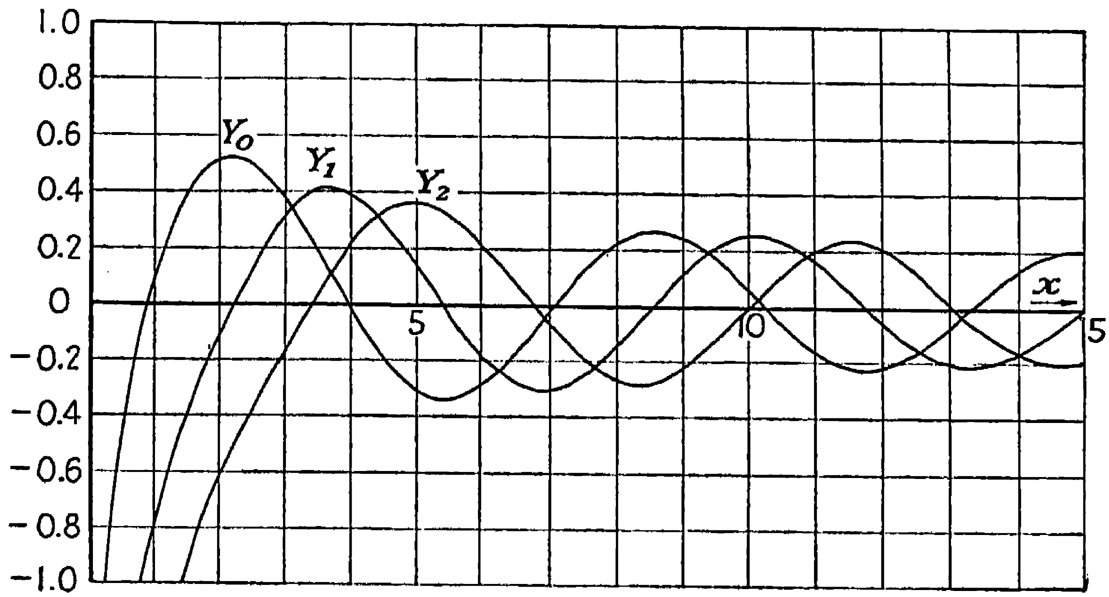

FIG. 3.2.—Bessel functions of the second kind.

The *Bessel functions of order ν of the third kind* or *Hankel functions of order ν* are defined as

$$\begin{aligned} H_\nu^{(1)}(x) &= J_\nu(x) + iY_\nu(x) \\ H_\nu^{(2)}(x) &= J_\nu(x) - iY_\nu(x) \end{aligned} \tag{3.7}$$

These functions are complex quantities.

In practice we must generally deal with an equation containing another parameter k [cf. Eq. (1.24)], *viz.*,

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(k^2 - \frac{\nu^2}{x^2}\right) y = 0 \tag{3.8}$$

This equation is again readily reduced to the standard form (2.1) by taking $\xi = kx$ as the independent variable. Then we have

$$\frac{d^2y}{d\xi^2} + \frac{1}{\xi}\frac{dy}{d\xi} + \left(1 - \frac{\nu^2}{\xi^2}\right) y = 0$$

and the general solution of (3.8) becomes

$$y = AJ_\nu(kx) + BY_\nu(kx) \tag{3.9}$$

4. Special Properties of Some Bessel Functions.—By taking the derivative of the left side of Eq. (1.2), we find

$$\frac{d^2y'}{dx^2} + \frac{1}{x}\frac{dy'}{dx} + \left(1 - \frac{1}{x^2}\right)y' = 0 \tag{4.1}$$

where $y' = dy/dx$. This equation is identical with Bessel's equation of the first order for the function y'. Its general solution is, therefore,

$$y' = \frac{dy}{dx} = AJ_1(x) + BY_1(x)$$

Since y itself is a solution of Bessel's differential equation of zero order [Eq. (1.2)], it appears that the derivatives of the Bessel functions of zero order can be expressed by the functions of first order. In fact, from the series expansions for $J_1(x)$ and $Y_1(x)$, it is found that

$$-\frac{d}{dx}J_0(x) = J_1(x)$$

and (4.2)

$$-\frac{d}{dx}Y_0(x) = Y_1(x)$$

Bessel Functions for which ν is Half an Odd Integer.—This is an important case, since these particular Bessel functions can be expressed in finite form by elementary functions. Consider, for instance, Eq. (2.1) with $\nu = \frac{1}{2}$:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{1}{4x^2}\right)y = 0 \tag{4.3}$$

The substitution

$$y = zx^{-\frac{1}{2}}$$

yields the differential equation for z

$$\frac{d^2z}{dx^2} + z = 0 \tag{4.4}$$

Hence, $z = A\cos x + B\sin x$, and the general solution of Eq. (4.3) becomes

$$y = \frac{1}{\sqrt{x}}(A\cos x + B\sin x) \tag{4.5}$$

Consequently, the functions $J_{\frac{1}{2}}(x)$ and $J_{-\frac{1}{2}}(x)$ must have the

form (4.5). They are given by

$$J_{1/2}(x) = \frac{\sin x}{\sqrt{\frac{1}{2}\pi x}} \qquad \text{and} \qquad J_{-1/2}(x) = \frac{\cos x}{\sqrt{\frac{1}{2}\pi x}} \tag{4.6}$$

The standard form for the function of the order $\frac{1}{2}$ of the second kind is, according to Eq. (3.3),

$$Y_{1/2}(x) = \frac{-1}{\sin \frac{1}{2}\pi} J_{-1/2}(x) = -J_{-1/2}(x)$$

In the same way when $\nu = n + \frac{1}{2}$ where n is any integer, the Bessel functions are reduced to sums of simple polynomials in $1/x$ multiplied by $\sin x$ and $\cos x$, respectively. For example,

$$J_{3/2}(x) = \frac{1}{\sqrt{\frac{1}{2}\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

$$J_{-3/2}(x) = \frac{1}{\sqrt{\frac{1}{2}\pi x}}\left(-\sin x - \frac{\cos x}{x}\right)$$

and (4.7)

$$J_{5/2}(x) = \frac{1}{\sqrt{\frac{1}{2}\pi x}}\left[\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right]$$

$$J_{-5/2}(x) = \frac{1}{\sqrt{\frac{1}{2}\pi x}}\left[\frac{3}{x}\sin x + \left(\frac{3}{x^2} - 1\right)\cos x\right]$$

A more complete list of these functions will be found in Gray, Mathews and MacRobert, "A Treatise on Bessel Functions and Their Application to Physics," page 17.

The standard form for the corresponding function of the second kind is [cf. Eq. (3.3)]

$$Y_{n+1/2}(x) = -\frac{1}{\sin (n + \frac{1}{2})\pi} J_{-(n+1/2)}(x) \tag{4.8}$$

Hence $J_{(n+1/2)}(x)$ and $J_{-(n+1/2)}(x)$ are two independent solutions of Bessel's equation of the order $n + \frac{1}{2}$.

5. Modified Bessel Functions.—If we put into Eq. (3.8) $k = i$, we are led to the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)y = 0 \tag{5.1}$$

A solution of this equation is found by replacing x in the power series representing $J_\nu(x)$ by ix. Hence, $J_\nu(ix)$ is a solution

of Eq. (5.1). The function $I_\nu(x) = i^{-\nu}J_\nu(ix)$ is taken as the standard form for one of the fundamental solutions of Eq. (5.1). It is known as the *modified Bessel function of the first kind of order* ν. It is a real function. Another fundamental solution of Eq. (5.1) is known as the *modified Bessel function of the second kind* and is generally defined by

$$K_\nu(x) = \frac{\pi/2}{\sin \nu\pi}[I_{-\nu}(x) - I_\nu(x)] \tag{5.2}$$

The general solution of Eq. (5.1) is

$$y = AI_\nu(x) + BK_\nu(x) \tag{5.3}$$

It can also be shown that we have

$$\frac{\pi}{2}i^{\nu+1}H_\nu^{(1)}(ix) = K_\nu(x) \tag{5.4}$$

These functions are not of the oscillating type; their behavior is similar to that of the exponential functions. This is shown by their asymptotic approximations; for $x \to \infty$ we have

$$\begin{aligned} I_0(x) &\cong \frac{e^x}{\sqrt{2\pi x}} \\ K_0(x) &\cong \sqrt{\frac{\pi}{2x}}e^{-x} \end{aligned} \tag{5.5}$$

For $x \to 0$, we have

$$I_0(0) = 1$$

and

$$K_0(x) \cong -\log\frac{x}{2} - \gamma \tag{5.6}$$

Values of $I_0(x)$ and $K_0(x)$ are plotted in Fig. 5.1.

We also have a simple relation between $I_0(x)$ and $I_1(x)$ similar to the relation between $J_0(x)$ and $J_1(x)$ [cf. Eq. (4.2)]:

$$\frac{dI_0(x)}{dx} = I_1(x) \tag{5.7}$$

6. Tables and Notations.—As we said, there is no unanimity in the literature either in the choice of the standard forms of the Bessel functions or in the choice of the symbols representing

them. For the reader's convenience we compiled in a comparative table the definitions and notations of the books listed at the end of this chapter. A few remarks for the use of this table appear necessary:

a. The *Bessel function of the first kind* is defined and denoted in the same way by nearly all authors.

b. For the *Bessel function of the second kind, i.e.*, for a second independent solution of Bessel's differential equation, Weber suggested the function that we defined by Eq. (3.3) and denoted by $Y_\nu(x)$. Neumann introduced as the second independent solution a function that is a linear combination of our $J_\nu(x)$ and

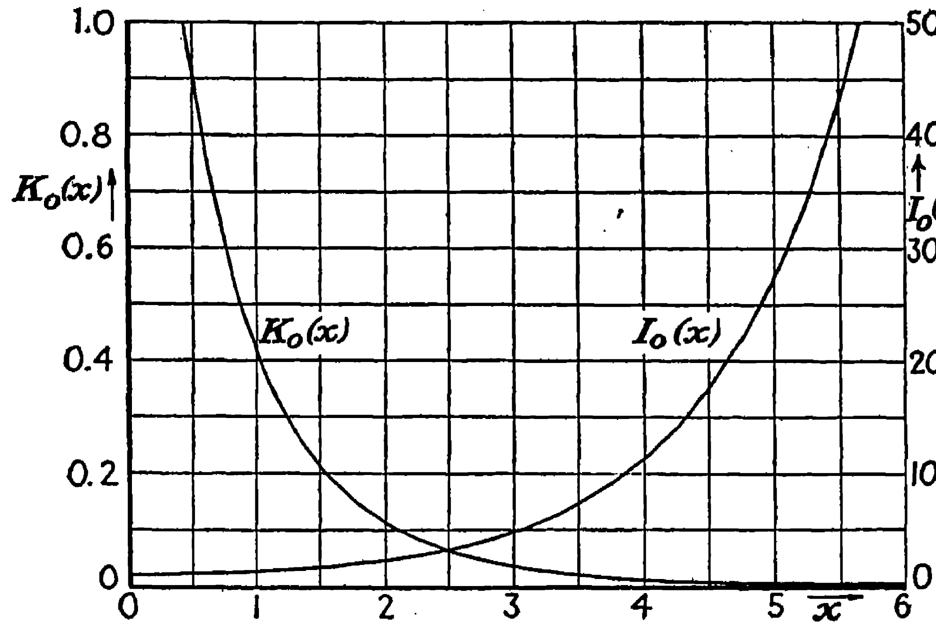

FIG. 5.1.—Modified Bessel functions of zero order.

$Y_\nu(x)$, *viz.*, $\frac{\pi}{2}Y_\nu(x) + (\log 2 - \gamma)J_\nu(x)$. However, some German authors use the term *Neumann's function* for any form of the second independent solution.

c. The German authors do not use the term *modified Bessel functions;* they consider them as Bessel or Hankel functions of an imaginary argument.

References 1 to 4 listed at the end of this chapter contain tables and charts for Bessel functions of different kinds; the tables contained in references 1 and 2 are especially comprehensive. The lack of standardized notation is an impediment in the use of such tables. We hope that the table of equivalence of symbols given below will be helpful for the reader.

TABLE OF EQUIVALENCE OF SYMBOLS FOR BESSEL FUNCTIONS

	This book	Jahnke-Emde	Mc-Lach-lan
Function of first kind......	$J_\nu(x)$	$J_\nu(x)$	$J_\nu(x)$
Function of second kind (Weber)................	$Y_\nu(x)$	$N_\nu(x)$	$Y_\nu(x)$
Function of second kind (Neumann).............	$\frac{\pi}{2}Y_\nu + (\log 2 - \gamma)J_\nu$		$\mathbf{Y}_\nu(x)$
Modified function of first kind..................	$I_\nu(x)$	$i^{-\nu}J_\nu(ix)$	$I_\nu(x)$
Modified function of second kind..................	$K_\nu(x)$	$\frac{1}{2}\pi i^{\nu+1}H_\nu^{(1)}(ix)$	$K_\nu(x)$

	Wat-son	Whit-taker Wat-son	Gray, Mathews, MacRobert
Function of first kind.............	$J_\nu(x)$	$J_\nu(x)$	$J_\nu(x)$
Function of second kind (Weber)...	$Y_\nu(x)$	$Y_\nu(x)$	$\frac{2}{\pi}[Y_\nu - (\log 2 - \gamma)J_\nu]$
Function of second kind (Neumann)	$Y^{(\nu)}(x)$	$Y^{(\nu)}(x)$	$Y_\nu(x)$
Modified function of first kind.....	$I_\nu(x)$	$I_\nu(x)$	$I_\nu(x)$
Modified function of second kind...	$K_\nu(x)$	$\dfrac{K_\nu(x)}{\cos \nu\pi}$	$K_\nu(x)$

7. Some Equivalent Forms of Bessel's Differential Equation.—The differential equation

$$\frac{d^2y}{dx^2} + \frac{a}{x}\frac{dy}{dx} + by = 0 \tag{7.1}$$

where a and b are constants, is equivalent to Bessel's differential equation (1.1). If we substitute

$$y = x^\nu z \tag{7.2}$$

we have, by differentiation of (7.2),

$$\begin{aligned} \frac{dy}{dx} &= \nu x^{\nu-1}z + x^\nu \frac{dz}{dx} \\ \frac{d^2y}{dx^2} &= \nu(\nu - 1)x^{\nu-2}z + 2\nu x^{\nu-1}\frac{dz}{dx} + x^\nu \frac{d^2z}{dx^2} \end{aligned} \tag{7.3}$$

Substituting (7.2) and (7.3) in Eq. (7.1), we obtain

$$x^\nu \frac{d^2z}{dx^2} + (a + 2\nu)x^{\nu-1}\frac{dz}{dx} + \{bx^\nu + [(a-1)\nu + \nu^2]\, x^{\nu-2}\}z = 0 \tag{7.4}$$

It is seen that if we put

$$a + 2\nu = 1$$

or

$$\nu = \frac{1-a}{2}$$

and divide Eq. (7.4) by x^ν, we obtain

$$\frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx} + \left(b - \frac{\nu^2}{x^2}\right)z = 0 \tag{7.5}$$

According to Eq. (3.9) the general solution of Eq. (7.5) is

$$z = Z_\nu(x\sqrt{b})$$

provided the symbol Z_ν denotes the general solution of Bessel's differential equation of νth order. Hence, the solution of Eq. (7.1) is given by

$$y = x^\nu Z_\nu(x\sqrt{b}) \tag{7.6}$$

where $\nu = \dfrac{1-a}{2}$. For example, when $a = 0$ then $\nu = \dfrac{1}{2}$, and the solution of the equation

$$\frac{d^2y}{dx^2} + by = 0$$

is given by

$$y = x^{1/2} Z_{1/2}(x\sqrt{b})$$

We remember that [cf. Eq. (4.5)]

$$Z_{1/2}(x) = \frac{1}{\sqrt{x}}(A \cos x + B \sin x)$$

and, therefore,

$$y = A' \cos x\sqrt{b} + B' \sin x\sqrt{b}$$

where A' and B' are constants.

Bessel's equation of zero order can be written in the form:

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + xy = 0 \tag{7.7}$$

We generalize this expression and write

$$\frac{d}{dx}\left(x^m \frac{dy}{dx}\right) + cx^n y = 0 \tag{7.8}$$

We shall show that Eq. (7.8) is equivalent to a Bessel's equation whose order depends on the exponents m and n. Let us introduce a new independent variable t by putting $x = t^\alpha$. Then

$$dx = \alpha t^{\alpha-1}\,dt$$

and we obtain

$$\frac{dy}{dx} = \frac{1}{\alpha t^{\alpha-1}}\frac{dy}{dt} \tag{7.9}$$

Introducing $x = t^\alpha$ and (7.9) into Eq. (7.8), this equation becomes

$$\frac{1}{\alpha t^{\alpha-1}}\frac{d}{dt}\left(\frac{1}{\alpha}\, t^{\alpha m+1-\alpha}\frac{dy}{dt}\right) + ct^{n\alpha}y = 0$$

or

$$\frac{d^2y}{dt^2} + \frac{\alpha(m-1)+1}{t}\frac{dy}{dt} + c\alpha^2 t^{\alpha(n-m+2)-2}y = 0 \tag{7.10}$$

It is evident that if we put $\alpha(n - m + 2) - 2 = 0$ or

$$\alpha = \frac{2}{n-m+2} \tag{7.11}$$

Equation (7.10) is reduced to the form (7.1), and we obtain

$$\frac{d^2y}{dt^2} + \frac{\alpha(m-1)+1}{t}\frac{dy}{dt} + c\alpha^2 y = 0 \tag{7.12}$$

According to Eq. (7.6), the solution of this equation is

$$y = t^\nu Z_\nu(t\alpha\sqrt{c})$$

where

$$\nu = \frac{\alpha}{2}(1-m) = \frac{1-m}{n-m+2}$$

Hence, returning to the independent variable x, by substituting $t = x^{\frac{1}{\alpha}}$, the solution of Eq. (7.8) is

$$y = x^{\frac{\nu}{\alpha}} Z_\nu(x^{\frac{1}{\alpha}}\alpha\sqrt{c}) \tag{7.13}$$

where

$$\nu = \frac{1-m}{n-m+2},$$

$$\frac{1}{\alpha} = \frac{n-m}{2} + 1, \qquad \text{and} \qquad \frac{\nu}{\alpha} = \frac{1-m}{2}^{*}$$

* This solution breaks down if $m = n + 2$, but in this case Eq. (7.8) becomes a differential equation with constant coefficients if we substitute $u = \log x$ instead of x as independent variable.

Consider, for example, the equation

$$\frac{d^2y}{dx^2} + xy = 0 \tag{7.14}$$

In this case $m = 0$, $n = 1$, and $c = 1$; therefore, $\alpha = \frac{2}{3}$, $\nu = \frac{1}{3}$, and the solution becomes

$$y = x^{1/2} Z_{1/3}(\tfrac{2}{3}x^{3/2}) \tag{7.15}$$

or

$$y = Ax^{1/2}J_{1/3}(\tfrac{2}{3}x^{3/2}) + Bx^{1/2}Y_{1/3}(\tfrac{2}{3}x^{3/2}) \tag{7.16}$$

This solution may also be found as follows: According to Eqs. (2.4) and (2.6), if ν is not an integer, Bessel's equation of the order ν has two independent solutions beginning with x^ν and $x^{-\nu}$. Taking into account that the independent variable is $x^{3/2}$, the solutions $Z_{1/3}(\frac{2}{3}x^{3/2})$ will have the form:

$$\eta_1 = x^{1/2}(a_0 + a_1x^3 + a_2x^6 + \cdots)$$

and

$$\eta_2 = x^{-1/2}(b_0 + b_1x^3 + b_2x^6 + \cdots)$$

Substituting these expressions in Eq. (7.15), we obtain the solutions of (7.14) in the form:

$$y_1 = x(a_0 + a_1x^3 + a_2x^6 + \cdots) \tag{7.17}$$

and

$$y_2 = b_0 + b_1x^3 + b_2x^6 + \cdots \tag{7.18}$$

The coefficients $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ can be determined directly from Eq. (7.14). Substituting (7.18), for example, in Eq. (7.14), we have

$$2 \cdot 3b_1 + b_0 = 0, \qquad 5 \cdot 6b_2 + b_1 = 0, \qquad \ldots$$

or

$$y_2 = b_0\left(1 - \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} - \frac{x^9}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} + \cdots\right) \tag{7.19}$$

In a similar way we obtain for the solution which starts with the first power of x

$$y_1 = a_0x\left(1 - \frac{x^3}{3 \cdot 4} + \frac{x^6}{3 \cdot 4 \cdot 6 \cdot 7} - \frac{x^9}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} + \cdots\right) \tag{7.20}$$

In some cases it will be convenient to determine the constants a_0 and b_0 directly from the boundary conditions, and

use the series (7.19) and (7.20). However, if x is large, the convergence of the series becomes poor, and it will be more practical to identify the solutions as Bessel functions and use the tabulated values.

Problems

1. Find the general solution of

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + k^2x^2y = 0$$

2. Plot the functions

$$y = J_0(x^2)$$
$$y = Y_0(x^2)$$

3. Solve graphically the equation

$$J_0(x) = \tfrac{1}{4}x$$

4. Find the general solution of the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - k^2y = a$$

where k and a are constants.

5. Find the general solution of

$$\frac{d^2y}{dx^2} + \frac{5}{x}\frac{dy}{dx} - 16x^4y = 0$$

Hint: Multiply the equation by a power of x so that it takes the form

$$\frac{d}{dx}\left(x^m\frac{dy}{dx}\right) + cx^ny = 0$$

6. Find the general solutions of

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + 4y = 0$$

and

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{1}{4}y = 0$$

7. Find the general solution of

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \frac{1}{x^2}y = 2x + 4$$

Hint: Try a particular solution of the form: $\alpha x^2 + \beta x^3$.

8. Expand the exponential functions $e^{\frac{1}{2}xt}$ and $e^{-\frac{1}{2}\frac{x}{t}}$ in a series of ascending

powers of $\frac{1}{2}xt$, $\frac{1}{2}\frac{x}{t}$, respectively, and show that the coefficients of $t^0, t, t^2, \ldots$ in the product of the two infinite series are equal to $J_0(x), J_1(x), J_2(x), \ldots$.

9. Verify that the differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 6y = 0$$

has a solution of the form $y = x^2 - a$. Determine the value of a.

10. Find the particular solutions of the differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

which have the values $y = 1$ for $x = 1$, $dy/dx = 0$ for $x = 0$, when $n = 0$, $n = 2$, and $n = 4$. Find the particular solutions of the same equations which have the values $y = 0$ for $x = 0$ and $y = 1$ for $x = 1$, when $n = 1$, $n = 3$, and $n = 5$.

Hint: Try a solution in the form of a power series expansion

$$y = a_0 + a_1x + a_2x^2 + \cdots,$$

and determine the coefficients by substitution in the differential equation. The series will be found to contain a finite number of terms, *i.e.*, the solutions asked for are polynomials. They are known as *Legendre's polynomials*.

11. For one of the natural oscillations of a circular membrane of radius a the deflection w is given by the formula

$$w = J_2\left(\alpha_2\frac{r}{a}\right)\sin 2\theta$$

where r and θ are polar coordinates in the plane of the membrane and α_2 is the second root of the equation $J_2(x) = 0$. Show that the nodal lines ($w = 0$) of the membrane are a circle and two perpendicular diameters. Plot the lines of constant height for $w = 0.1, 0.2, 0.3, 0.4, -0.1, -0.2, -0.3$, and -0.4, using the tables for Bessel functions of the first kind and second order.

References

1. WATSON, G. N.: "Theory of Bessel Functions," Cambridge University Press, London, 1922.
2. JAHNKE, E., and F. EMDE: "Tables of Functions," B. G. Teubner, Leipzig, 1933.
3. MCLACHLAN, N. W.: "Bessel Functions for Engineers," Oxford University Press, New York, 1934.
4. GRAY, A., G. B. MATHEWS, and J. M. MACROBERT: "A Treatise on Bessel Functions and Their Applications to Physics," Macmillan & Company, Ltd., London, 1931.
5. WHITTAKER, E. J., and G. N. WATSON: "Modern Analysis," Cambridge University Press, London, 4th ed., 1927.

CHAPTER III

FUNDAMENTAL CONCEPTS OF DYNAMICS

> In this sense rational mechanics will be the science of motions resulting from any forces whatsoever, and of the forces required to produce any motions, accurately proposed and demonstrated.
>
> Cambridge, College of the Holy Trinity, 1686.
>
> Is. Newton.

Introduction.—In this chapter the fundamental notions and principles of the *mechanica rationalis* are reviewed first from the viewpoint of Newton's laws of motion. The equations of motion are applied to a single mass point, to a system of mass points, and to rigid bodies. As an example of the dynamics of rigid bodies the fundamentals of the theory of the gyroscope are treated. In the second part of the chapter the equilibrium and the motion of arbitrary systems are considered from the viewpoints of the principle of virtual displacement and the principle of d'Alembert. In the last section Lagrange's equations of motion are deduced. A section on the elements of vector algebra is also included in this chapter.

1. Newton's Laws of Motion.—*Newton's first law* of motion is often called the *law of inertia.* It was stated by Newton in the following form:

"Every body persists in its state of rest or of uniform motion in a straight line except insofar as it may be compelled by force to change that state."

The fundamental principle that is announced by this law and underlies our whole conception of dynamics was first discovered by Galileo Galilei (1564–1642). Before him, the opinion prevailed that motion could be maintained only by a permanent application of force. Most previous thinkers connected the force with the speed or the velocity of the moving body; Galilei first connected it with the change of the velocity, *i.e.*, with the acceleration. Newton (1643–1727) specified the relation between the force and the rate of change of velocity in the following way. He defines the *quantity of motion,* or,

using modern terminology, the *momentum*, of a body as the product of its mass and its velocity. Then he announces that the rate of change of momentum is proportional to the force applied to the body, and it takes place in the direction of the straight line in which the force acts. This is known as *Newton's second law.*

The third law of motion is called the *law of action and reaction.* It refers to the forces acting between bodies which belong to the same mechanical system and states that the force exerted by a body A on a body B is equal and opposite in direction to the force exerted by B on A, and both forces act along the same line.

2. Addition and Multiplication of Vectors.—In modern terminology the velocity and the force are *vectors;* the mass is a *scalar.* A *scalar* is a quantity that is completely determined by its magnitude. A quantity that is determined by magnitude and direction is called a *vector.* A vector quantity is represented geometrically by a line segment whose length is equal, in some convenient scale, to the magnitude and whose direction—indicated by an arrow—is given by the direction of the vector quantity. Multiplication of a vector by a scalar changes the magnitude of the vector but does not change its direction. The momentum is the product of mass and velocity; hence, the velocity and the momentum are vectors having the same direction. Newton's second law states that the rate of change of the momentum vector per unit time is equal to the force. Or, written in vectorial form,

$$\frac{d}{dt}(m\bar{v}) = \bar{F} \tag{2.1}$$

where m is the mass, $\bar{v}$ is the velocity, and $\bar{F}$ is the force vector.

The rate of change of the momentum is the limit of the ratio of the change of the vector $m\bar{v}$ during the time between t and $t + \Delta t$ to the time interval Δt:

$$\frac{d}{dt}(m\bar{v}) = \lim_{\Delta t \to 0} \frac{\Delta(m\bar{v})}{\Delta t} \tag{2.2}$$

Hence, we must first define the change of the vector $m\bar{v}$; it is the vectorial difference between the vector $m\bar{v}$ observed at the time $t + \Delta t$ and the same vector observed at the time t. We call the vector whose components are equal to $a_x - b_x$, $a_y - b_y$,

$a_z - b_z$, where a_x, a_y, a_z and b_x, b_y, b_z are the components of $\bar{a}$ and $\bar{b}$, respectively, the difference $\bar{a} - \bar{b}$. It is seen from Fig. (2.1) that if the vector $\bar{a}$ is represented by $\overline{OA}$ and the vector $\bar{b}$ by $\overline{OB}$, the difference $\bar{c} = \bar{a} - \bar{b}$ is represented by the line $\overline{BA}$. We call the vector whose components are equal to the sum of the components of two vectors the sum of the two vectors. For example, $\bar{a} = \bar{b} + \bar{c}$. We obtain the line representing the vector $\bar{a}$ by drawing the line $\overline{OB}$, representing the vector $\bar{b}$, and then drawing the line $\overline{BA}$, representing $\bar{c}$, from the end point of the vector $\overline{OB}$. The line connecting the origin of $\bar{b}$ with the end point of $\bar{c}$ gives the vector $\bar{a} = \bar{b} + \bar{c}$. The addition of vectors is a commutative operation: $\bar{a} = \bar{b} + \bar{c} = \bar{c} + \bar{b}$.

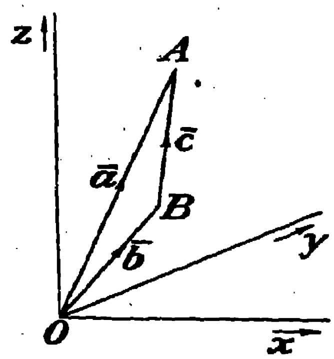

FIG. 2.1.—Subtraction of vectors: $\bar{c} = \bar{a} - \bar{b}$.

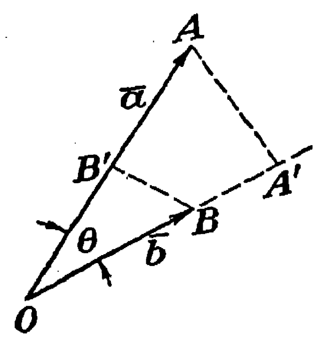

FIG. 2.2.—The scalar product of two vectors: $\bar{a} \cdot \bar{b} = \overline{OA} \cdot \overline{OB} \cdot \cos\theta$.

If $\overline{OB}$ is the momentum vector at the time t and $\overline{OA}$ the momentum vector at the time $t + \Delta t$, the difference $\overline{BA}$ is the change of the momentum vector in the time interval Δt, and, therefore, $\lim\limits_{\Delta t \to 0} \frac{\overline{BA}}{\Delta t} = \frac{d}{dt}(m\bar{v})$. This quantity is equal to the force $\bar{F}$. The vector $\overline{BA}$ is approximately equal to $\bar{F}\,\Delta t$ if Δt is small.

It is seen that if the magnitude of the momentum vector remains constant and only its direction changes, the force is normal to the momentum. If the direction of the momentum remains constant and its magnitude changes, force and momentum have the same direction.

In further developments of this chapter we shall encounter products of vectors. We distinguish between the *scalar product* and the *vector product*. The scalar product $\bar{a} \cdot \bar{b}$ is a scalar equal

to the magnitude of $\bar{a}$ multiplied by the projection OB' of $\bar{b}$ upon $\bar{a}$, or to the magnitude of $\bar{b}$ multiplied by the projection OA' of $\bar{a}$ upon $\bar{b}$. If we denote the angle between $\bar{a}$ and $\bar{b}$ by θ (Fig. 2.2), both definitions give the value $|a||b| \cos \theta$ for $\bar{a} \cdot \bar{b}$. It is seen that $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$, *i.e.*, the scalar multiplication is also a commutative operation; for $\bar{a} \cdot \bar{a}$ we write $(\bar{a})^2$.

The vector product $\bar{a} \times \bar{b}$ is a vector which is normal to the plane passing through $\overline{OA}$ and $\overline{OB}$, where $\bar{a} = \overline{OA}$ and $\bar{b} = \overline{OB}$ (Fig. 2.3). Its direction is determined such that for an observer looking in the direction of the vector $\bar{a} \times \bar{b}$ a clockwise turn through an angle less than 180° brings the vector $\bar{a}$ into the direction $\bar{b}$. Its magnitude is equal to twice the area of the triangle OAB or equal to OA multiplied by the normal distance BB' between the point B and the line $\overline{OA}$, or to OB multiplied by the normal distance AA' between the point A and the line $\overline{OB}$. It is seen that according to this definition $\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$, *i.e.*, $\bar{a} \times \bar{b}$ and $\bar{b} \times \bar{a}$ have the same magnitudes but opposite directions.

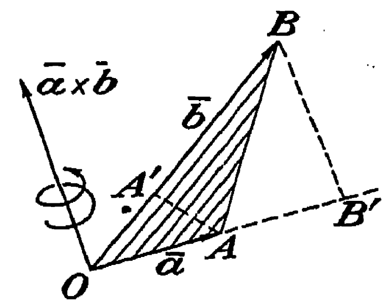

FIG. 2.3.—The vector product $\bar{a} \times \bar{b}$ is a vector normal to $\bar{a}$ and $\bar{b}$; its magnitude is twice the area OAB.

The scalar product of two parallel vectors is equal to the product of the magnitudes of the vectors. The scalar product of two vectors that are perpendicular to each other is zero. The vector product of two parallel vectors is zero, and the magnitude of the vector product of two perpendicular vectors is equal to the product of the magnitudes of the vectors.

The three components of $\bar{a} \times \bar{b}$ are $a_yb_z - a_zb_y$, $a_zb_x - a_xb_z$, and $a_xb_y - a_yb_x$. The following method is useful for the computation of these components: We denote the "unit vector" in the x-direction by $\bar{\imath}$, in the y-direction by $\bar{\jmath}$, in the z-direction by $\bar{k}$. Since $\bar{\imath}$, $\bar{\jmath}$ and $\bar{k}$ are normal to each other,

$$\bar{\imath} \cdot \bar{\jmath} = \bar{\jmath} \cdot \bar{k} = \bar{k} \cdot \bar{\imath} = 0, \qquad \text{and} \qquad (\bar{\imath})^2 = (\bar{\jmath})^2 = (\bar{k})^2 = 1;$$

furthermore, $\bar{\imath} \times \bar{\jmath} = \bar{k}$, $\bar{\jmath} \times \bar{k} = \bar{\imath}$, and $\bar{k} \times \bar{\imath} = \bar{\jmath}$. Taking these relations into account, we can carry out formally all arithmetical operations involving additions and multiplications of vectors. For example,

$$\begin{aligned} \bar{a} \cdot \bar{b} &= (a_x\bar{\imath} + a_y\bar{\jmath} + a_z\bar{k}) \cdot (b_x\bar{\imath} + b_y\bar{\jmath} + b_z\bar{k}) \\ &= a_xb_x + a_yb_y + a_zb_z \end{aligned} \tag{2.3}$$

The vector product

$$\bar{a} \times \bar{b} = (a_x\bar{\imath} + a_y\bar{\jmath} + a_z\bar{k}) \times (b_x\bar{\imath} + b_y\bar{\jmath} + b_z\bar{k})$$
$$= (a_yb_z - a_zb_y)\bar{\imath} + (a_zb_x - a_xb_z)\bar{\jmath} + (a_xb_y - a_yb_x)\bar{k} \quad (2.4$$

We can prove in this way without difficulty the following rul for triple vector products:

$$\bar{a} \times (\bar{b} \times \bar{c}) = \bar{b}(\bar{a} \cdot \bar{c}) - \bar{c}(\bar{a} \cdot \bar{b}) \quad (2.5$$

Using the concept of the vector product we obtain an impor tant result concerning the rate of change of an arbitrary vector $\bar{a}$ An infinitesimal change of the vector $\bar{a}$ consists of two compo nents: one in the direction of the vector and one normal to it The first component is equal tc the change of the magnitude $|\bar{a}$ of the vector, the second can be derived by an infinitesimal rotation around an axis which is perpendicular to $\bar{a}$. If the angle of rotation is $\Delta\alpha$, we call $\lim_{\Delta t \to 0} \Delta\alpha/\Delta t = \bar{\omega}$ the *angular velocity*. We consider the angular velocity $\bar{\omega}$ as a vector whose direction coincides with the axis of rotation such that to an observer looking in the direction of the vector the rotation appears clockwise. The rate of change of $\bar{a}$ due to the rotation is equal to $\bar{\omega} \times \bar{a}$. This result will often be used later in this chapter.

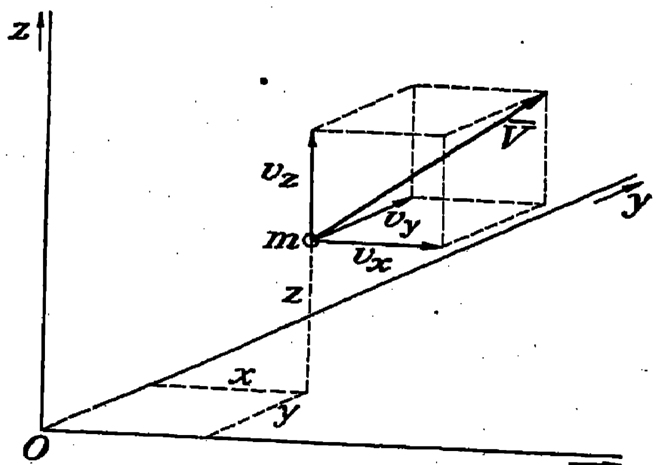

FIG. 3.1.—The three components of the velocity vector.

3. Motion of a Single Mass Point.—Let us apply Newton's second law first to a single mass point, then to a system of n mass points.

If we denote the mass of a mass point by m, its coordinates by x, y, and z, its velocity components by v_x, v_y, and v_z (Fig. 3.1), the components of the momentum of the mass point, according to the above definition, are mv_x, mv_y, and mv_z, and Newton's second law states that

$$\begin{aligned} \frac{d}{dt}(mv_x) &= X \\ \frac{d}{dt}(mv_y) &= Y \\ \frac{d}{dt}(mv_z) &= Z \end{aligned} \qquad (3.1)$$

where X, Y, and Z are the components of the force applied to m.

The velocity components v_x, v_y, and v_z are equal to the derivatives of x, y, and z with respect to the time; hence, Eq. (3.1) can also be written in the form:

$$\begin{aligned} \frac{d}{dt}\left(m\frac{dx}{dt}\right) &= X \\ \frac{d}{dt}\left(m\frac{dy}{dt}\right) &= Y \\ \frac{d}{dt}\left(m\frac{dz}{dt}\right) &= Z \end{aligned} \tag{3.2}$$

If the mass is invariable with time, the Eqs. (3.1) and (3.2) become

$$\begin{aligned} m\frac{dv_x}{dt} &= m\frac{d^2x}{dt^2} = X \\ m\frac{dv_y}{dt} &= m\frac{d^2y}{dt^2} = Y \\ m\frac{dv_z}{dt} &= m\frac{d^2z}{dt^2} = Z \end{aligned} \tag{3.3}$$

The vector with the components dv_x/dt, dv_y/dt, dv_z/dt is called the *acceleration vector*. Equation (3.3) states that in the case of constant mass the force is equal to the product of mass and acceleration.

To write the Eqs. (3.1), (3.2), and (3.3) in vector form we introduce the *radius vector* $\bar{r}$, whose components are x, y, z; then the velocity vector $\bar{v} = d\bar{r}/dt$. According to Eq. (3.1) we have

$$\frac{d}{dt}(m\bar{v}) = \frac{d}{dt}\left(m\frac{d\bar{r}}{dt}\right) = \bar{F} \tag{3.4}$$

If m is constant, we obtain

$$m\frac{d\bar{v}}{dt} = m\frac{d^2\bar{r}}{dt^2} = \bar{F} \tag{3.5}$$

The vector $d\bar{v}/dt = d^2\bar{r}/dt^2$ is the acceleration vector, whose components are d^2x/dt^2, d^2y/dt^2, d^2z/dt^2. If the magnitude of $\bar{v}$ is constant, the acceleration vector is normal to the velocity vector.

We now define the moment of a force $\bar{F}$ with respect to an arbitrary point P (Fig. 3.2) by the vector product $\overline{PA} \times \bar{F}$

where A is any point on the line of action of the force $\bar{F}$. According to this definition the moment of the force $\bar{F}$ with respect to P is a vector normal to the plane that goes through the line of action of $\bar{F}$ and the point P. Its magnitude is equal to the force multiplied by the normal distance n between P and the line of action of $\bar{F}$. It is seen that the moment thus defined is independent of the choice of A.

For example, the moment of the force $\bar{F}$ with respect to the origin of the coordinate system is equal to the vector product

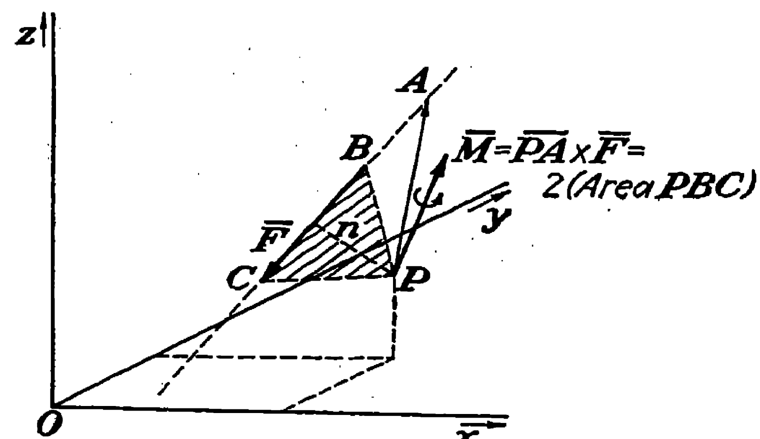

FIG. 3.2.—The moment of the force F with respect to a point P.

$\bar{r} \times \bar{F}$, where $\bar{r}$ is the radius vector of an arbitrary point on the line of action of the force. The components of this vector product, $M_x = yZ - zY$, $M_y = zX - xZ$, and $M_z = xY - yX$, are called the components of the moment $\bar{M}$ or the moments of the force with respect to the coordinate axes indicated by the subscripts.

By analogy with the moment of a force, we call the vector product of the radius vector and the momentum vector the moment of momentum of the mass point with respect to the origin. Denoting the moment of momentum by $\bar{H}$, we have $\bar{H} = \bar{r} \times m\bar{v}$.

Let us now calculate the rate of change of the moment of momentum. We obtain:

$$\frac{d\bar{H}}{dt} = \frac{d\bar{r}}{dt} \times m\bar{v} + \bar{r} \times \frac{d}{dt}(m\bar{v})$$

Since $\frac{d\bar{r}}{dt} = \bar{v}$, we have $\frac{d\bar{r}}{dt} \times \bar{v} = 0$; on the other hand, $\frac{d}{dt}(m\bar{v}) = \bar{F}$ [Eq. (3.4)], and thus we obtain

$$\frac{d\bar{H}}{dt} = \bar{r} \times \bar{F} \tag{3.6}$$

or, in other words, the rate of change of the moment of momentum of a mass point with respect to the origin O is equal to the moment of the force with respect to the same point.

Let us assume that $\bar{M} = 0$, *i.e.*, the moment of the force $\bar{F}$ with respect to the origin O vanishes. This is the case if $\bar{F}$ is a so-called *central force* whose line of action passes through the origin. According to Eq. (3.6), under the action of central forces the moment of momentum of a mass point with respect to the force center is invariable. It follows that the path described by a mass point under action of a central force is always a plane curve; it lies in the plane perpendicular to the moment of momentum vector. The theorem of the conservation of the moment of momentum is sometimes called the *Law of Areas* because the magnitude of the moment of momentum is equal to the product of the mass and twice the area swept over in unit time by the radius vector $\overline{OP}$. The law of areas—with reference to the motion of the planets around the sun—was first discovered by Kepler (1571–1630) (*Kepler's first law*), based on the analysis of observations made by Tycho Brahe (1546–1601).

4. Application of Newton's Laws to a System of Mass Points.— Denote the masses of n mass points by $m_1, m_2, \ldots, m_n$, the mass of the ith mass point by m_i, its radius vector by $\bar{r}_i$, and its velocity by $\bar{v}_i$. Furthermore, denote the force applied to m_i from outside the mechanical system by $\bar{F}_i$, and the force exerted by the mass point m_k on the mass point m_i by $\bar{F}_{ik}$. Then, according to Newton's second law, the equation of motion of the ith mass point is

$$\frac{d}{dt}(m_i\bar{v}_i) = \frac{d}{dt}\left(m_i\frac{d\bar{r}_i}{dt}\right) = \bar{F}_i + \sum_k \bar{F}_{ik} \tag{4.1}$$

We call $\bar{F}_i$ an external force, and the $\bar{F}_{ik}$'s, the internal forces.

The application of Newton's third law leads us to important conclusions which hold for an arbitrary system of mass points.

a. First, we add the Eqs. (4.1). Since, according to Newton's third law, $\bar{F}_{ik} = -\bar{F}_{ki}$, the internal forces drop out, and we obtain

$$\frac{d}{dt}\left(\sum m_i\bar{v}_i\right) = \frac{d}{dt}\left(\sum m_i\frac{d\bar{r}_i}{dt}\right) = \sum \bar{F}_i \tag{4.2}$$

We call the vector sum of all momentum vectors the total or resultant momentum of the system. Equation (4.2) announces the following theorem: *The rate of change of the total momentum of the system is equal to the resultant of the external forces and is independent of the internal forces.*

The vector sum of all external forces applied to the system is called the resultant of the external forces.

b. Second, we calculate the rate of change of the moment of momentum of an arbitrary mass point m_i. This, according to Eq. (3.6), is equal to the moment of the external and internal forces applied to m_i. The moments, being defined as vectors, follow the addition law of vectors, and we call the vector sum of two or more moments the *resultant moment.* Thus the rate of change of the resultant moment of momentum of the whole system of n mass points is equal to the resultant moment of all the external and internal forces. In accordance with Newton's third law the internal forces appear twice but with opposite signs. Hence, we obtain the theorem that *the rate of change of the resultant moment of momentum of a system of mass points with respect to an arbitrary point is equal to the resultant moment of the external forces and is independent of the internal forces.*

In mathematical form:

$$\frac{d}{dt}\sum m_i(\bar{r}_i \times \bar{v}_i) = \sum \bar{r}_i \times \bar{F}_i \tag{4.3}$$

In summarizing the results of this section, it appears that both the resultant momentum and the resultant moment of momentum of an arbitrary system are *independent of the internal forces.*

If we apply Eqs. (4.2) and (4.3) to the equilibrium of arbitrary mass systems, it follows that the resultant and the resultant moment of the external forces must vanish if the system is in equilibrium. If, for example, three forces are acting on an arbitrary system, they have to pass through one point if the system is to be in equilibrium.

5. Mass Center (Center of Gravity).—The point C, defined by its coordinates,

$$x_c = \frac{\Sigma m_i x_i}{\Sigma m_i}, \qquad y_c = \frac{\Sigma m_i y_i}{\Sigma m_i}, \qquad z_c = \frac{\Sigma m_i z_i}{\Sigma m_i} \tag{5.1}$$

or by its radius vector

$$\bar{r}_c = \frac{\Sigma m_i \bar{r}_i}{\Sigma m_i} \tag{5.2}$$

is called the *mass center*, or *center of gravity*, of the system of n mass points. Evidently x_c, y_c, z_c represent certain mean values of the coordinates x_i, y_i, z_i where every point enters into the averaging process with a weight proportional to its mass. The expression *center of gravity* refers to the fact that in a homogeneous gravity field the resultant of the gravity forces acting on the masses $m_1, m_2, \ldots, m_n$ passes through the mass center.

We shall assume that the masses $m_1, \ldots, m_n$ are constant and denote the total mass Σm_i by m. Introducing the coordinates of the mass center into Eq. (4.2), we obtain the following equation:

$$\sum \frac{d}{dt}\left(m_i \frac{d\bar{r}_i}{dt}\right) = m\frac{d^2\bar{r}_c}{dt^2} = \sum \bar{F}_i \tag{5.3}$$

The motion of the mass center follows the law of motion of a single mass point having a mass equal to the total mass of the system under the action of the resultant of all external forces applied to the system.

We now introduce the coordinates of the mass center into the expression for the moment of momentum of a mass system.

The resultant moment of momentum of the system is equal to

$$\bar{H} = \Sigma(\bar{r}_i \times m_i\bar{v}_i) \tag{5.4}$$

We substitute into this expression $\bar{r}_i = \bar{r}_c + \bar{r}'_i$, where $\bar{r}_c$ is the radius vector of the mass center and $\bar{r}'_i$ is the radius vector drawn from the mass center to the mass point m_i. Then we obtain

$$\bar{H} = \bar{r}_c \times \Sigma m_i\bar{v}_i + \Sigma(\bar{r}'_i \times m_i\bar{v}_i) \tag{5.5}$$

We call the second term the *moment of momentum* of the system with respect to the mass center and denote it by $\bar{H}_c$. Thus Eq. (5.5) becomes

$$\bar{H} = \bar{H}_c + \bar{r}_c \times \Sigma m_i\bar{v}_i \tag{5.6}$$

By differentiation

$$\frac{d\bar{H}}{dt} = \frac{d\bar{H}_c}{dt} + \frac{d\bar{r}_c}{dt} \times \sum m_i\bar{v}_i + \bar{r}_c \times \frac{d}{dt}\left(\sum m_i\bar{v}_i\right) \tag{5.7}$$

According to the definition of the center of gravity

$$\frac{d\bar{r}_c}{dt}\sum m_i = \sum m_i\bar{v}_i$$

and, therefore, the vector product

$$\frac{d\bar{r}_c}{dt} \times \sum m_i\bar{v}_i = 0$$

We are left with

$$\frac{d\bar{H}}{dt} = \frac{d\bar{H}_c}{dt} + \bar{r}_c \times \frac{d}{dt}\left(\sum m_i\bar{v}_i\right) \tag{5.8}$$

or since

$$\frac{d}{dt}\left(\sum m_i\bar{v}_i\right) = \sum \bar{F}_i$$

$$\frac{d\bar{H}}{dt} = \frac{d\bar{H}_c}{dt} + \bar{r}_c \times \sum \bar{F}_i \tag{5.9}$$

On the other hand, the resultant moment of the external forces is equal to

$$\bar{M} = \Sigma(\bar{r}_i \times \bar{F}_i) = \Sigma[(\bar{r}_c + \bar{r}_i') \times \bar{F}_i] \tag{5.10}$$

We denote the sum $\Sigma(\bar{r}_i' \times \bar{F}_i)$ by $\bar{M}_c$; it represents the moment of the forces with respect to the mass center. By comparison of (5.9) and (5.10) we obtain

$$\frac{d\bar{H}_c}{dt} = \bar{M}_c \tag{5.11}$$

Hence, the rate of change of the resultant moment of momentum with respect to the mass center is equal to the resultant moment of the external forces with respect to the same point. The importance of this theorem consists in the fact that using the center of gravity we can refer the moments to a point moving with the system instead of a point fixed in space. This simplifies the calculation in cases in which the motion consists of both translation and rotation of the system.

Let us now refer the motion of the system to a coordinate system ξ, η, ζ, which performs a pure translation. Its origin is denoted by Q. For instance, we can identify Q with one of the mass points of the system, but not necessarily with the mass center C. We denote the velocity vector of Q by $\bar{v}_Q$, the radius

vector of an arbitrary mass point m in the moving system by $\bar{r}$, its velocity relative to the moving system by $\bar{v} = d\bar{r}/dt$. Then the velocity of m relative to a fixed system is equal to $\bar{v}_Q + \bar{v}$.

Hence, according to Newton's first law of motion,

$$m \frac{d\bar{v}_Q}{dt} + m \frac{d\bar{v}}{dt} = \bar{F}$$

or

$$m \frac{d\bar{v}}{dt} = \bar{F} - m \frac{d\bar{v}_Q}{dt} \tag{5.12}$$

Therefore, Newton's law holds for a motion relative to the moving coordinate system, if we introduce an additional force equal to the negative product of the mass and the acceleration of the origin Q of the moving coordinate system. Consequently, if we refer coordinates, velocities, accelerations, and moments to the moving system, all the conclusions of this and the previous sections concerning mass systems remain valid provided we assume that an additional force equal to the negative product of the mass and the acceleration of the system of reference acts on each mass point.

Since these additional forces are all parallel and their magnitudes are proportional to the masses on which they act, their resultant is equal to the negative product of the total mass and the acceleration of the origin Q. Hence, the motion of the mass center is determined by the equation

$$\left(\sum m\right) \frac{d\bar{v}_c}{dt} = \sum \bar{F} - \left(\sum m\right) \frac{d\bar{v}_Q}{dt} \tag{5.13}$$

where $\bar{r}_c$ and $\bar{v}_c = \dfrac{d\bar{r}_c}{dt}$ denote the radius vector and velocity of the mass center relative to the moving coordinate system.

The rate of change of the moment of momentum $\bar{H}$ of the mass system with respect to Q is given by the equation

$$\frac{d\bar{H}}{dt} = \bar{M} - \left(\sum m\right)\left(\bar{r}_c \times \frac{d\bar{v}_Q}{dt}\right) \tag{5.14}$$

where $\bar{M}$ is the moment of the forces with respect to the origin Q.

It is seen from (5.14) that the rate of change of the moment of momentum with respect to the moving origin is equal to the moment of the external forces either if the mass center is chosen as origin ($\bar{r}_c = 0$), or if the coordinate system is translated with a velocity of constant magnitude and direction ($d\bar{v}_Q/dt = 0$).

These results are of fundamental importance in problems of *relative motion.* If a rotating coordinate system is used as a system of reference, further additional forces must be introduced. They are equal to the negative products of the masses and the accelerations caused by the rotation of the coordinate system.

Let us assume that the coordinate system rotates with a constant angular velocity $\bar{\Omega}$ around a fixed axis. We denote the radius vector of a mass point m drawn from the origin in the rotating coordinate system by $\bar{r}$, the relative velocity of the mass point by $d\bar{r}/dt$. Then the absolute velocity of the mass point is equal to $\bar{v}_a = \dfrac{d\bar{r}}{dt} + \bar{\Omega} \times \bar{r}$. The absolute acceleration $\bar{a}$ is obtained in the simplest way by drawing the vector $\bar{v}_a$ from the origin and computing the absolute velocity of its end point:

$$\bar{a} = \frac{d\bar{v}_a}{dt} + \bar{\Omega} \times \bar{v}_a$$

We have, therefore,

$$\begin{aligned}\bar{a} &= \frac{d}{dt}\left(\frac{d\bar{r}}{dt} + \bar{\Omega} \times \bar{r}\right) + \bar{\Omega} \times \left(\frac{d\bar{r}}{dt} + \bar{\Omega} \times \bar{r}\right) \\ &= \frac{d^2\bar{r}}{dt^2} + \bar{\Omega} \times \frac{d\bar{r}}{dt} + \bar{\Omega} \times \frac{d\bar{r}}{dt} + \bar{\Omega} \times (\bar{\Omega} \times \bar{r})\end{aligned}$$

The first term is the acceleration relative to the rotating coordinate system; the rest of the expression is the acceleration due to the rotation of the system of reference. Hence, to comply with Newton's law we have to assume that a force equal to

$$-m\left[2\left(\bar{\Omega} \times \frac{d\bar{r}}{dt}\right) + \bar{\Omega} \times (\bar{\Omega} \times \bar{r})\right]$$

acts on every mass point m. The term $-2m\left(\bar{\Omega} \times \dfrac{d\bar{r}}{dt}\right)$ is known as the *Coriolis force;* it is perpendicular to the angular velocity of the coordinate system and the relative velocity of the point. The term $-m[\bar{\Omega} \times (\bar{\Omega} \times \bar{r})]$ is known as the *centrifugal force;* it is perpendicular to the axis of rotation, directed outward, and equal to the product of the distance of the point from the axis of rotation and the square of the angular velocity Ω.

6. Motion of a Rigid Body.—The Eqs. (4.2) and (4.3) are in general not sufficient to determine the motion of the system of

mass points. To determine the path of each mass we must take into account the forces acting between the masses. However, in the case of a rigid system they determine the motion completely. We call a system of n points rigid if the mutual distances between the points are invariable. The motion of such a system can be described by a translation of the center of gravity and a rotation around the same point. The motion of the center of gravity is determined by Eq. (5.3); the rotation, by Eq. (5.11).

Let us first consider the so-called *plane motion* of a rigid system or a *rigid body*. In this case all points of the body move in parallel planes, *e.g.*, in planes parallel to the xy plane of a fixed coordinate system (Fig. 6.1). To describe the motion of the body we introduce a coordinate system ξ, η, ζ, which remains parallel to x, y, z, but whose origin is fixed to the center of gravity of the body. Then the motion is completely determined if we know the velocity components $v_{xc} = dx_c/dt$ and $v_{yc} = dy_c/dt$ and the angular velocity $\omega = d\varphi/dt$, with which the body rotates around the ζ-axis.

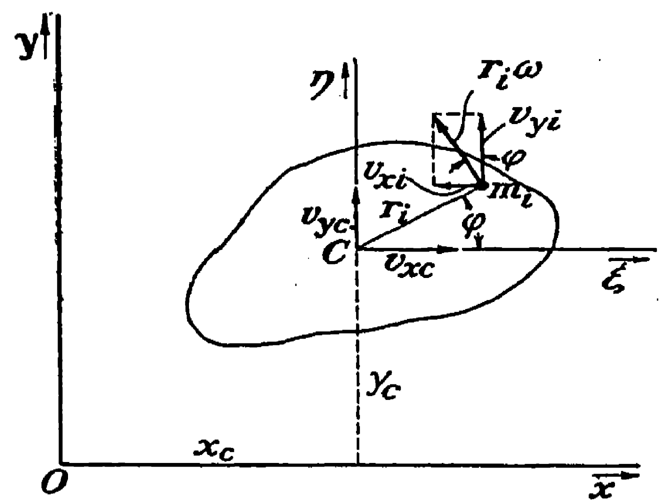

FIG. 6.1.—Plane motion of a rigid body referred to a coordinate system whose origin is fixed at the center of gravity.

To obtain the equations of motion we calculate the components of the momentum and of the moment of momentum. The x- and y-components of the momentum are mv_{xc} and mv_{yc}, respectively, where m is the mass of the body; the z-component is equal to zero. The only component of the moment of momentum which is different from zero is the z-component. We choose the mass center as point of reference. Then the z-component of the moment of momentum is equal to $H_\zeta = \Sigma r_i m_i r_i \omega = \omega \Sigma m_i r_i^2$, where r_i is the distance of an arbitrary point from the ζ-axis. We call the quantity $I_\zeta = \Sigma m_i r_i^2$ the *moment of inertia* of the body with respect to the ζ-axis.

The equations of motion are

$$m \frac{dv_{xc}}{dt} = X$$

$$m \frac{dv_{yc}}{dt} = Y \tag{6.1}$$
$$I_\zeta \frac{d\omega}{dt} = M_c$$

In these equations X and Y denote the components of the resultant force, M_c the moment of the forces with respect to the ζ-axis.

The *rotation around a fixed axis* is a special case of plane motion. In this case it is more convenient to choose the fixed axis as axis of reference for the moment of momentum and for the moment of the forces. The equation of motion is given by

$$I_z \frac{d\omega}{dt} = M_z \tag{6.2}$$

i.e., the angular acceleration is equal to the moment of the forces with respect to the axis of rotation divided by the inertia moment with respect to the same axis.

Another important example, which belongs to this section, is the *motion of a rigid body around a fixed point.* In this case we choose the fixed point as the origin of the coordinate system. The moment of momentum $\bar{H}$, with respect to this point, is equal to

$$\bar{H} = \Sigma m_i(\bar{r}_i \times \bar{v}_i) \tag{6.3}$$

The motion of a rigid body around a fixed point may be considered at any instant as a mere rotation around an instantaneous axis passing through the fixed point. To be sure, the position of this axis is, in general, variable both in space and in relation to the body. However, if we denote the angular velocity of the rotation around the instantaneous axis by $\bar{\omega}$, then the velocity of an arbitrary point P is given by

$$\bar{v}_i = \bar{\omega} \times \bar{r}_i \tag{6.4}$$

In other words, the velocity vector is equal to the vector product of the $\bar{\omega}$ vector and the radius vector.

Introducing the expression (6.4) into Eq. (6.3), and applying the rule given by Eq. (2.5), we obtain

$$\bar{H} = \Sigma m_i[\bar{r}_i \times (\bar{\omega} \times \bar{r}_i)] = \bar{\omega}\Sigma m_i r_i^2 - \Sigma m_i \bar{r}_i(\bar{r}_i \cdot \bar{\omega})$$

or, in components,

$$\begin{aligned} H_x &= I_x\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_y &= -I_{yx}\omega_x + I_y\omega_y - I_{yz}\omega_z \\ H_z &= -I_{zx}\omega_x - I_{zy}\omega_y + I_z\omega_z \end{aligned} \tag{6.5}$$

where

$$
\begin{aligned}
I_x &= \Sigma m_i(y_i^2 + z_i^2) & I_{xy} &= I_{yx} = \Sigma m_i x_i y_i \\
I_y &= \Sigma m_i(z_i^2 + x_i^2) & I_{yz} &= I_{zy} = \Sigma m_i y_i z_i \\
I_z &= \Sigma m_i(x_i^2 + y_i^2) & I_{zx} &= I_{xz} = \Sigma m_i z_i x_i
\end{aligned}
\tag{6.6}
$$

I_x, I_y, I_z are called the *moments of inertia* with respect to the axes indicated by the subscripts. They are equal to the sum of the masses multiplied by the squares of their distances from the respective axes of reference. I_{xy}, I_{yz}, I_{zx} are called the *products of inertia.*

The motion of the body is determined by the vector equation $d\bar{H}/dt = \bar{M}$ or, written in components,

$$
\frac{dH_x}{dt} = M_x, \qquad \frac{dH_y}{dt} = M_y, \qquad \frac{dH_z}{dt} = M_z \tag{6.7}
$$

where $\bar{M}$ is the moment of the forces with respect to the fixed point.

However, the practical use of the Eqs. (6.7) is limited, because the inertia moments are referred to axes fixed in space and are, therefore, variable. Euler transformed the Eqs. (6.7) in such a way that both the components of the angular velocity $\bar{\omega}$ and the inertia moments are referred to axes moving with the body. The reader interested in *Euler's equation* may consult any book on analytical mechanics.

7. The Gyroscope.—Let us investigate an example of special interest for engineering, *viz.*, the case of a body with axial symmetry. Denoting the so-called *body axes, i.e.*, the coördinate axes rigidly connected with the body, by the subscripts 1, 2, 3, the inertia moments with reference to such axes, I_1, I_2, I_3, and the products of inertia, I_{12}, I_{23}, I_{31}, are constants of the body depending on the location of the fixed point, on the orientation of the system of axes, and on the mass distribution in the body. It can be proved that for an arbitrary distribution of the mass it is always possible to find three axes perpendicular to each other such that the products of inertia vanish. We call such axes *principal inertia axes* and the corresponding inertia moments A, B, and C the *principal moments of inertia.*

Let us assume that two of the principal inertia moments are equal, for example, $A = B$. We call a rigid body of such mass distribution rotating around a fixed point a *symmetrical top,* or

symmetrical gyroscope. In this case the third principal axis is called the *axis of symmetry* of the top.

The most important type of motion of a symmetrical top is called *precession.* Such a motion can be produced by making the body rotate around its axis of symmetry with the angular velocity Ω, and by making the axis of symmetry rotate at the same time with the angular velocity μ around an axis fixed in space, *e.g.*, around the z-axis. The axis of symmetry describes in this case a conical surface whose half apex angle is denoted by θ. The vector $\bar{\omega}$ is the resultant of Ω and μ, and, as seen from Fig. 7.1, it rotates around the z-axis.

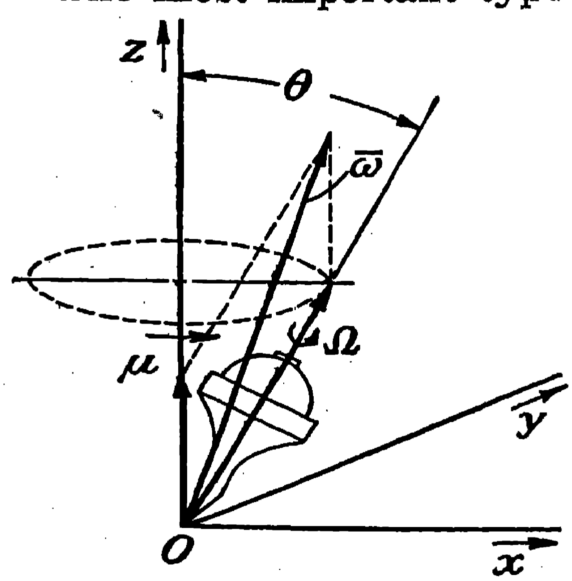

FIG. 7.1.—Precession of a symmetrical top. Diagram of angular velocities.

Our first problem is to consider the possibility of such motion without the action of external forces, for example, if the center of gravity of the top coincides with the fixed point, and no other

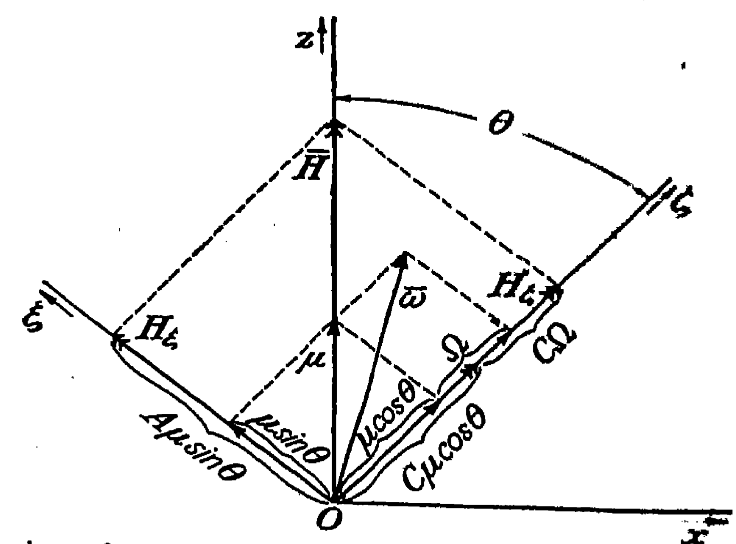

FIG. 7.2.—Precession of a symmetrical top. Diagram of moment of momentum.

forces are applied to the top. According to Eqs. (6.7), if the moment of the external forces around the fixed point is zero, the moment of momentum of the body remains constant. We first calculate the components of the moment of momentum referred to the body axes.

The moment of momentum due to the rotation Ω around the axis of symmetry is equal to $C\Omega$, where C is the inertia moment

with respect to the axis of symmetry. We introduce a coordinate system ξ, η, ζ (Fig. 7.2) such that the origin is at the fixed point, the ζ-axis is the axis of symmetry of the top, and the ξ-axis is normal to ζ and lies in the $z\zeta$ plane. Then the angular velocity μ, whose axis is the z-axis, gives us one component in the ζ-direction, equal to $\mu \cos \theta$, and one component in the ξ-direction equal to $\mu \sin \theta$. The corresponding contributions to the angular momentum are $C\mu \cos \theta$ and $A\mu \sin \theta$. Hence, the two components of the resultant angular momentum are $C(\Omega + \mu \cos \theta)$ and $A\mu \sin \theta$.

The axis of symmetry and, therefore, the $z\zeta$ plane rotate in the space around the z-axis. As was mentioned above, the condition that the motion be possible for the case when the moment of the forces vanishes is that the resultant moment of momentum is constant as far as both its magnitude and its direction are concerned. This can occur only if the $\bar{H}$ vector coincides with the z-axis, *i.e.*, the sum of the components normal to z vanishes. Otherwise, the $\bar{H}$ vector would also rotate around the z-axis. This condition yields the relation

$$C(\Omega + \mu \cos \theta) \sin \theta - A\mu \sin \theta \cos \theta = 0 \quad (7.1)$$

or

$$\mu = -\frac{\Omega}{\cos \theta} \frac{C}{C - A} \quad (7.2)$$

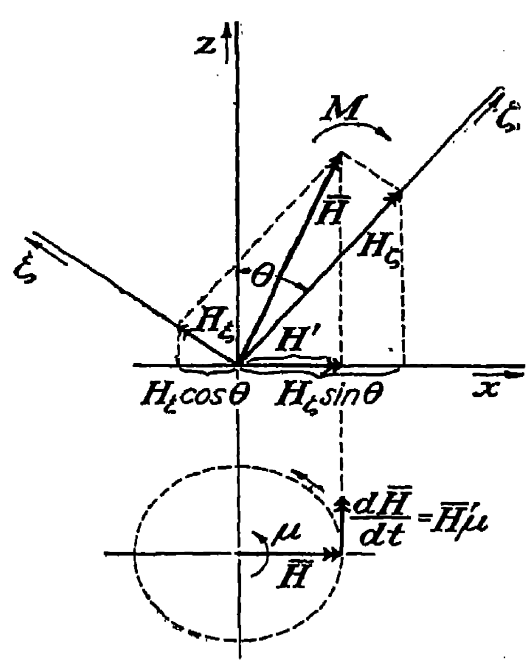

FIG. 7.3.—Precession of a symmetrical top. Rate of change of the moment of momentum.

Equation (7.1) is called the condition of *free precession*. We notice that, for $\theta < 90°$, if $A > C$, μ and Ω have the same sign; whereas, if $A < C$, their signs are opposite. For $\theta \to 90°$, $\mu \to \infty$.

The second problem is to calculate the moment M which is necessary to maintain a precession with a certain angular velocity μ, which does not satisfy the condition (7.1). This moment is equal to the rate of change of the vector $\bar{H}$. If the top performs a pure precession, the vector $\bar{H}$ rotates with the angular velocity μ around the z-axis, its magnitude being constant. Hence, $d\bar{H}/dt$, *i.e.*, the rate of change of $\bar{H}$, is equal to $\bar{\mu} \times \bar{H}$ (Fig. 7.3).

To calculate the magnitude of this vector product we take into account that the vector $\bar{H}$ is composed of the component $H_\zeta = C(\Omega + \mu \cos \theta)$ in the ζ-direction and the component $H_\xi = A\mu \sin \theta$ in the ξ-direction. We form the vector products of $\bar{\mu}$ and these two components and obtain by addition

$$M = C(\Omega + \mu \cos \theta)\mu \sin \theta - A\mu \sin \theta \cdot \mu \cos \theta \qquad (7.3)$$

The moment M acts around the η-axis, which is normal to the $z\zeta$ plane.

For $\theta = 90°$, we obtain

$$M = C\Omega\mu \qquad (7.4)$$

i.e., if the symmetry axis of the top is normal to the axis of precession, the moment necessary to maintain the precession is equal to the product of the inertia moment of the top, the angular velocity of the rotation of the top around its symmetry axis, and the angular velocity of the precession. The moment $-M$ is called the *moment of precession* or the *gyroscopic moment.*

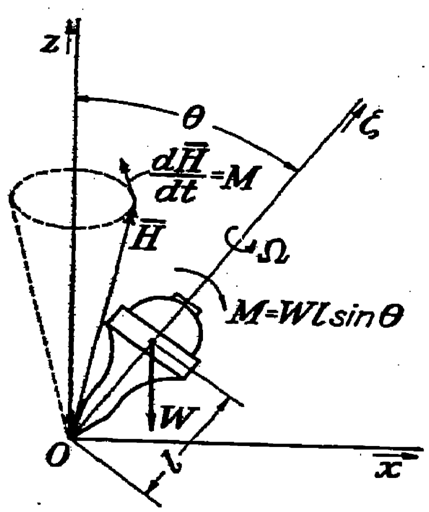

FIG. 7.4.—Precession of a heavy top.

For example, if a shaft carrying a rotating disk is constrained to turn in a given plane, a moment M, according to Eq. (7.4), must be exerted to keep the shaft in that plane. In other words, the disk will exert a moment $-M$ on the bearings of the shaft unless the axis of rotation is allowed to take the inclination corresponding to the free precession. This effect is used in many engineering applications, such as the gyroscopic compass, the turn and bank indicator, the automatic pilot for airplanes, and the gyroscopic stabilizer for ships, torpedoes, etc.

If the center of gravity of the top is at a distance l (Fig. 7.4) from the fixed point and its weight is W, the moment due to the weight is equal to $M = Wl \sin \theta$.

Introducing M in Eq. (7.3), we obtain

$$C\mu(\Omega + \mu \cos \theta) - A\mu^2 \cos \theta = Wl \qquad (7.5)$$

Equation (7.5) determines the speed of precession μ, under action of gravity, as function of Ω and θ. If Eq. (7.5) yields two real roots for μ, there are two different precessions possible, both

corresponding to the same inclination θ. They are called the *fast* and the *slow* precessions.

If $\theta = 90°$, *i.e.*, the axis of precession and the axis of symmetry are normal to each other,

$$C\mu\Omega = Wl \tag{7.6}$$

In this case (Fig. 7.5), the moment of the weight of the top is balanced by the gyroscopic moment which is equal to the product of the inertia moment of the top around the ζ-axis, the angular velocity of the top Ω around the same axis, and the angular velocity μ of the precession.

If $\theta = 0$, Ω and μ cannot be separated. The motion in the neighborhood of $\theta = 0$ is, however, of great importance for many applications and will be investigated in Chapter VI.

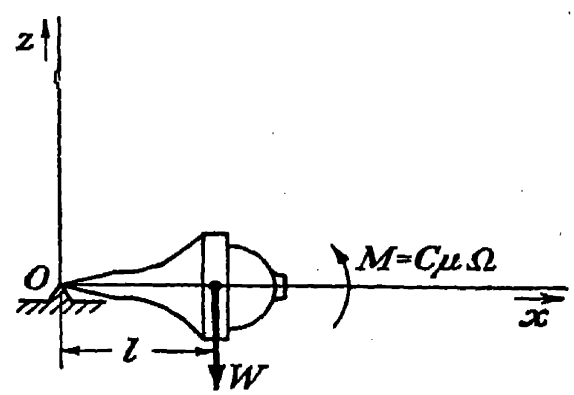

FIG. 7.5.—Balance between the gyroscopic moment of a heavy top precessing in a horizontal plane and the moment of its weight.

8. Work and Energy.—Consider the motion of a mass point under action of a force $\bar{F}$ whose components are X, Y, and Z. If the mass point moves in the time dt from the point x, y, z to the point $x + dx$, $y + dy$, $z + dz$, the *work done by the force* is defined by

$$dW = X\,dx + Y\,dy + Z\,dz = \bar{F} \cdot d\bar{s} \tag{8.1}$$

where $d\bar{s}$ is the vector whose components are dx, dy, dz. Hence, the work is the scalar product of the force and the displacement; in other words, it is equal to the product of the force and the component of the displacement in the direction of the force (or the product of the displacement and the component of the force in the direction of the displacement).

Substituting X, Y, and Z from Eq. (3.3), it is seen that the work done is equal to

$$dW = m\left(\frac{d^2x}{dt^2}\,dx + \frac{d^2y}{dt^2}\,dy + \frac{d^2z}{dt^2}\,dz\right) \tag{8.2}$$

or

$$dW = m\left(\frac{d^2x}{dt^2}\frac{dx}{dt} + \frac{d^2y}{dt^2}\frac{dy}{dt} + \frac{d^2z}{dt^2}\frac{dz}{dt}\right)dt \tag{8.3}$$

We call the product mass times one-half the square of the velocity the *kinetic energy* T of the mass point:

$$T = \frac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] \tag{8.4}$$

Then comparing (8.3) and (8.4) it is seen that

$$dW = dT \tag{8.5}$$

This equation states that the change in the kinetic energy of the mass point is equal to the work done by the force applied to it. The kinetic energy increases if the work is positive, decreases if the work is negative. Let us assume the mass point moves from the point A, whose coordinates are x_A, y_A, and z_A, to the point B, whose coordinates are x_B, y_B, and z_B; the work done by the force $\bar{F}$ is given by the integral

$$W = \int_A^B \bar{F} \cdot d\bar{s}$$

or

$$W = \int_A^B (X\,dx + Y\,dy + Z\,dz) \tag{8.6}$$

If the work depends on the coordinates of the end points only, we call the force acting on the mass point a *conservative force*. For example, the force due to gravity is conservative, since the work done is equal to the weight of the mass point multiplied by the difference in height between A and B, independently of the path followed. Obviously, we can assume that when located at A the mass point possesses a certain capacity for doing work called its *potential energy;* if it descends from A to B, its capacity for doing further work is diminished by the amount of work done between A and B. Hence, if the potential energy of the mass point at A is equal to U_A, and at B to U_B, the work is equal to the difference $U_A - U_B$.

In general for the case of conservative forces we define the potential energy U by

$$dU = -dW = -(X\,dx + Y\,dy + Z\,dz) \tag{8.7}$$

It follows that $X = -\partial U/\partial x$, $Y = -\partial U/\partial y$, $Z = -\partial U/\partial z$, *i.e.*, the components of the force $\bar{F}$ are equal to the derivatives of the potential energy, taken with the negative sign. It is seen that if X, Y, and Z are given functions of x, y, and z, an arbitrary

constant can be added to U. That is, the zero level of the potential energy can be arbitrarily chosen. For example, if U_A is chosen arbitrarily,

$$U_B = U_A - \int_A^B (X\,dx + Y\,dy + Z\,dz) \tag{8.8}$$

From (8.5) and (8.7) it follows that

$$d(T + U) = 0, \qquad \text{or} \qquad T + U = \text{const.} \tag{8.9}$$

The quantity $T + U$ is called the *total energy* of the mass point. If the mass point is subjected to conservative forces only, *the sum of its potential and kinetic energies is constant.* This theorem is called the *law of conservation of energy.*

If the force $\bar{F}$ is *nonconservative, i.e.*, if the work done between two points A and B depends on the path followed, we are unable to assign a definite value of the potential energy U to the position of the mass point, since, for example, from Eq. (8.8) we would obtain different values for U_B, depending on the path of integration. We say that in this case the differential dW is not a *perfect differential, i.e.*, X, Y, and Z cannot be expressed as derivatives of a function of x, y, and z.

Our next aim is to calculate the work and the energy balance for a system of n mass points subjected to the action of external and internal forces.

We define the total work of the external forces done in the time dt by

$$dW_e = \Sigma \bar{F}_i \cdot d\bar{r}_i$$

where $\bar{F}_i$ is the external force acting on the mass point m_i, and $\bar{r}_i$ is the radius vector drawn from the origin to the position of m_i.

Substituting the value of $\bar{F}_i$ from Eq. (4.1), we obtain for constant masses m_i

$$dW_e = \sum_{i=1}^{n} m_i \frac{d^2\bar{r}_i}{dt^2} \cdot d\bar{r}_i - \sum_{i=1}^{n}\sum_{k=1}^{n} \bar{F}_{ik} \cdot d\bar{r}_i \tag{8.10}$$

The first sum on the right side is equal to

$$\sum_{i=1}^{n} m_i \frac{d^2\bar{r}_i}{dt^2} \cdot \frac{d\bar{r}_i}{dt}\,dt = \frac{d}{dt}\sum_{i=1}^{n} \frac{1}{2} m_i \left(\frac{d\bar{r}_i}{dt}\right)^2 dt$$

or, if we call

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\frac{d\bar{r}_i}{dt}\right)^2 = \frac{1}{2}\sum_{i=1}^{n} m_i\bar{v}_i^2$$

the kinetic energy of the mass system,

$$\sum_{i=1}^{n} m_i \frac{d^2\bar{r}_i}{dt^2} \cdot d\bar{r}_i = dT \tag{8.11}$$

To evaluate the double summation, we remember that

$$\bar{F}_{ik} = -\bar{F}_{ki},$$

as a result of Newton's third law. Hence, we have terms of the form $\bar{F}_{ik} \cdot (d\bar{r}_i - d\bar{r}_k)$. If we denote the vector connecting m_i and m_k by $\bar{r}_{ik}$, $d\bar{r}_k - d\bar{r}_i = d\bar{r}_{ik}$, and, therefore,

$$-\sum_{i=1}^{n}\sum_{k=1}^{n} \bar{F}_{ik} \cdot d\bar{r}_i = \tfrac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} \bar{F}_{ik} \cdot d\bar{r}_{ik} \tag{8.12}$$

Substituting (8.11) and (8.12) in Eq. (8.10), we obtain

$$dW_e = dT + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} \bar{F}_{ik} \cdot d\bar{r}_{ik} \tag{8.13}$$

The differential $d\bar{r}_{ik}$ of the vector is equal to the change of the distance between m_i and m_k. Hence, (8.12) is equal to the sum of the products of the internal forces and the changes of the distances between the mass points. If the forces depend only on the distances between the mass points, we can introduce a function U_i of the distances, such that

$$\tfrac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} \bar{F}_{ik} \cdot d\bar{r}_{ik} = -dU_i \tag{8.14}$$

and call U_i the internal potential energy of the system. Then Eq. (8.13) can be written in the form:

$$dW_e = dT + dU_i \tag{8.15}$$

Now, if we suppose that the external forces are conservative, and that, therefore, their work is equal to the decrease of a

suitably defined external potential energy U_e, it follows from (8.15) that

$$d(U_e + T + U_i) = 0 \tag{8.16}$$

Equation (8.16) announces the *theorem of conservation of energy under assumption of conservative external and internal forces.*

The engineer is especially concerned with the internal energy of compressible fluids and of elastic solid bodies. In such problems the internal forces will be replaced by continuously distributed stresses. The change in the distances between discrete points will be replaced by certain quantities that measure the deformation of a continuous medium, *e.g.*, the change of density of a fluid, the strain in an elastic body, etc. However, the general theorems presented in this and in the following sections do apply also—with certain changes in the terminology—to such continuous systems. It is evident that in the case of a rigid system $dU_i = 0$, because all distances $\bar{r}_{ik}$ are constant. The work of the stresses occurring in a rigid body is equal to zero.

Kinetic Energy of a Rigid Body.—The kinetic energy of a rigid body can be expressed, as we have done for the momentum and the moment of momentum, by introducing a coordinate system whose origin moves with the center of gravity. Let us take, for example, a so-called *plane motion* (cf. section 6); in this case the kinetic energy is given by

$$T = \tfrac{1}{2}\Sigma m(\dot{x}^2 + \dot{y}^2) \tag{8.17}$$

where the symbols $\dot{x}$ and $\dot{y}$ mean time derivatives. On the other hand, the velocity components of an arbitrary point are (cf. Fig. 6.1)

$$\begin{aligned} \dot{x} &= \dot{x}_c - \eta\omega \\ \dot{y} &= \dot{y}_c + \xi\omega \end{aligned} \tag{8.18}$$

where $\dot{x}_c$ and $\dot{y}_c$ are the velocity components of the center of gravity C, and ξ and η are the coordinates of the point in a coordinate system whose origin is at C. Introducing (8.18) in (8.17) and taking into account that $\Sigma m\xi = \Sigma m\eta = 0$, we obtain, with $\Sigma m(\xi^2 + \eta^2) = I$,

$$T = \tfrac{1}{2}(\Sigma m)v_c^2 + \tfrac{1}{2}I\omega^2 \tag{8.19}$$

In the case of a body rotating around a fixed axis, the kinetic energy for arbitrary mass distribution is equal to $\frac{1}{2}I\omega^2$, where I is the moment of inertia with respect to the fixed axis. The reader will also verify, using Eqs. (6.4) and (6.6), that the kinetic energy of a rigid body rotating around a fixed point is equal to

$$T = \tfrac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 - 2I_{xy}\omega_x\omega_y - 2I_{yz}\omega_y\omega_z - 2I_{zx}\omega_z\omega_x) \qquad (8.20)$$

or (cf. Eq. (6.5))

$$T = \tfrac{1}{2}(H_x\omega_x + H_y\omega_y + H_z\omega_z) = \tfrac{1}{2}\bar{H} \cdot \bar{\omega}$$

where the components of the angular velocity and the inertia moments are referred to fixed axes x, y, z. If they are referred to the principal axes moving with the body,

$$T = \tfrac{1}{2}(A\omega_A^2 + B\omega_B^2 + C\omega_C^2) \qquad (8.21)$$

For instance, the kinetic energy of a symmetrical top in the case of regular precession is equal to

$$T = \tfrac{1}{2}[C(\Omega + \mu\cos\theta)^2 + A\mu^2\sin^2\theta] \qquad (8.22)$$

9. The Theorem of Virtual Displacements.—Equations (4.1) represent $3n$ simultaneous differential equations for the $3n$ coordinates of the n mass points. If the external forces are numerically known or are given functions of the positions of the points and the internal forces are given functions of their relative positions, Eqs. (4.1) determine the motion of the system, provided the initial positions and velocities of the points are known. A system of mass points which are not subjected to any geometrical restrictions in their motion is called a *free system*. If the mass points must comply with certain geometrical conditions, we must introduce certain forces which enforce these conditions. Such geometrical conditions are known as *geometrical constraints*; the corresponding forces are called *reactions*. As a matter of fact, we have already dealt with a class of systems subjected to geometrical constraints; *viz.*, with rigid bodies. A rigid body is a system of mass points subjected to the constraint that the distance between two arbitrary points is constant. However, in this case it is not necessary to introduce internal reactions, since the six equations for the total momentum and the total

moment of momentum are sufficient to determine the motion of the system.

In the following discussions we consider systems consisting of mass points and rigid bodies subjected to a number of geometrical constraints. If the number of such constraints is m, we introduce into the equations of motion m unknown reactions. Thus, we increase the number of the unknown variables by the number of the geometrical relations representing the constraints. With these relations the number of equations is again equal to the number of unknown quantities, and we can solve first the equations of motion and then eliminate the unknown reactions by means of the geometrical conditions.

The method of *virtual displacements* enables us to set up the equations of equilibrium and motion in such a way that the reactions do not appear at all. This is possible in the case of so-called *frictionless reactions.* We call a reaction frictionless if it does not do work when the system moves in a way compatible with the geometrical constraints.

Let us assume, for example, that a reaction R is introduced to constrain a point to a given surface. Since the work of a force is equal to its scalar product by the displacement of its point of action, the work of R vanishes if R is normal to the surface. In this case R is a frictionless reaction. If the reaction has a component parallel to the surface, we generally assume that this component is directed opposite to the motion of the point and its magnitude depends on the magnitude of the normal component of the reaction. The tangential component of the reaction represents the frictional resistance of the mechanism. The ratio between the tangential and the normal components of a reaction is called the *coefficient of friction.*

We shall assume for the following discussion that all reactions are frictionless.

Let us apply the method of virtual displacements first to the problem of equilibrium.

We consider a system of n mass points subjected to the action of certain given external forces and certain geometrical constraints, which are defined by m relations of the form:

$$f_r(x, y, z, \cdots, x_n, y_n, z_n) = 0 \tag{9.1}$$

where $r = 1, 2, \cdots, m$. The given forces are denoted by

X_i, Y_i, Z_i $(i = 1, 2, \cdots, n)$, and the components of the reactions due to the constraints by X_{ir}, Y_{ir}, Z_{ir}. Then the n points of the system are in equilibrium when the resultant of the forces acting on any of the n points vanishes, *i.e.*,

$$X_i + \sum_r X_{ir} = 0, \qquad Y_i + \sum_r Y_{ir} = 0, \qquad Z_i + \sum_r Z_{ir} = 0 \quad (9.2)$$

It is obvious that if δx_i, δy_i, δz_i are arbitrary displacements of the ith mass point, we have

$$\left(X_i + \sum_r X_{ir}\right)\delta x_i + \left(Y_i + \sum_r Y_{ir}\right)\delta y_i + \left(Z_i + \sum_r Z_{ir}\right)\delta z_i = 0 \quad (9.3)$$

In fact, if δx_i, δy_i, and δz_i are arbitrary, (9.3) is merely a way of writing the three equations (9.2) in the form of one equation. Consequently, the sum of n such expressions, the summation being extended over all the n points of the system, is also equal to zero. Hence,

$$\sum_{i=1}^{n}\left[\left(X_i + \sum_r X_{ir}\right)\delta x_i + \left(Y_i + \sum_r Y_{ir}\right)\delta y_i + \left(Z_i + \sum_r Z_{ir}\right)\delta z_i\right] = 0 \quad (9.4)$$

Now the work done by each of the reactions and, therefore, also the total work done by all the reactions is equal to zero, provided the displacements are compatible with the geometrical constraints. We call such displacements *virtual displacements*. The term *virtual* means that the displacements are arbitrary with the only restriction that they comply with the geometrical constraints.

The work done by the reactions is given by the expression

$$\delta W_r = \sum_{i=1}^{n}\left(\delta x_i \sum_r X_{ir} + \delta y_i \sum_r Y_{ir} + \delta z_i \sum_r Z_{ir}\right) \quad (9.5)$$

Introducing $\delta W_r = 0$ into Eq. (9.4), this equation becomes

$$\sum_{i=1}^{n}\left(X_i\,\delta x_i + Y_i\,\delta y_i + Z_i\,\delta z_i\right) = 0 \quad (9.6)$$

The expression (9.6) is called the *virtual work* δW_v. It represents the work done by the external forces if the displacements given to the system satisfy the geometrical constraints but are otherwise arbitrary.

The exact meaning of Eq. (9.6) is the following: the sum in (9.6) is a linear expression in the variations δx_i, δy_i, and δz_i, where the coefficients are constants or given functions of x_i, y_i, and z_i. However, due to the geometrical constraints (9.1), the variations δx_i, δy_i, and δz_i are not independent but must satisfy the m relations:

$$\sum_{i=1}^{n}\left(\frac{\partial f_r}{\partial x_i}\,\delta x_i + \frac{\partial f_r}{\partial y_i}\,\delta y_i + \frac{\partial f_r}{\partial z_i}\,\delta z_i\right) = 0 \tag{9.7}$$

($r = 1, \cdots, m$), which we obtain by differentiation of the conditions (9.1). If we express m of the variations δx_i, δy_i, and δz_i in terms of the remaining $3n - m$ variations and substitute them into Eq. (9.6), this equation will contain $3n - m$ perfectly arbitrary variations only. Then $\delta W_v = 0$ means that the coefficients of these variations vanish. Hence, Eq. (9.6) is equivalent to $3n - m$ equations and, together with the m equations (9.1) (*i.e.*, the geometrical constraints), determines the $3n$ unknown coordinates of the equilibrium position.

If the external forces are conservative, the work δW_v is equal to $-\delta U$, where δU is the variation of the potential energy corresponding to the virtual displacements. Hence, the variation of the potential energy of a conservative system must vanish in the equilibrium position for arbitrary virtual displacements, *i.e.*, for variations of the coordinates which comply with the geometrical constraints. If the variation of U is equal to zero for a certain position of a system, we say that the value of U is stationary in this position. If U in addition has a minimum, the equilibrium is stable. The stable equilibrium can be exactly defined in the following way: the position x_i, y_i, and z_i, of a mechanical system is a *stable equilibrium position* if the system put in motion with sufficiently small initial velocities remains for all times in an arbitrarily close neighborhood of the equilibrium position. If we now assume that the potential energy in the position in question has a smaller value than anywhere else in the neighborhood and the initial value of the kinetic energy

of the system is sufficiently small, it follows from the theorem of the conservation of energy that the system must remain within the prescribed neighborhood of the equilibrium position. The equilibrium is thus stable and the system will oscillate around the equilibrium position.

Equation (9.6) and all following conclusions can be extended to the case of internal forces. If the internal forces are conservative, the theorem of virtual work states that the work of the external forces is equal to the increase of the internal energy provided the variations of the coordinates represent virtual displacements, *i.e.*,

$$\delta W_v = \delta U_i \tag{9.8}$$

This theorem is often announced in the following form: The difference between the variation of the internal energy and the virtual work is equal to zero. If both internal and external forces are conservative, the theorem of virtual work requires that the variation of the sum of the external and internal potential energies vanish for virtual displacements, *i.e.*, for variations of the coordinates which satisfy the geometrical constraints.

10. d'Alembert's Principle.—The principle of virtual displacement can be extended to problems of dynamics, *i.e.*, to the determination of the actual motion. The extension is based on *d'Alembert's principle*, which states that every state of motion may be considered at any instant as a state of equilibrium if appropriate inertia forces are introduced. Newton's first law states that the force acting on a mass is equal to the mass times the acceleration. Instead of this we can say that the force impressed is in equilibrium with the *inertia force* which is defined as the product of the mass and the negative acceleration. In fact, starting with the simplest example, the equation

$$m\frac{d^2x}{dt^2} = X \tag{10.1}$$

can be written in the form

$$0 = -m\frac{d^2x}{dt^2} + X . \tag{10.2}$$

Then we call $-m\dfrac{d^2x}{dt^2}$ the x-component of the inertia force

and in analogous way $-m\dfrac{d^2y}{dt^2}$, $-m\dfrac{d^2z}{dt^2}$ its y- and z-components. When this concept of the inertia force is used, the equations of motion are equivalent to the statement that for every instant of the motion the resultant of the forces proper and the inertia forces vanishes.

If we include in this way the inertia forces in our force system and the motion is subjected to certain constraints, we can use the method of virtual displacements to eliminate the reactions. Then we can set up a system of equations for the motion in the same way as we did for the equilibrium. Introducing the notations $dx/dt = \dot{x}$ and $d^2x/dt^2 = \ddot{x}$ for derivatives with respect to t, we have

$$\sum_{i=1}^{n} [(m_i\ddot{x}_i - X_i)\,\delta x_i + (m_i\ddot{y}_i - Y_i)\,\delta y_i + (m_i\ddot{z}_i - Z_i)\,\delta z_i] = 0 \qquad (10.3)$$

This equation is analogous to Eq. (9.6), except that Eq. (10.3) leads to a system of differential equations with the time as independent variable, while (9.6) gives a system of finite equations for the coordinates.

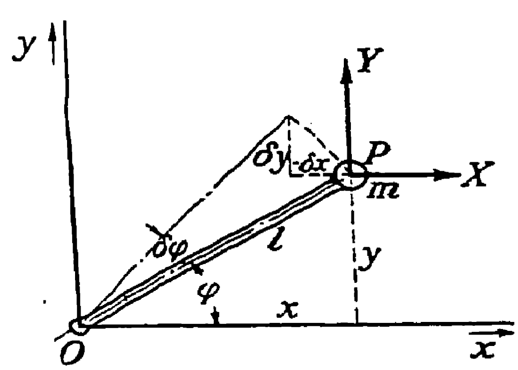

FIG. 10.1.—Mass point constrained to a circular path.

In order to make clear the meaning of (10.3), we apply this equation to a very elementary case. Assume that a mass point P is rigidly connected to a fixed point O by a rod of length l. The motion of the rod is restricted to the xy plane (Fig. 10.1). Its mass is neglected. If we denote the mass of the mass point by m and its coordinates by x and y, d'Alembert's principle announces that

$$(m\ddot{x} - X)\,\delta x + (m\ddot{y} - Y)\,\delta y = 0 \qquad (10.4)$$

provided that δx and δy constitute a system of virtual displacements. The geometrical constraint is expressed by the relation

$$x^2 + y^2 = l^2 \qquad (10.5)$$

The displacements δx and δy are compatible with (10.5) when

$$x\,\delta x + y\,\delta y = 0 \qquad (10.6)$$

Eliminating δy by means of (10.6), *i.e.*, putting

$$\delta y = -\frac{x}{y}\,\delta x \tag{10.7}$$

and introducing (10.7) in (10.4), we obtain

$$\left(m\ddot{x} - X - m\ddot{y}\,\frac{x}{y} + Y\,\frac{x}{y}\right)\delta x = 0 \tag{10.8}$$

Hence, the equation of motion is

$$m(\ddot{x}y - \ddot{y}x) - (Xy - Yx) = 0 \tag{10.9}$$

We recognize the physical meaning of Eq. (10.9) by rewriting it in the form:

$$m\,\frac{d}{dt}(\dot{x}y - \dot{y}x) = Xy - Yx \tag{10.10}$$

Then the left side of the equation represents the rate of change of the moment of momentum; the right side, the moment of the external forces.

Equation (10.10), together with the geometrical condition (10.5), leads to the solution of the problem. However, it is seen that the application of d'Alembert's equation in this manner is rather cumbersome. The reason is that we used two coordinates to describe a mechanical system with one degree of freedom, *i.e.*, a system whose configuration is completely determined by one parameter, for instance, by the angle between the connecting rod l and the x-axis. Denoting this angle by φ, we express x and y by the angle φ:

$$x = l\cos\varphi, \qquad y = l\sin\varphi \tag{10.11}$$

and the displacements δx and δy by the variation of the angle φ:

$$\begin{aligned} \delta x &= -l\sin\varphi\cdot\delta\varphi \\ \delta y &= l\cos\varphi\cdot\delta\varphi \end{aligned} \tag{10.12}$$

The variation, or angular displacement, $\delta\varphi$ is now arbitrary. Introducing (10.12) into d'Alembert's equation (10.4), we obtain

$$[(m\ddot{x} - X)l\sin\varphi - (m\ddot{y} - Y)l\cos\varphi]\,\delta\varphi = 0 \tag{10.13}$$

By differentiating (10.11) we express $\ddot{x}$ and $\ddot{y}$ by $\ddot{\varphi}$ and obtain

$$\begin{aligned} \ddot{x} &= -l \cos \varphi \cdot \dot{\varphi}^2 - l \sin \varphi \cdot \ddot{\varphi} \\ \ddot{y} &= -l \sin \varphi \cdot \dot{\varphi}^2 + l \cos \varphi \cdot \ddot{\varphi} \end{aligned} \tag{10.14}$$

Hence, Eq. (10.13) becomes

$$[ml^2\ddot{\varphi} + (Xl \sin \varphi - Yl \cos \varphi)]\, \delta\varphi = 0 \tag{10.15}$$

The variation $\delta\varphi$ being arbitrary, the expression in the bracket must vanish, and we obtain

$$ml^2\ddot{\varphi} = -Xl \sin \varphi + Yl \cos \varphi \tag{10.16}$$

or, denoting the moment of the external forces by M,

$$ml^2\ddot{\varphi} = M \tag{10.17}$$

It is seen that we obtained a considerable simplification by using the parameter φ instead of the Cartesian coordinates. However, we still carried out some superfluous calculations. We notice that the terms containing $\dot{\varphi}^2$ drop out of d'Alembert's equation because $ml\dot{\varphi}^2$ represents the centrifugal force, which is normal to the virtual displacement and, therefore, does not contribute to the work. In the above simple problem we could have avoided the labor of this calculation by applying the theorem of the moment of momentum. However, it is desirable to have a general method of applying d'Alembert's principle to arbitrary systems with a minimum of labor. Such a method was given by Lagrange using the notions of generalized coordinates and generalized forces.

11. Generalized Coordinates. Lagrange's Equations.—Let us consider a system of r degrees of freedom; for example, let us assume that the configuration of n mass points is given by r independent parameters. We call them the r *generalized coordinates* $q_1, q_2, \ldots, q_r$ and assume that we are able to express the Cartesian coordinates of the n points in terms of the r quantities $q_1, q_2, \ldots, q_r$ by $3n$ relations of the form:

$$\begin{aligned} x_i &= x_i(q_1, q_2, \cdots, q_r) \\ y_i &= y_i(q_1, q_2, \cdots, q_r) \\ z_i &= z_i(q_1, q_2, \cdots, q_r) \end{aligned} \tag{11.1}$$

where $i = 1, 2, \cdots, n$. Then we are also able to express the displacements $\delta x_1, \delta y_1, \delta z_1; \delta x_2, \delta y_2, \delta z_2; \ldots$ in terms of the

variations $\delta q_1, \delta q_2, \ldots, \delta q_r$ of the generalized coordinates by differentiation of the Eqs. (11.1). For instance, we have

$$\delta x_i = \sum_{k=1}^{r} \frac{\partial x_i}{\partial q_k} \delta q_k \tag{11.2}$$

and obtain analogous formulas for δy_i and δz_i. Similarly, we express the derivatives with respect to time $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$ in terms of the derivatives $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_r$ of $q_1, q_2, \ldots, q_r$, writing

$$\dot{x}_i = \sum_{k=1}^{r} \frac{\partial x_i}{\partial q_k} \dot{q}_k \tag{11.3}$$

and so on. It is seen that the original velocity components $\dot{x}_i$, $\dot{y}_i$, and $\dot{z}_i$ are linear functions of the new *generalized velocities* $\dot{q}_k$. Consequently,

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \frac{\partial x_i}{\partial q_k} \tag{11.4}$$

We shall now substitute the new variables in d'Alembert's equation (10.3):

$$\sum_{i=1}^{n} [(m_i\ddot{x}_i - X_i)\,\delta x_i + (m_i\ddot{y}_i - Y_i)\,\delta y_i + (m_i\ddot{z}_i - Z_i)\,\delta z_i] = 0 \tag{11.5}$$

We collect the terms containing the forces. Their sum is equal to $-\delta W$, where δW is the expression for the work done by the forces, *i.e.*,

$$\delta W = \sum_{i=1}^{n} (X_i\,\delta x_i + Y_i\,\delta y_i + Z_i\,\delta z_i)$$

Using the Eq. (11.2), we obtain

$$\delta W = \sum_{k=1}^{r} \left[\sum_{i=1}^{n} \left(X_i \frac{\partial x_i}{\partial q_k} + Y_i \frac{\partial y_i}{\partial q_k} + Z_i \frac{\partial z_i}{\partial q_k} \right) \right] \delta q_k \tag{11.6}$$

We call the expression $\displaystyle\sum_{i=1}^{n} \left(X_i \frac{\partial x_i}{\partial q_k} + Y_i \frac{\partial y_i}{\partial q_k} + Z_i \frac{\partial z_i}{\partial q_k} \right) = Q_k$

the *generalized force* corresponding to the generalized coordinate q_k. Then (11.6) takes the form:

$$\delta W_v = \sum_{k=1}^{r} Q_k\, \delta q_k \tag{11.7}$$

The generalized force Q_k has the dimension of a force when q_k is a length, and the dimension of a moment when q_k is an angular quantity. In general, the dimension of Q_k is determined by the rule that the product $Q_k \cdot q_k$ has the dimension of work.

Let us now calculate the remaining terms on the left side of (11.5), *viz.*,

$$\sum_{i=1}^{n} m_i(\ddot{x}_i\, \delta x_i + \ddot{y}_i\, \delta y_i + \ddot{z}_i\, \delta z_i) \tag{11.8}$$

We transform this expression by introducing the components of the momentum $m_i\dot{x}_i$, $m_i\dot{y}_i$, and $m_i\dot{z}_i$. These quantities can be deduced from the kinetic energy by differentiation with respect to the velocity components. The expression of the kinetic energy is

$$T = \tfrac{1}{2}\sum_{i=1}^{n} (m_i\dot{x}_i^2 + m_i\dot{y}_i^2 + m_i\dot{z}_i^2) \tag{11.9}$$

Then it is seen that

$$m_i\dot{x}_i = \frac{\partial T}{\partial \dot{x}_i}, \qquad m_i\dot{y}_i = \frac{\partial T}{\partial \dot{y}_i}, \qquad m_i\dot{z}_i = \frac{\partial T}{\partial \dot{z}_i} \tag{11.10}$$

Hence, the expression (11.8) can be written in the form:

$$\sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)\delta x_i + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_i}\right)\delta y_i + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}_i}\right)\delta z_i\right]$$

and substituting the values of δx_i, δy_i, and δz_i from (11.2), it becomes

$$\sum_{i=1}^{n}\sum_{k=1}^{r}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)\frac{\partial x_i}{\partial q_k} + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_i}\right)\frac{\partial y_i}{\partial q_k} + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}_i}\right)\frac{\partial z_i}{\partial q_k}\right]\delta q_k \tag{11.11}$$

Let us now consider the expression which multiplies δq_k in (11.11).

It is seen that according to elementary rules of differentiation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)\frac{\partial x_i}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\frac{\partial x_i}{\partial q_k}\right) - \frac{\partial T}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial q_k} \tag{11.12}$$

and, therefore, the expression which multiplies δq_k in (11.11) can be written in the following form:

$$\sum_{i=1}^{n}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\frac{\partial x_i}{\partial q_k} + \frac{\partial T}{\partial \dot{y}_i}\frac{\partial y_i}{\partial q_k} + \frac{\partial T}{\partial \dot{z}_i}\frac{\partial z_i}{\partial q_k}\right)$$
$$- \sum_{i=1}^{n}\left(\frac{\partial T}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial q_k} + \frac{\partial T}{\partial \dot{y}_i}\frac{\partial \dot{y}_i}{\partial q_k} + \frac{\partial T}{\partial \dot{z}_i}\frac{\partial \dot{z}_i}{\partial q_k}\right) \tag{11.13}$$

Using Eq. (11.4), the first part of the expression (11.13) is written in the form:

$$\sum_{i=1}^{n}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial \dot{q}_k} + \frac{\partial T}{\partial \dot{y}_i}\frac{\partial \dot{y}_i}{\partial \dot{q}_k} + \frac{\partial T}{\partial \dot{z}_i}\frac{\partial \dot{z}_i}{\partial \dot{q}_k}\right) \tag{11.14}$$

We can easily show that if we consider T as a function of the q_r's and the $\dot{q}_r$'s, the expression between brackets in (11.14) is equal to $\partial T/\partial \dot{q}_k$. Namely, T in Cartesian coordinates is a function of the $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$ only and independent of x_i, y_i, z_i. Therefore,

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_{i=1}^{n}\left(\frac{\partial T}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial \dot{q}_k} + \frac{\partial T}{\partial \dot{y}_i}\frac{\partial \dot{y}_i}{\partial \dot{q}_k} + \frac{\partial T}{\partial \dot{z}_i}\frac{\partial \dot{z}_i}{\partial \dot{q}_k}\right) \tag{11.15}$$

which is identical with the expression between brackets in (11.14). We call the quantity $\partial T/\partial \dot{q}_k$ the *generalized momentum* corresponding to the generalized coordinate q_k. The second sum in (11.13) is evidently equal to $\partial T/\partial q_k$.

Hence, the whole expression which multiplies δq_k in (11.11) will simply be equal to $d/dt\ (\partial T/\partial \dot{q}_k) - \partial T/\partial q_k$, and d'Alembert's equation (11.5) can be written in the form:

$$\sum_{k=1}^{r}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} - Q_k\right]\delta q_k = 0 \tag{11.16}$$

The variations δq_k are arbitrary, because the $q_1, q_2, \ldots, q_r$ are free coordinates without geometrical constraint. Hence,

Eq. (11.16) is equivalent to r equations of the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k \tag{11.17}$$

These equations are called *Lagrange's equations* of motion. The quantity $\partial T/\partial \dot{q}_k$ was introduced as generalized momentum. Therefore, in generalized coordinates the *rate of change of the generalized momentum is in general not equal to the generalized force*, but equal to the force plus the derivative of the kinetic energy with respect to the corresponding coordinate.

That such a term necessarily must be included in the equation of motion if the kinetic energy is a function not only of the velocities, but also of the coordinates, can be shown by an elementary example. Consider the motion of a mass point in a plane, using polar coordinates r and φ as generalized coordinates. Then the kinetic energy is equal to

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) \tag{11.18}$$

and, as it is seen, is a function of $\dot{r}, \dot{\varphi}$ and of the coordinate r.

The components of the generalized momentum are $\partial T/\partial \dot{r} = m\dot{r}$, and $\partial T/\partial \dot{\varphi} = mr^2\dot{\varphi}$. Let us assume that the generalized forces are R and M, where R is the radial component of the external force and M is the moment of the external forces with respect to the origin. Then using Lagrange's equations (11.17), putting $q_1 = r$, $q_2 = \varphi$, $Q_1 = R$, $Q_2 = M$,

$$\begin{aligned} \frac{d}{dt}(m\dot{r}) - mr\dot{\varphi}^2 &= R \\ \frac{d}{dt}(mr^2\dot{\varphi}) &= M \end{aligned} \tag{11.19}$$

The physical significance of these equations is evident: $m\dot{r}$ is the radial component of the momentum of the mass point, $mr^2\dot{\varphi}$ is the moment of its momentum. The left side of the first equation is equal to the rate of change of the momentum in the direction of the radius vector $\bar{r}$ of the moving mass point. This rate of change consists of two portions: the rate of change of the magnitude of $m\dot{r}$, which is given by $d/dt(\partial T/\partial \dot{r})$, and the portion $mr\dot{\varphi}^2$, which is a contribution originating from the change in the direction of the component $mr\dot{\varphi}$. The term $mr\dot{\varphi}^2$ is known as

the centrifugal force. In Lagrange's equations it is brought in by the term $-\frac{\partial T}{\partial r}$.

If the external forces are conservative, *i.e.*, the work done by the forces is equal to the change of the potential energy $U(q_1, q_2, \ldots, q_n)$ of the system $\delta W_v = -\delta U$, and, therefore, $Q_k = -\frac{\partial U}{\partial q_k}$. Then Lagrange's equations can be written in the following form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial (T - U)}{\partial q_k} = 0 \tag{11.20}$$

Taking into account that U does not depend on the $\dot{q}_k$, *i.e.*, $\frac{\partial U}{\partial \dot{q}_k} = 0$, we can write (11.20) in the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \tag{11.21}$$

where $L = T - U$, *i.e.*, the difference between kinetic and potential energies. L is called the *Lagrangian function of the system.*

In Chapters V and VI we shall apply Lagrange's equations to the problem of small oscillations. In this case the kinetic energy will be approximated by a quadratic function of the generalized velocities with constant coefficients, so that $\partial T/\partial q_k$ vanishes and Lagrange's equations are reduced to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = Q_k \tag{11.22}$$

Problems

1. A rope passes over a frictionless and weightless pulley. Two monkeys M_1 and M_2 are grasping the freely hanging ends of the rope. Determine the motion of M_1 and M_2:

a. If they have the same weight, M_1 climbs at the rate of 1 ft./sec., relative to the rope, and M_2 merely hangs on the rope.
b. If the weight of M_1 is twice the weight of M_2 and both climb at the rate of 1 ft./sec., relative to the rope.
c. If the weight of M_1 is twice the weight of M_2 and M_2 climbs at the rate of 1 ft./sec., relative to the rope, and M_1 merely hangs on.

The motion starts for all cases with zero initial velocity at $t = 0$.

2. A single-cylinder engine is mounted so that it can move freely in the horizontal direction. The piston moves horizontally, its weight is 2.3 lb.,

and the stroke is 6.5 in.; the ratio between the length r of the crank and the length l of the connecting rod is $r/l = \frac{1}{8}$. We assume that the mass of the connecting rod can be replaced by a mass of 0.7 lb. at the piston and 0.8 lb. at the crankpin, and that the latter mass and that of the crank are balanced by a counterweight. This leaves as the only unbalance the mass of 2.3 + 0.7 lb. moving with the piston.

Find the amplitude of the horizontal motion of the machine frame if the total weight of the system is equal to 110 lb.

Determine the additional counterweight to be mounted at a distance of 4 in. from the axis of the crankshaft so that the amplitude of the horizontal motion of the machine would be zero if $l = \infty$. What is the actual remaining amplitude due to the fact that $r/l = \frac{1}{8}$?

3. The two ends A and B of a beam of length l can slide without friction on a horizontal circle of radius r (Fig. P.3). A dog starts from rest at A and runs with a constant velocity toward B to reach a dish of food. Determine the location on the circle of the end B at the instant the dog reaches the food.

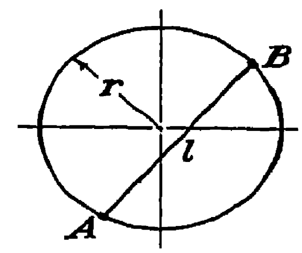

FIG. P.3.

Hint: The total moment of momentum of the beam and the dog with respect to the center of the circle is zero initially and remains equal to zero at all instants. In the limiting case where the mass of the beam is negligible, it will be found that the dog returns to his initial absolute position.

4. A rigid mathematical pendulum of length l and mass m can swing around a horizontal axis (Fig. P.4) which is mounted on a disk driven with a constant angular velocity Ω about a vertical axis. The vertical axis passes through the point of suspension of the pendulum. For what values of Ω is a motion possible such that θ is constant and different from zero? What is the relation between θ and Ω?

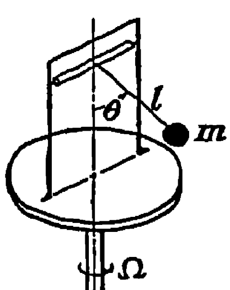

FIG. P.4.

5. Assume that the disk described in Prob. 4 is not driven but can rotate freely around the vertical axis. At the instant $t = 0$, the pendulum is in the vertical position and is given an initial velocity v_0; at the same time the disk is given an initial angular velocity Ω_0. Set up and discuss the equations of motion for the system. Determine the interaction between the motion of the pendulum and the rotation of the disk.

Hint: Use the theorems of conservation of moment of momentum and of energy.

6. The propeller of an airplane rotates at 1,500 r.p.m. clockwise when seen from the cockpit. The airplane turns to the right in a horizontal plane with a constant angular velocity corresponding to a 360° turn in 15 sec. Find the gyroscopic moment of the propeller. The diameter of the two-bladed propeller is 10 ft., its weight is 100 lb., and the weight can be assumed approximately linearly distributed between a maximum at the axis and zero at the tips. Show that the gyroscopic pitching moment of a two-bladed propeller varies during a revolution, whereas that of a three- or four-bladed propeller is constant.

Hint: Use a coordinate system turning with the airplane and apply the rules of relative motion (section 5).

7. A top consists of a circular disk mounted on a spindle that has one fixed point and is free to assume all directions around that point (Fig. P.7). The disk revolves about the spindle at 800 r.p.m., it weighs 2 lb., its diameter is 10 in., and its distance from the fixed point is 8 in. If the spindle is assumed to make an angle of 30° with the vertical, what are the two speeds of precession? What are these speeds if the spindle is horizontal?

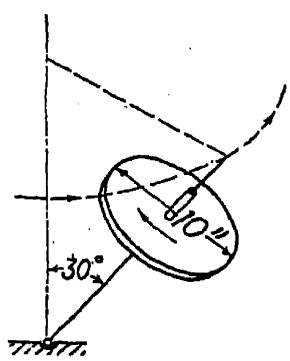

FIG. P.7.

8. Two beads can slide without friction on two rods lying in a vertical plane forming an X, whose legs are inclined at 45° with the vertical. One of the beads weighs 2 lb., the other one 3 lb. They are connected by a weightless rigid rod and can slide through the point of intersection of the X. Find the equilibrium positions of the beads:

a. Using the principle of virtual displacements.
b. By determining the path described by the center of gravity of the two beads.

9. A uniform steel beam is suspended from a hoisting hook by a cable fastened at the points A and B (Fig. P.9). The cable is free to slip without friction on the hook. The distance between A and B is equal to l, the length of the cable is L, and the center of gravity of the beam lies at the mid-point between A and B. Show that without friction the horizontal position of the beam is unstable.

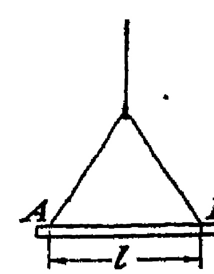

FIG. P.9.

10. A uniform beam of length l and weight W is supported at its two ends. At the instant $t = 0$, one of the supports is suddenly removed. Determine, using d'Alembert's principle, the load that acts at the same instant on the remaining support.

11. Investigate the stability of the vertical equilibrium position of the pendulum described in Prob. 4 when the mass of the pendulum is below the point of suspension.

Hint: Consider the motion of the pendulum relative to the rotating disk. Then the impressed forces are (*a*) gravity and (*b*) centrifugal force. Hence, the potential energy of the mass of the pendulum in an arbitrary position is equal to the potential energy of gravity minus the work done by the centrifugal force. If r is the distance from the axis of rotation, the work of the centrifugal force is given by $m\int_0^r r\Omega^2\,dr = mr^2\Omega^2/2$, *i.e.*, is equal to the kinetic energy of the mass. Hence, the pendulum is stable in the vertical position, if $mgl\cos\theta - \dfrac{mr^2\Omega^2}{2} = mgl\cos\theta - \dfrac{ml^2\sin^2\theta\,\Omega^2}{2}$ has a minimum for $\theta = 0$.

12. Investigate the stability of the same pendulum in its deflected equilibrium positions.

13. A cylindrical drum of gasoline falls with its flat bottom toward the ground. The total weight of the drum is 800 lb., the weight of the gasoline 700 lb., the area of the bottom 3.5 sq. ft., and the coefficient of air resistance C referred to this area 0.9 (the total air resistance is equal to $CS\frac{\gamma v^2}{2g}$ where γ is the specific weight of air, S the area, and v the velocity of fall). Calculate the pressure difference in the liquid between the bottom and the top during the fall up to the terminal velocity. Calculate the same pressure difference neglecting air resistance.

14. Set up Lagrange's equation for the mechanism shown in the accompanying figure. The necessary data are the following:

m_1 is the mass of the piston.
m_2 is the mass of the connecting rod AB.
I_2 is the moment of inertia of the connecting rod about the point A.
I_3 is the moment of inertia of the crank OB, the shaft, and the flywheel.
S is the area of the piston.
p is the pressure acting on the piston.
l is the length of the connecting rod.
s is the distance between A and the C.G. of the connecting rod.
r is the length of the crank OB.
Q is the torque acting on the shaft.

Assume for simplicity that if θ is the angle of the crank with the horizontal, the displacement of the piston is $x = r\cos\theta$. Use θ as generalized coordinate. Friction is neglected.

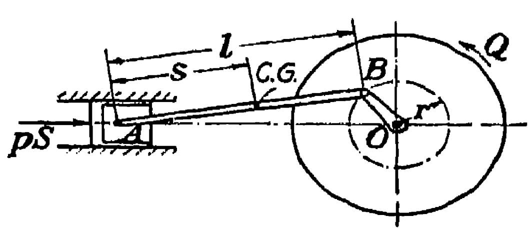

FIG. P.14.

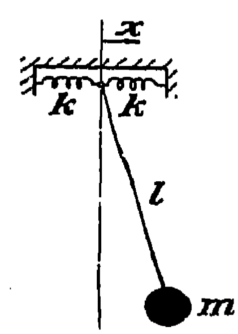

FIG. P.15.

15. The point of suspension of a mathematical pendulum of length l and mass m in the accompanying figure is restrained to a fixed base by springs, which develop a restoring force equal to $-kx$ if the displacement of the point of suspension is x. Set up the equations of motion using Lagrange's equations. Show that for small displacements the system is equivalent to a mathematical pendulum of length $l + \frac{mg}{k}$.

16. A and B are fixed particles located on the same horizontal level. Each exerts an attraction on a mass point C according to the law K/r, *i.e.*, the force is inversely proportional to the distance. The weight of C is mg. Determine the possible equilibrium positions of C, and investigate their stability.

References

General:

1. OSGOOD, W. F.: "Mechanics," The Macmillan Company, New York, 1937.
2. WEBSTER, A. G.: "The Dynamics of Particles and of Rigid, Elastic and Fluid Bodies," B. G. Teubner, Leipzig, 1925.
3. LAMB, H.: "Higher Mechanics," Cambridge University Press, London, 1920.
4. JEANS, J. H.: "An Elementary Treatise on Theoretical Mechanics," Ginn and Company, Boston, 1907.

Gyroscope:

5. KLEIN, F. and A. SOMMERFELD: "Theorie des Kreisels," vol. IV (contains engineering applications), B. G. Teubner, Leipzig, 1910.
6. FERRY, E. S.: "Applied Gyrodynamics," John Wiley & Sons, Inc., New York, 1932.

Vector algebra and analysis:

7. WILLS, A. P.: "Vector Analysis with an Introduction to Tensor Analysis," Prentice-Hall, Inc., 1931.
8. COFFIN, J. F.: "Vector Analysis," John Wiley & Sons, Inc., New York, 1911.

CHAPTER IV

ELEMENTARY PROBLEMS IN DYNAMICS

> An intelligent being who knew for a given instant all forces by which nature is animated and possessed complete information on the state of matter of which nature consists—provided his mind were powerful enough to analyze these data—could express in the same equation the motion of the largest bodies of the universe and the motion of the smallest atoms. Nothing would be uncertain for him, and he would see the future as well as the past at one glance.
>
> LE MARQUIS DE LAPLACE,
> "Théorie Analytique des Probabilités" (1820)

Introduction.—In this chapter we shall deal with problems of motion that require the integration of differential equations with given initial conditions. Most of the problems refer to the motion of a single particle under the action of forces originating from gravity, elastic resilience, resistance of the medium, etc. There are two *excursions* into the field of mathematics proper included in this chapter: section 4 gives some information on elliptic integrals and elliptic functions from the practical viewpoint of the engineer and physicist. The treatment is carried only far enough to enable the reader to recognize the type of the elliptic integral which he might encounter in his work, and to reduce it to a form suitable for numerical evaluation by tables or by series. The second mathematical topic (section 11) is concerned with the singularities of differential equations of the first order. The various types of singularities are geometrically illustrated.

1. Linear Motion of a Particle in a Resisting Medium.—Let us first consider the motion of a particle along a straight line. The distance of the particle from its initial position at the time $t = 0$ is denoted by x. The particle is subjected to a force that is assumed to be a function of the velocity $v = dx/dt$.

Denoting the mass of the particle by m, the force by $-f(v)$, the equation of motion is given by

$$m \frac{dv}{dt} = -f(v) \tag{1.1}$$

The negative sign is chosen for the reason that in most of the practical cases such a force depending on the velocity has the character of a resistance which is opposed to the motion. Separating the variables, Eq. (1.1) becomes

$$dt = -m \frac{dv}{f(v)}$$

and denoting the initial value of the velocity by v_0, we obtain

$$t = -\int_{v_0}^{v} \frac{m\,dv}{f(v)} = m \int_{v}^{v_0} \frac{dv}{f(v)} \tag{1.2}$$

Then the distance x reached by the particle at the time t is given by

$$x = \int_0^t v\,dt = -\int_{v_0}^{v} \frac{mv\,dv}{f(v)} = m \int_{v}^{v_0} \frac{v\,dv}{f(v)} \tag{1.3}$$

Three cases can be easily calculated:

a. Constant friction: $f(v) = F$.

b. Resistance proportional to the velocity: $f(v) = \beta v$.

c. Resistance proportional to the square of the velocity: $f(v) = cv^2$.

Evaluating the integrals occurring in Eqs. (1.2) and (1.3), the following results are obtained:

	t	x
a	$\dfrac{m(v_0 - v)}{F}$	$\dfrac{m(v_0^2 - v^2)}{2F}$
b	$\dfrac{m}{\beta} \log \dfrac{v_0}{v}$	$\dfrac{m}{\beta}(v_0 - v)$
c	$\dfrac{m}{c}\left(\dfrac{1}{v} - \dfrac{1}{v_0}\right)$	$\dfrac{m}{c} \log \dfrac{v_0}{v}$

Substituting $v = 0$ into the expression given in the table, we obtain the time required for the particle to come to rest and the total distance traversed by the particle. It is seen that the time is finite in the case a only, whereas the total distance is finite

in the two cases a and b, *viz.*, $l = mv_0^2/2F$ and $l = mv_0/\beta$, respectively. In the case of the quadratic resistance law, both the time and distance are infinite.

2. Linear Motion of a Particle under Action of a Force Depending on the Position of the Particle.—It is assumed that the force acting on a particle is a function $g(x)$ of the distance x measured from an arbitrary origin O. Then the equation of the motion will be

$$m \frac{d^2x}{dt^2} = g(x) \tag{2.1}$$

By multiplication of both sides of Eq. (2.1) by dx/dt, we obtain

$$m \frac{dx}{dt} \frac{d^2x}{dt^2} = g(x) \frac{dx}{dt}$$

and by integration we find upon substituting the initial values $x = x_0$, $dx/dt = v_0$,

$$\frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 - v_0^2\right] = \int_{x_0}^{x} g(x)\, dx \tag{2.2}$$

This relation is a so-called *first integral* of the differential equation of the second order (2.1) (cf. Chapter I, section 7). It represents the mathematical expression for the theorem of the conservation of energy; the left side represents the change in the kinetic energy, the right side the work done by the force $g(x)$, or the decrease of the potential energy. In fact, if we define the potential energy by $G(x) = -\int_0^x g(x)\, dx$, Eq. (2.2) can be written in the form:

$$\left(\frac{dx}{dt}\right)^2 = v_0^2 + \frac{2[G(x_0) - G(x)]}{m}$$

Separating the variables, we obtain

$$dt = \frac{dx}{\sqrt{v_0^2 + \dfrac{2}{m}[G(x_0) - G(x)]}}$$

or

$$t = \int_{x_0}^{x} \frac{dx}{\sqrt{v_0^2 + \dfrac{2}{m}[G(x_0) - G(x)]}} \tag{2.3}$$

Equation (2.3) is the general solution of the equation of motion (2.1).

As an example let us assume $g(x) = -kx$, $G(x) = kx^2/2$. The force $-kx$ is called the *elastic restoring force*. It represents for example the action of a spring or any other force which has the tendency to pull the mass point back toward the *equilibrium position* $x = 0$ (Fig. 2.1). In most cases such a force can be

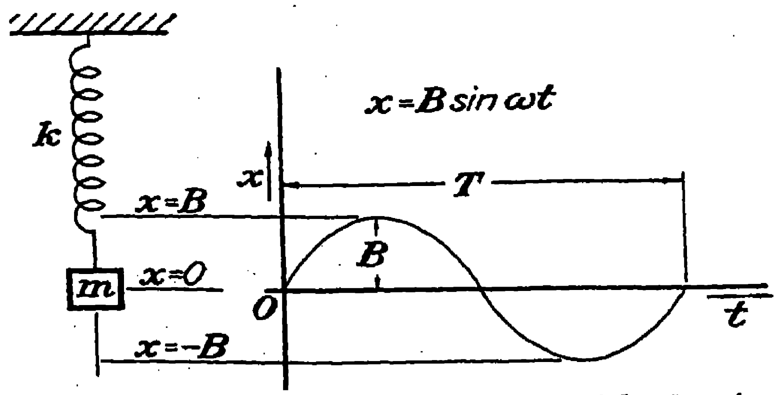

FIG. 2.1.—Harmonic oscillations of a mass with elastic restraint.

approximated for small values of x by a linear expression of the form $g(x) = -kx$.

Introducing $G(x) = kx^2/2$ in Eq. (2.3), and assuming $x_0 = 0$, it follows that

$$t = \int_0^x \frac{dx}{\sqrt{v_0^2 - \frac{k}{m}x^2}}$$

or

$$t = \frac{1}{v_0}\int_0^x \frac{dx}{\sqrt{1 - \frac{kx^2}{mv_0^2}}}$$

Carrying out the integration, we obtain

$$t = \sqrt{\frac{m}{k}}\sin^{-1}\left(\sqrt{\frac{k}{m}}\frac{x}{v_0}\right) \tag{2.4}$$

or

$$x = v_0\sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}t \tag{2.5}$$

Obviously (2.5) can be derived more directly by integration of the equation of motion

$$m\frac{d^2x}{dt^2} = -kx \tag{2.6}$$

solution of Eq. (2.6) is

$$x = A \cos \sqrt{\frac{k}{m}}t + B \sin \sqrt{\frac{k}{m}}t$$

ιts of integration are determined by the initial values ; $= v_0$ at $t = 0$. This gives $A = 0$ and $B = v_0\sqrt{m/k}$ .ce with Eq. (2.5). The motion is represented in

ion characterized by Eq. (2.5) is called a *simple otion;* $v_0\sqrt{m/k}$ is the *amplitude,* $T = 2\pi\sqrt{m/k}$ the $\frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ the *frequency* of the motion. We call ty $\sqrt{k/m} = \omega$ the *angular frequency.* Then the tween frequency, period, and angular frequency are e formula: $\omega = 2\pi\nu = 2\pi/T$.

ı **of a Pendulum.**—Let us consider a pendulum of moment of inertia I about its point of suspension O We denote the distance from f suspension to the center of y r, the angular displacement OC from the vertical by θ and ation of gravity by g. Then on of motion becomes (cf. ., section 6)

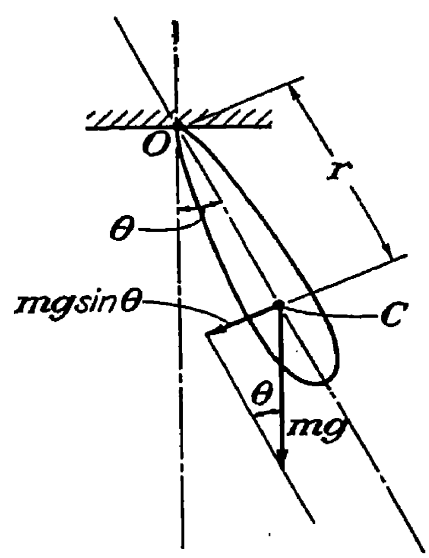

Fig. 3.1.—The compound pendulum.

$$I \frac{d^2\theta}{dt^2} = -mgr \sin \theta \qquad (3.1)$$

ion states that the rate of he angular momentum about suspension is equal to the the weight about the same st integral of Eq. (3.1) can again be obtained immedi- multiplying both sides by $d\theta/dt$. Then we have

$$I \frac{d^2\theta}{dt^2} \cdot \frac{d\theta}{dt} = -mgr \sin \theta \frac{d\theta}{dt}$$

gration

$$\frac{I}{2}\left(\frac{d\theta}{dt}\right)_0^2 - \frac{I}{2}\left(\frac{d\theta}{dt}\right)^2 = mgr(\cos \theta_0 - \cos \theta) \qquad (3.2)$$

or

$$\frac{d\theta}{dt} = \sqrt{\left(\frac{d\theta}{dt}\right)_0^2 + \frac{2mgr}{I}(\cos\theta - \cos\theta_0)}$$

Finally, we obtain

$$t = \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\left(\frac{d\theta}{dt}\right)_0^2 + \frac{2mgr}{I}(\cos\theta - \cos\theta_0)}} \tag{3.3}$$

Assuming the following initial conditions for $t = 0$; $\theta_0 = 0$, and $(d\theta/dt)_0 = \omega_0$, we obtain

$$t = \frac{1}{\omega_0}\int_0^{\theta} \frac{d\theta}{\sqrt{1 - \frac{2mgr}{I\omega_0^2}(1 - \cos\theta)}}$$

or

$$t = \frac{1}{\omega_0}\int_0^{\theta} \frac{d\theta}{\sqrt{1 - \frac{4mgr}{I\omega_0^2}\sin^2\frac{\theta}{2}}} \tag{3.4}$$

The behavior of the integral (3.4) is different depending upon whether $4mgr/I\omega_0^2 \lesseqgtr 1$ (Fig. 3.2).

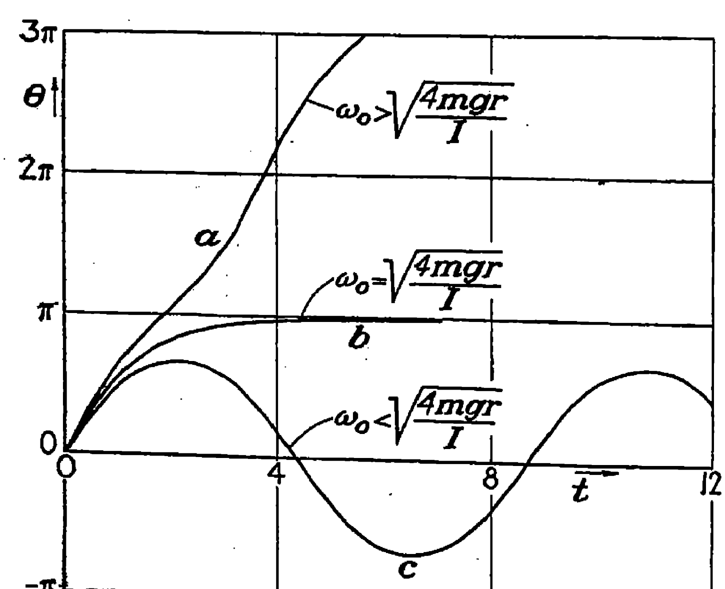

FIG. 3.2.—Various types of motion of a pendulum: (a) rotation with periodically varying speed; (b) asymptotic approach to $\theta = 180°$; (c) periodic oscillation.

a. Assume $4mgr/I\omega_0^2 < 1$. In this case the angle increases indefinitely, because the expression under the square root sign is always positive. In other words, if the initial angular velocity

$\omega_0 > \sqrt{4mgr/I}$, the pendulum *continues its rotational motion in one direction indefinitely* without return (curve a in Fig. 3.2). To be sure, the body rotates with variable velocity. The condition $4mgr/I\omega_0^2 < 1$ is self-explanatory in the form $I\omega_0^2/2 > 2mgr$; it means that the amount of kinetic energy available is larger than the work necessary to raise the center of gravity from its lowest to its highest position.

b. Assume $4mgr/I\omega_0^2 = 1$. In this case the value of the integral in Eq. (3.4) for $\theta = \pi$ as upper limit becomes divergent; this means that the pendulum approaches the top position without reaching it in finite time (b, Fig. 3.2).

c. Assume $4mgr/I\omega_0^2 > 1$. In this case the expression under the radical in Eq. (3.4) is real only if the value of θ is restricted between $\pm\alpha$, where $\sin^2 \alpha/2 = I\omega_0^2/4mgr$. Hence, θ will increase to α and decrease again (c in Fig. 3.2). We obtain an oscillatory motion with the amplitude α. Introducing $\sin^2 \alpha/2$ into Eq. (3.2), we obtain

$$t = \frac{1}{\omega_0} \int_0^\theta \frac{d\theta}{\sqrt{1 - \dfrac{\sin^2 \dfrac{\theta}{2}}{\sin^2 \dfrac{\alpha}{2}}}} \tag{3.5}$$

The integral (3.5) extended from 0 to α gives one-quarter of the period of a complete oscillation. However, for the practical calculation of the period, it is more convenient to have an integral with the limits 0 and $\pi/2$, than with the limits 0 and α. We apply the transformation

$$\frac{\sin \dfrac{\theta}{2}}{\sin \dfrac{\alpha}{2}} = \sin \varphi$$

Then we have by differentiation

$$\frac{1}{2} \frac{\cos \dfrac{\theta}{2}\, d\theta}{\sin \dfrac{\alpha}{2}} = \cos \varphi \, d\varphi$$

and Eq. (3.5) becomes

$$t = \sqrt{\frac{I}{mgr}} \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - \sin^2 \frac{\alpha}{2} \sin^2 \varphi}} \tag{3.6}$$

The limit $\theta = \alpha$ corresponds now to $\varphi = \pi/2$, and one-quarter of the total period is equal to

$$\frac{T}{4} = \sqrt{\frac{I}{mgr}} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \sin^2 \frac{\alpha}{2} \sin^2 \varphi}} \tag{3.7}$$

The integrals on the right side of Eqs. (3.6) and (3.7) cannot be resolved in terms of elementary functions. They are called *elliptic integrals of the first kind.* Some information concerning their practical computation and their application to our problem is given in the next section.

For small values of α, however, we obtain by use of the binomial theorem,

$$\left(1 - \sin^2 \frac{\alpha}{2} \sin^2 \varphi\right)^{-1/2} = 1 + \frac{1}{2} \sin^2 \frac{\alpha}{2} \sin^2 \varphi + \cdots$$

the following expansion:

$$T = 4\sqrt{\frac{I}{mgr}} \left(\int_0^{\pi/2} d\varphi + \frac{1}{2} \sin^2 \frac{\alpha}{2} \int_0^{\pi/2} \sin^2 \varphi \, d\varphi + \cdots \right)$$

After carrying out the integrations

$$T = 2\pi\sqrt{\frac{I}{mgr}} \left(1 + \frac{1}{4} \sin^2 \frac{\alpha}{2} + \cdots \right) \tag{3.8}$$

It is seen that in the first approximation (for infinitely small oscillations) the period is independent of the amplitude α and is equal to

$$T_0 = 2\pi\sqrt{\frac{I}{mgr}} \tag{3.9}$$

The second approximation is

$$T = 2\pi\sqrt{\frac{I}{mgr}} \left(1 + \frac{\alpha^2}{16}\right) \tag{3.10}$$

In Fig. 3.3 are plotted the exact value of T/T_0 given by Eq. (3.7) and the second approximation given by (3.10).

The first approximation can be obtained in a more direct way from the equation of motion

$$I\frac{d^2\theta}{dt^2} = -mgr\sin\theta \tag{3.11}$$

For small values of θ, we have approximately $\sin\theta = \theta$, and the solution of (3.11) is

$$\theta = A\sin\sqrt{\frac{mgr}{I}}t + B\cos\sqrt{\frac{mgr}{I}}t$$

Hence, $T_0\sqrt{mgr/I} = 2\pi$, in accordance with Eq. (3.9).

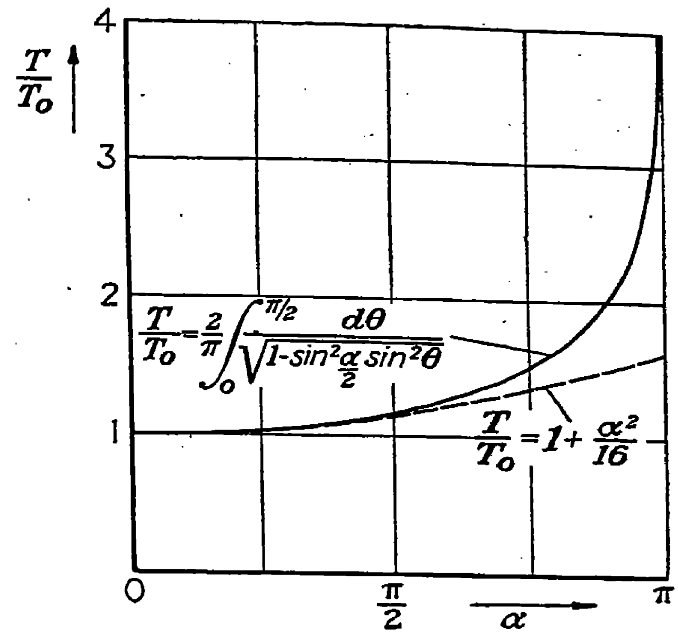

FIG. 3.3.—The ratio T/T_0 as a function of the amplitude α.

4. Some Information Concerning Elliptic Integrals and Elliptic Functions.—The solution of dynamical problems often requires the calculation of integrals of the form $\int dx/\sqrt{Q(x)}$ where $Q(x)$ is a polynomial in x. If $Q(x)$ is of the second degree, it is known that the integral can be reduced to one of the following fundamental integrals: $\int dz/\sqrt{1-z^2}$, $\int dz/\sqrt{1+z^2}$, and $\int dz/\sqrt{z^2-1}$. In fact, if $Q(x)$ is a quadratic function of x, it can be reduced by multiplication by a positive constant either to the form

$$Q_1(x) = a + 2bx + x^2$$

or $Q_2(x) = a + 2bx - x^2$, where a and b are real numbers. The function $Q(x)$ is assumed to be positive between the limits of integration. Using the linear substitution $x = p + z$, the integral becomes

$$I_1 = \int\frac{dz}{\sqrt{(a+2bp+p^2)+2(b+p)z+z^2}} \tag{4.1}$$

or

$$I_2 = \int\frac{dz}{\sqrt{(a+2bp-p^2)+2(b-p)z-z^2}} \tag{4.2}$$

respectively. By choosing $p = -b$ or $p = b$, respectively, we eliminate the linear term. Then we easily obtain the fundamental integrals mentioned on page 119.

In the case of I_1, we see that $a + 2bp + p^2 = a - b^2$.

a. Assume that $a - b^2 > 0$, then with

$$\zeta = \frac{z}{\sqrt{a - b^2}}, \qquad I_1 = \int \frac{d\zeta}{\sqrt{1 + \zeta^2}}$$

b. Assume that $a - b^2 < 0$, then with

$$\zeta = \frac{z}{\sqrt{b^2 - a}}, \qquad I_1 = \int \frac{d\zeta}{\sqrt{\zeta^2 - 1}}$$

c. With $a - b^2 = 0$, the integral degenerates into

$$I_1 = \int \frac{dz}{z} = \log z + \text{const.}$$

In the case of I_2, $a + 2bp - p^2 = a + b^2$; this expression must be positive if $Q(x) > 0$, as it was assumed. Therefore, we put $\zeta = z/\sqrt{a + b^2}$ and obtain

$$I_2 = \int \frac{d\zeta}{\sqrt{1 - \zeta^2}}$$

The three fundamental integrals obtained above are the so-called *circular* and *hyperbolic integrals.* They define the *inverse circular and hyperbolic functions** thus:

$$\int_0^\zeta \frac{d\zeta}{\sqrt{1 + \zeta^2}} = \sinh^{-1} \zeta, \qquad \int_1^\zeta \frac{d\zeta}{\sqrt{\zeta^2 - 1}} = \cosh^{-1} \zeta,$$
$$\int_0^\zeta \frac{d\zeta}{\sqrt{1 - \zeta^2}} = \sin^{-1} \zeta, \qquad \int_\zeta^1 \frac{d\zeta}{\sqrt{1 - \zeta^2}} = \cos^{-1} \zeta$$

As $\int_0^1 d\zeta/\sqrt{1 - \zeta^2} = \pi/2$, we have $\cos^{-1} \zeta = \pi/2 - \sin^{-1} \zeta$. The definite integral $\int_0^1 d\zeta/\sqrt{1 - \zeta^2} = \pi/2$ can be called the *complete circular integral.*

Elliptic Integrals of the First Kind.—We have detailed these elementary calculations in order to point out the analogy between the class of the circular and hyperbolic integrals and the class of the elliptic integrals.

* The notation $\sinh^{-1} \zeta$ means that if $u = \sinh^{-1} \zeta$, then $\zeta = \sinh u$.

Let us now assume that $Q(x)$ is a polynomial of the 3rd or 4th degree. In this case, we call the integral

$$I = \int \frac{dx}{\sqrt{Q(x)}} \tag{4.3}$$

an *elliptic integral*. It was shown by Legendre that the integral (4.3) can be reduced by simple substitutions to the form:

$$I = \text{const.} \int \frac{d\zeta}{\sqrt{\pm(1 \pm \zeta^2)(1 \pm c^2\zeta^2)}}$$

and by further substitutions to

$$I = \text{const.} \int \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \tag{4.4}$$

The definite integral $\int_0^\varphi d\varphi/\sqrt{1 - k^2 \sin^2 \varphi}$ is called *Legendre's standard form of the elliptic integral of the first kind.* Tables exist for the value of this integral as a function of the *amplitude* φ and of the *modulus* k. In order to use these tables, it is necessary to know how an integral of the form (4.3) can be reduced to the form (4.4).

Let us assume that $Q(x)$ is of the *4th degree;* then it can be written in the form $Q(x) = \pm(x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$, where $\alpha, \beta, \gamma, \delta$ are the roots of the algebraic equation $Q(x) = 0$. The roots $\alpha, \beta, \gamma, \delta$ can be all real, or α, β can be real, and γ, δ, complex conjugates; finally, α, β and γ, δ can each be a pair of complex conjugate quantities. Instead of the substitution $x = p + z$, we now use the substitution $x = \dfrac{p + qz}{1 + z}$. Then we obtain

$$I = \int \frac{dx}{\sqrt{Q(x)}} = \int \frac{(q - p)\, dz}{\sqrt{\pm G(z)}} \tag{4.5}$$

where

$$G(z) = [(p - \alpha) + (q - \alpha)z][(p - \beta) + (q - \beta)z][(p - \gamma) + (q - \gamma)z][(p - \delta) + (q - \delta)z] \tag{4.6}$$

We choose p and q in such a way that the terms with odd powers in $G(z)$ vanish. If (4.6) is expanded and the terms with z and z^3 collected, the reader will verify that the coefficients of both z and z^3 vanish if

$$(p - \alpha)(q - \beta) + (q - \alpha)(p - \beta) = 0$$
$$(p - \gamma)(q - \delta) + (q - \gamma)(p - \delta) = 0$$

or

$$pq - (p + q)\left(\frac{\alpha + \beta}{2}\right) + \alpha\beta = 0$$
$$pq - (p + q)\left(\frac{\gamma + \delta}{2}\right) + \gamma\delta = 0$$

From these two equations we obtain

$$p + q = -\frac{2(\gamma\delta - \alpha\beta)}{\alpha + \beta - \gamma - \delta} \tag{4.7}$$
$$pq = \frac{\alpha\beta(\gamma + \delta) - \gamma\delta(\alpha + \beta)}{\alpha + \beta - \gamma - \delta}$$

The Eqs. (4.7) are satisfied if p and q are the roots of the equation

$$Lv^2 + Mv + N = 0 \tag{4.8}$$

where $L = \alpha + \beta - \gamma - \delta$
$M = 2(\gamma\delta - \alpha\beta)$
$N = \alpha\beta(\gamma + \delta) - \gamma\delta(\alpha + \beta)$

The coefficients of (4.8) are real in all three cases mentioned on page 121. The roots of (4.8) will be real if

$$(\gamma\delta - \alpha\beta)^2 - (\alpha + \beta - \gamma - \delta)\,[\alpha\beta(\gamma + \delta) - \gamma\delta(\alpha + \beta)] > 0 \tag{4.9}$$

The expression (4.9) can be reduced to

$$(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) > 0$$

and it can be verified that it is satisfied whether the roots of $Q(x) = 0$ are real or complex.

Hence, the coefficients p and q of the transformation always turn out to be real. An exceptional case is $\alpha + \beta = \gamma + \delta$. In this case we have to use the substitution $x = p + z$, where $p = \frac{\alpha + \beta}{2}$*.

If p and q are determined in such a way that they satisfy (4.8), the integral I will appear in the form:

$$I = \text{const.} \int \frac{dz}{\sqrt{\pm(1 \pm g^2z^2)(1 \pm h^2z^2)}} \tag{4.10}$$

* See E. Goursat, "A Course in Mathematical Analysis," p. 226 (see Ref. 8).

Now, if h^2 denotes the larger of the two quantities g^2 and h^2, we put $z = \zeta/h$ and $g^2/h^2 = c^2$, *where c^2 is less than one.* Then the integral becomes

$$I = \text{const.}\,\frac{1}{h}\int \frac{d\zeta}{\sqrt{\pm(1 \pm \zeta^2)(1 \pm c^2\zeta^2)}} \qquad (4.11)$$

It is seen that up to this point the steps in the substitutions are quite analogous to those occurring in the reduction of the circular or the hyperbolic class of integrals. However, although for these classes the value of p is easily found, for the class of elliptic integrals in general it is difficult to find the values of p and q in a direct way, because the roots α, β, γ, δ must be known. Fortunately, in most of the practical cases $Q(x)$ already appears in a *factorized form* so that it is easy to find the proper substitution which leads to (4.10) or (4.11).

The case of $Q(x)$, being of the third degree, may be considered as a limiting case assuming that one of the roots, say $\alpha = \infty$. The substitution $x = \dfrac{p + qz}{1 + z}$ transfers the root $x = \infty$ to $z = -1$, and we obtain a polynomial of the *fourth degree for* $G(z)$.

In order to reduce (4.11) to the *standard form,* we have to consider eight possible cases:

a. Assume the first factor is $1 - \zeta^2$ and $\zeta^2 < 1$, then

(1) $I = \displaystyle\int \frac{d\zeta}{\sqrt{(1 - \zeta^2)(1 - c^2\zeta^2)}}$ with $\zeta = \sin\varphi$ becomes

$$I = \int \frac{d\varphi}{\sqrt{1 - c^2 \sin^2 \varphi}}$$

(2) $I = \displaystyle\int \frac{d\zeta}{\sqrt{(1 - \zeta^2)(1 + c^2\zeta^2)}}$ with $\zeta = \cos\varphi$ becomes

$$I = -\frac{1}{\sqrt{1 + c^2}}\int \frac{d\varphi}{\sqrt{1 - \dfrac{c^2}{1 + c^2}\sin^2 \varphi}}$$

The case $(1 - \zeta^2)(c^2\zeta^2 - 1)$ must be excluded because if $\zeta^2 < 1$, then $c^2\zeta^2 - 1 < 0$, and the product under the square root sign would be negative.

b. Assume now that the first factor is $1 + \zeta^2$, then

(3) $I = \int \frac{d\zeta}{\sqrt{(1 + \zeta^2)(1 + c^2\zeta^2)}}$ with $\zeta = \tan \varphi$ becomes

$$I = \int \frac{d\varphi}{\sqrt{1 - (1 - c^2) \sin^2 \varphi}}$$

(4) $I = \int \frac{d\zeta}{\sqrt{(1 + \zeta^2)(1 - c^2\zeta^2)}}$ with $\zeta = \frac{\cos \varphi}{c}$ becomes

$$I = -\frac{1}{\sqrt{1 + c^2}} \int \frac{d\varphi}{\sqrt{1 - \frac{1}{1 + c^2} \sin^2 \varphi}}$$

(5) $I = \int \frac{d\zeta}{\sqrt{(1 + \zeta^2)(c^2\zeta^2 - 1)}}$ with $\zeta = \frac{1}{c \cos \varphi}$ becomes

$$I = \frac{1}{\sqrt{1 + c^2}} \int \frac{d\varphi}{\sqrt{1 - \frac{c^2}{1 + c^2} \sin^2 \varphi}}$$

c. Finally, assume that the first factor is $\zeta^2 - 1$, and that $\zeta^2 > 1$ then,

(6) $I = \int \frac{d\zeta}{\sqrt{(\zeta^2 - 1)(c^2\zeta^2 - 1)}}$ with $\zeta = \frac{1}{c \sin \varphi}$ becomes

$$I = -\int \frac{d\varphi}{\sqrt{1 - c^2 \sin^2 \varphi}}$$

(7) $I = \int \frac{d\zeta}{\sqrt{(\zeta^2 - 1)(1 + c^2\zeta^2)}}$ with $\zeta = \frac{1}{\cos \varphi}$ becomes

$$I = \frac{1}{\sqrt{1 + c^2}} \int \frac{d\varphi}{\sqrt{1 - \frac{1}{1 + c^2} \sin^2 \varphi}}$$

(8) $I = \int \frac{d\zeta}{\sqrt{(\zeta^2 - 1)(1 - c^2\zeta^2)}}$. This case requires special consideration. We put $\sqrt{\zeta^2 - 1} = m \cos \varphi$, $\sqrt{1 - c^2\zeta^2} = n \sin \varphi$, where m and n are constant factors to be determined. Evidently, $\zeta^2 = 1 + m^2 \cos^2 \varphi = \frac{1}{c^2} - \frac{n^2}{c^2} \sin^2 \varphi$; hence, $1 + m^2 = \frac{1}{c^2}$, $m^2 = \frac{n^2}{c^2}$, and $\zeta^2 = 1 + \frac{1 - c^2}{c^2} \cos^2 \varphi = \frac{1}{c^2} - \frac{1 - c^2}{c^2} \sin^2 \varphi$. Intro-

ducing this substitution for ζ into the integral, we obtain

$$I = -\int \frac{d\varphi}{\sqrt{1 - (1 - c^2)\sin^2\varphi}}.$$

The above analysis shows that the integrals (4.10) and (4.11) can always be reduced to the form:

$$\int \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}.$$

where $k^2 < 1$.

The definite integral

$$u = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} = F(k,\varphi) \tag{4.12}$$

or its equivalent obtained by the substition $\zeta = \sin\varphi$,

$$u = \int_0^{\zeta} \frac{d\zeta}{\sqrt{(1 - \zeta^2)(1 - k^2\zeta^2)}} \tag{4.13}$$

is called the *elliptic integral of the first kind*. $F(k,\varphi)$ is a function of the *modulus* k and of the upper limit φ.*

The integral (4.13) is analogous to the circular integral

$$u = \int_0^{\zeta} \frac{d\zeta}{\sqrt{1 - \zeta^2}} = \sin^{-1}\zeta$$

which defines a periodic function ζ of u, *viz.*,

$$\zeta = \sin u$$

whose quarter period is given by the complete circular integral

$$\frac{\pi}{2} = \int_0^1 \frac{d\zeta}{\sqrt{1 - \zeta^2}}$$

In the same way, the elliptic integral (4.13) defines a function ζ of u. We first consider the upper limit φ of the integral (4.12) as a function of u and use the notation

$$\varphi = \text{am}\, u \text{ (amplitude of } u)$$

Then we have

$$\zeta = \sin(\text{am}\, u)$$

* Tables of elliptic functions are available, for example, in B. O. Peirce, "A Short Table of Integrals"; E. Jahnke and F. Emde, "Tables of Functions." In these tables the parameter α defined by $\sin\alpha = k$ is used instead of k. A simple introduction to elliptic functions for practical use with many graphs and tables will be found in H. Hancock, "Elliptic Integrals," Mathematical Monographs.

This is generally written in the form:

$$\zeta = \operatorname{sn} u \tag{4.14}$$

The function $\operatorname{sn} u$ is one of *Jacobi's elliptic functions*. It is a periodic function of u. Its amplitude is equal to unity. Its period is not a fixed constant as in the case of the circular functions, but depends on the value of the modulus k.

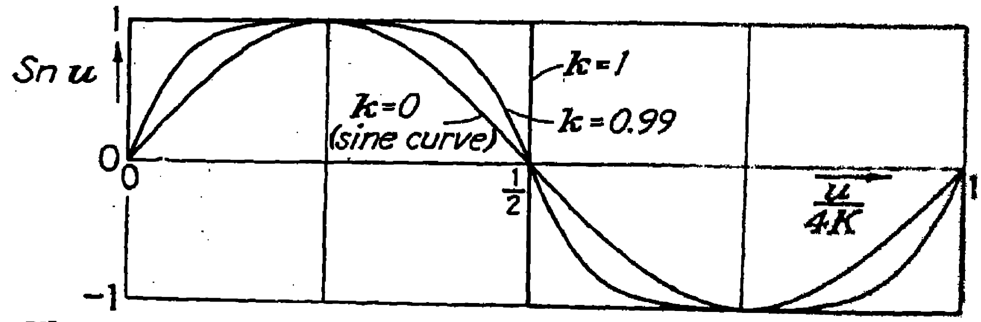

FIG. 4.1.—The elliptic function sn u plotted as function of $u/4K$ for different values of the modulus k.

The quarter period denoted by K is expressed by the following integral:

$$K = \int_0^1 \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}$$
$$= \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = F(k,\pi/2) \tag{4.15}$$

This integral is called the *complete elliptic integral of the first kind*.

The function $\zeta = \operatorname{sn} u$ is plotted in Fig. 4.1 for $k = 0$, $k = 0.99$ and $k = 1$. The abscissa is the value of u divided by the period $4K$, so that the functions are reduced to the same period, and only the shape varies with k.

We note that for $k = 0$ we have $\operatorname{sn} u = \sin u$ and $K = \dfrac{\pi}{2}$. If $k \to 1$, the period tends to infinity, because for $k = 1$ the integral (4.15) becomes divergent. The function asymptotically approaches unity as $u \to \infty$.

The Pendulum.—As an example of the application of the elliptic integrals of the first kind, let us return to the problem of the pendulum. We found for $4mgr/T\omega_0^2 \geqq 1$ the following expression for the time t as function of the angular displacement [cf. Eq. (3.6)]:

$$t = \sqrt{\frac{I}{mgr}} \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-\sin^2\dfrac{\alpha}{2}\sin^2\varphi}} \tag{4.16}$$

where $\sin\varphi$ is defined by $\sin\varphi = \dfrac{\sin\theta/2}{\sin\alpha/2}$ and α is the maximum angular displacement. The integral in Eq. (4.16) is an elliptic integral of the first kind with the modulus $k = \sin\alpha/2$ and the amplitude φ. Hence,

$$t = \sqrt{\frac{I}{mgr}}\,F\left(\sin\frac{\alpha}{2}, \varphi\right)$$

Considering φ as function of the time, we write

$$\varphi = \mathrm{am}\left(t\sqrt{\frac{mgr}{I}}\right)$$

Hence, the angular displacement θ is given by

$$\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\sin\mathrm{am}\left(t\sqrt{\frac{mgr}{I}}\right)$$

or

$$\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\,\mathrm{sn}\left(t\sqrt{\frac{mgr}{I}}\right) \tag{4.17}$$

The period of a full oscillation is

$$T = 4K\sqrt{\frac{I}{mgr}} \tag{4.18}$$

Therefore, we may write (4.17) in the form:

$$\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\,\mathrm{sn}\left(4K\frac{t}{T}\right) \tag{4.19}$$

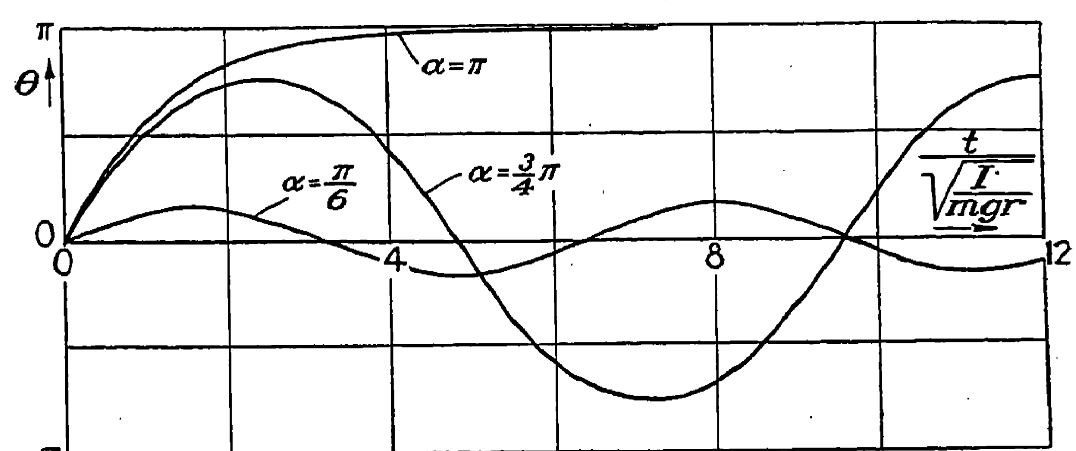

FIG. 4.2.—Angular displacement of a pendulum as function of time for various amplitudes α.

The value of θ is plotted in Fig. 4.2 for three values of the amplitude α, *viz.*, $\alpha = \pi/6$, $\alpha = \frac{3}{4}\pi$, and $\alpha = \pi$. In the last case the pendulum approaches the inverted vertical position; the period in that case is infinite.

For small oscillations we may write approximately

$$\sin\frac{\theta}{2} = \frac{\theta}{2}, \qquad \sin\frac{\alpha}{2} = \frac{\alpha}{2}, \qquad \operatorname{sn} u = \sin u, \qquad K = \frac{\pi}{2}$$

We have, therefore,

$$\theta = \alpha \sin\frac{2\pi t}{T}$$

where

$$T = 2\pi\sqrt{\frac{I}{mgr}}$$

Elliptic Integral of the Second Kind—Length of an Arc of an Ellipse.—The term elliptic integral originates from the problem of the *rectification* of an ellipse, *i.e.*, from the problem of determining the length of an elliptic arc.

An ellipse may be represented by the parametric equations

$$x = a\cos\varphi$$
$$y = b\sin\varphi$$

The constants a and b are the two semiaxes of the ellipse (Fig. 4.3).

The length of the elliptic arc AB is given by

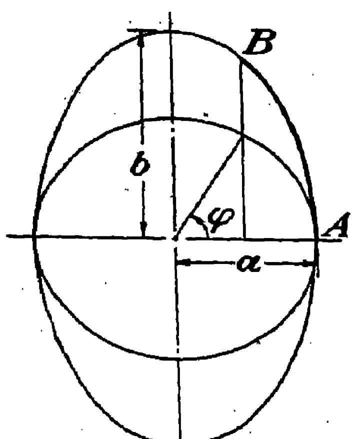

FIG. 4.3.—Derivation of an ellipse from a circle.

$$l = \int_A^B \sqrt{dx^2 + dy^2}$$
$$= \int_0^\varphi \sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}\, d\varphi$$

Assuming $b^2 > a^2$,

$$l = b\int_0^\varphi \sqrt{1 - \frac{b^2 - a^2}{b^2}\sin^2\varphi}\, d\varphi$$

Putting $k^2 = \dfrac{b^2 - a^2}{b^2}$, we write

$$E(k,\varphi) = \int_0^\varphi \sqrt{1 - k^2\sin^2\varphi}\, d\varphi \tag{4.20}$$

This integral is called the *elliptic integral of the second kind.* The complete elliptic integral of the second kind is defined by

$$E = \int_0^{\pi/2} \sqrt{1 - k^2\sin^2\varphi}\, d\varphi \tag{4.21}$$

Tables for the values of $E(k,\varphi)$ as functions of k and φ and of E as function of k will be found in Jahnke-Emde and in Peirce. The parameter α is sometimes used instead of the modulus k, where $\sin \alpha = k$.

By the substitution $\zeta = \sin \varphi$, we have

$$E(k,\varphi) = \int_0^\zeta \frac{\sqrt{1 - k^2\zeta^2}}{\sqrt{1 - \zeta^2}}\, d\zeta \tag{4.22}$$

The integral (4.22) can be written in the form:

$$E(k,\varphi) = \int_0^\zeta \frac{1 - k^2\zeta^2}{\sqrt{(1 - \zeta^2)(1 - k^2\zeta^2)}}\, d\zeta$$

In general, any integral of the form $\int \frac{P(z)}{R(z)\sqrt{Q(z)}}\, dz$, where $P(z)$, $R(z)$ are polynomials in z and $Q(z)$ is a polynomial of the third or fourth degree, is called an *elliptic integral*. It can be shown that such an integral always can be reduced to a sum of integrals of the following forms:

a. Elliptic integrals of the first and second kind, as defined in this section.

b. Integrals of the form $\int_0^z \frac{dz}{(c + z^2)\sqrt{(1 - k^2z^2)(1 - z^2)}}$

c. Integrals which can be expressed by elementary functions. It might be the case that computing an integral of the class considered, we encounter representatives of all groups *a*, *b*, and *c*, or that one or two of the groups appear. The elliptic integrals under *a* are tabulated. The integral

$$\int_0^z \frac{dz}{(c + z^2)\sqrt{(1 - k^2z^2)(1 - z^2)}}$$

is called an *elliptic integral of the third kind*. It is seen that it contains, besides the modulus k and the upper limit z, an additional parameter c. General tables for such integrals are not available, but the complete elliptic integral (upper limit equal to unity) of the third kind can be expressed by incomplete integrals of the first and second kind and by elementary functions.*

* Louis V. King, "On the Direct Numerical Calculation of Elliptic Functions (Ref. 10)."

However, when elliptic integrals of the third kind are involved in an engineering problem, the reduction to standard forms is sometimes so cumbersome that graphical or numerical integration is preferred.

5. Linear Motion of a Particle with Elastic Restraint and Damping.—This case is usually referred to as the problem of damped free oscillations. The problem is formulated here for the motion of a particle restricted to move in one dimension. The simplest example of such a system is a mass connected to a spring and a dashpot (Fig. 5.1). We have seen that the spring force can be written as $-kx$, and the simplest way to represent a damping force depending on the velocity is to write it as $-\beta v = -\beta \dfrac{dx}{dt}$. This corresponds to viscous damping and is generally a good approximation, at least for small velocities. The scheme and results of the following calculations are applicable to the small oscillations of any damped system with one degree of freedom.

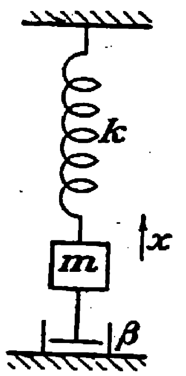

FIG. 5.1.—A mass connected to a spring and a dashpot.

The equation of motion has the form:

$$m \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + kx = 0 \tag{5.1}$$

and its general solution is

$$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \tag{5.2}$$

where λ_1 and λ_2 are the roots of the characteristic equation (cf. Chapter I, section 10):

$$m\lambda^2 + \beta\lambda + k = 0 \tag{5.3}$$

Its roots are

$$\lambda_1 = -\frac{\beta}{2m} + \sqrt{\left(\frac{\beta}{2m}\right)^2 - \frac{k}{m}}$$

and

$$\lambda_2 = -\frac{\beta}{2m} - \sqrt{\left(\frac{\beta}{2m}\right)^2 - \frac{k}{m}} \tag{5.4}$$

We assume $\beta \neq 0$ and consider two different cases:

a. $\beta^2 > 4km$. In this case λ_1 and λ_2 are both real and negative, hence, the right side of (5.2) consists of two exponential

terms. It is easily shown that if the mass starts, for instance, from the position $x = 0$ at $t = 0$, it cannot pass through the position $x = 0$ again (cf. the example in Fig. 5.2). If the mass

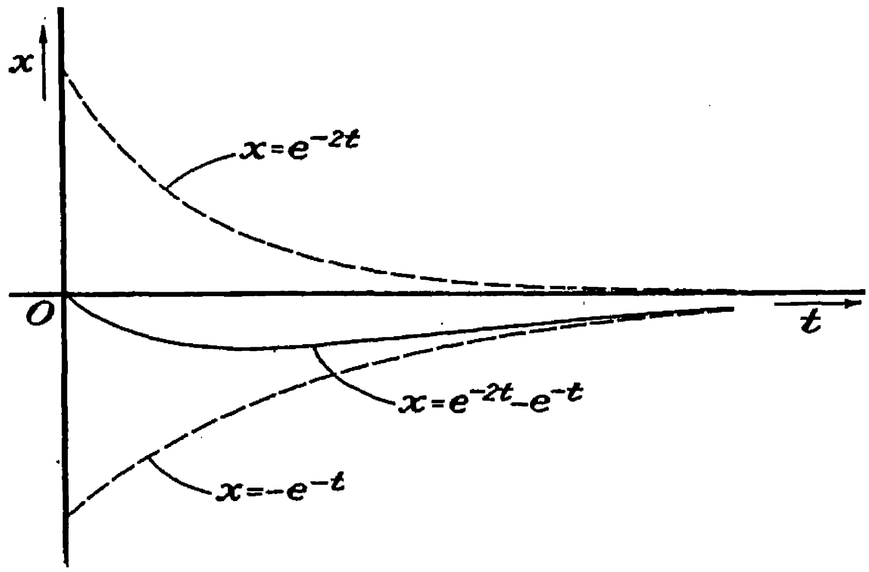

FIG. 5.2.—Example of the motion of a mass connected to a spring and a dashpot, put in motion from the equilibrium position.

starts from an arbitrary position, it cannot pass through $x = 0$ more than once (Fig. 5.3). In each case, x tends to 0 as the time t becomes infinite. This type of motion is called a *subsidence.*

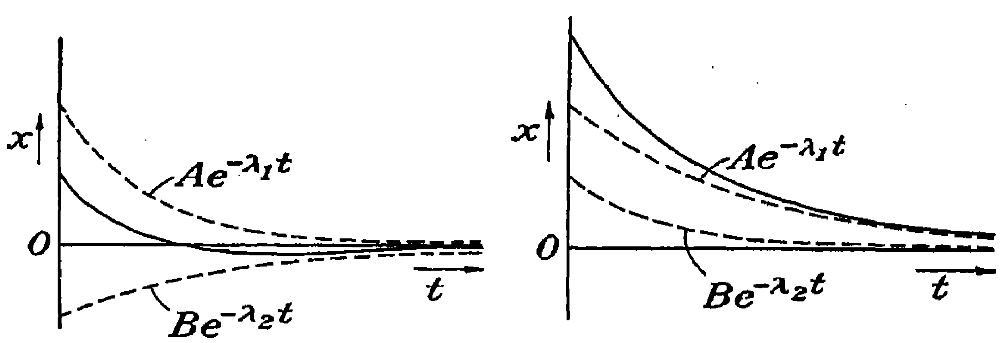

FIG. 5.3.—Examples of subsidence for a mass moving under action of a spring and a dashpot.

b. $\beta^2 < 4km$. In this range λ_1 and λ_2 are complex and conjugate. The solution can be written in the following form containing real quantities only:

$$x = Ce^{-\frac{\beta}{2m}t} \cos\left(\sqrt{\frac{k}{m} - \frac{\beta^2}{4m^2}}\, t\right) + De^{-\frac{\beta}{2m}t} \sin\left(\sqrt{\frac{k}{m} - \frac{\beta^2}{4m^2}}\, t\right) \quad (5.5)$$

The expression (5.5) corresponds to damped oscillations with an infinite number of passages through $x = 0$ (Fig. 5.4). The period, defined by the time between two passages in the same

direction, is equal to $T = 2\pi \frac{1}{\sqrt{(k/m) - (\beta^2/4m^2)}}$. The frequency $\nu = 1/T$ is plotted in dimensionless form as a function

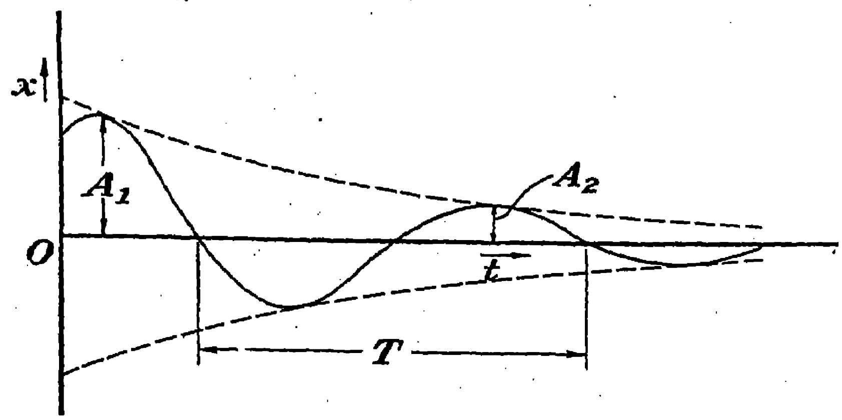

FIG. 5.4.—Oscillatory type of damped motion.

of the damping factor β in Fig. 5.5. The frequency $\nu_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ corresponds to zero damping, whereas the damping factor $2\sqrt{mk}$, called *critical damping factor*, corresponds to the damping for which the motion changes from an oscillatory type to a pure subsidence. It is easily shown by differentiation of (5.5) that the maxima of x occur at equidistant values of t with the same interval as the period T; hence, the ratio between two consecutive maxima is equal to $\Delta = e^{-\frac{\beta}{2m}t}/e^{-\frac{\beta}{2m}(t+T)} = e^{\frac{\beta}{2m}T}$. The quantity $\log \Delta = \frac{\beta}{2m} T$ is called the *logarithmic decrement*.

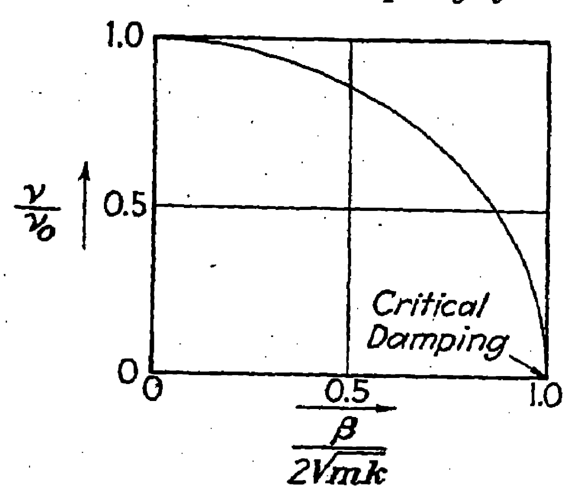

FIG. 5.5.—Variation of the frequency ν as function of the damping factor β.

In the case of critical damping, *i.e.*, for $\beta = 2\sqrt{mk}$, $\lambda_1 = \lambda_2$, and it is necessary to find a second linearly independent solution. Using the method given in Chapter I, section 10, we obtain

$$x = Ae^{\lambda_1 t} + Bte^{\lambda_2 t}$$

where $\lambda_1 = \lambda_2 = -\sqrt{k/m}$. The second term also represents a subsident motion. An example of such a motion is represented in Fig. 5.6.

6. Motion of a Mass with Elastic Restraint under Action of a Periodic External Force. Resonance.—We now take up the

problem of *forced vibrations* without damping. The equation of motion is

$$m\frac{d^2x}{dt^2} + kx = F_0 \sin \omega t \tag{6.1}$$

where $F_0 \sin \omega t$ is a periodic impressed force and ω its angular frequency. This equation applies, for instance, to the case in

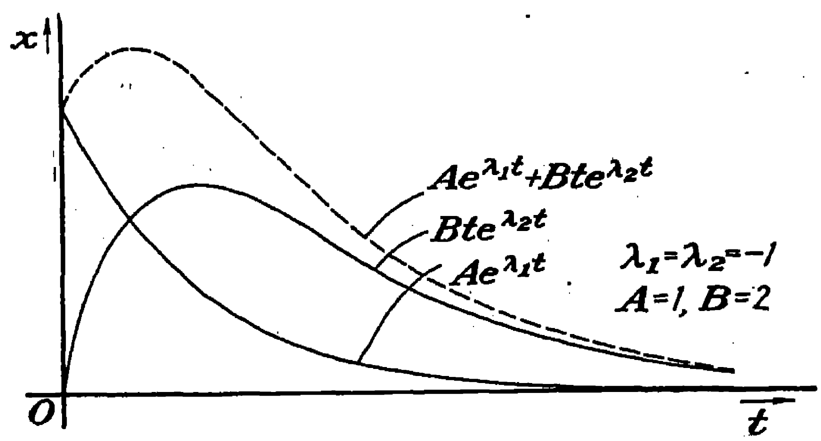

FIG. 5.6.—Damped motion in case of critical damping.

which a body is elastically connected to an oscillating base (Fig. 6.1). In this case, let l represent the length of the spring when the body is in static equilibrium; x, the extension of the

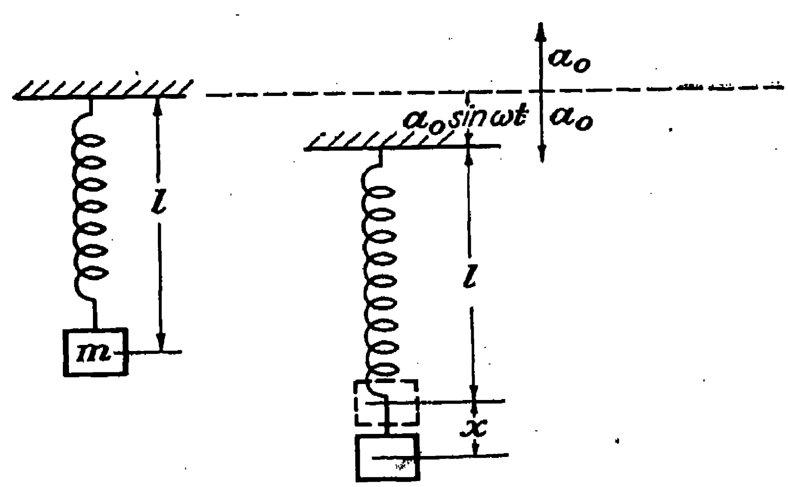

FIG. 6.1.—Forced vibration of a mass suspended by a spring due to the motion of the point of support.

spring; and $a_0 \sin \omega t$, the periodic displacement of the base. Then the coordinate of the mass with respect to a fixed origin is $a_0 \sin \omega t + l + x$. According to Newton's second law, the equation of motion is

$$m\frac{d^2}{dt^2}(a_0 \sin \omega t + l + x) + kx = 0$$

or

$$m\frac{d^2x}{dt^2} + kx = ma_0\omega^2 \sin \omega t \tag{6.2}$$

Thus we see that the influence of the motion of the base with the amplitude a_0 and the frequency ω is equivalent to the action of a periodic force $F_0 \sin \omega t$, where $F_0 = ma_0\omega^2$.* The solution of (6.1) is composed of the general solution of the homogeneous equation and one arbitrary particular solution of the nonhomogeneous equation. We find a particular solution by substituting $x = C \sin \omega t$, and obtain:

$$-mC\omega^2 + kC = F_0 \tag{6.3}$$

or

$$C = \frac{F_0}{k - m\omega^2}$$

Denoting by ω_0 the angular frequency of the free oscillation, where $\omega_0^2 = k/m$, we obtain

$$x = \frac{F_0}{m}\frac{\sin \omega t}{\omega_0^2 - \omega^2} = \frac{F_0}{k}\frac{\sin \omega t}{1 - \dfrac{\omega^2}{\omega_0^2}} \tag{6.4}$$

The oscillation represented by the expression (6.4) is of special interest. If ω/ω_0 is small compared with unity, we have approximately

$$x = \frac{F_0 \sin \omega t}{k}$$

i.e., the deflection x at any instant is equal to the static deflection corresponding to the instantaneous value of the force. If ω/ω_0 is not negligible, the deflection is increased by the factor

$$\frac{1}{1 - \dfrac{\omega^2}{\omega_0^2}} \tag{6.5}$$

This factor is known as the *resonance factor*. It is infinite if $\omega = \omega_0$, *i.e.*, if the frequency of the impressed force and the frequency of the free oscillation coincide.

* Equation (6.2) can also be obtained from the rule that if the motion is referred to a moving coordinate system, we have to introduce an additional force equal to the product of the mass and the negative acceleration of the moving coordinate system (Chapter III, section 5).

Since the factor (6.5) is positive when $\omega < \omega_0$ and is negative when $\omega > \omega_0$, the force and the deflection have the same sign below the resonance frequency and opposite signs above the resonance frequency. Let us denote the period of a complete oscillation by T, where $T = 2\pi/\omega$. Then the work done in the time interval T is given by $W = \int_0^T F_0 \sin \omega t \, dx$, or, substituting dx from (6.4), $W = \frac{F_0^2}{k} \frac{\omega}{1 - \frac{\omega^2}{\omega_0^2}} \int_0^T \sin \omega t \cos \omega t \, dt.$

Putting $\omega t = \varphi$, we obtain

$$W = \frac{F_0^2}{k} \frac{1}{1 - \frac{\omega^2}{\omega_0^2}} \int_0^{2\pi} \sin \varphi \cos \varphi \, d\varphi \tag{6.6}$$

The integral in (6.6) is equal to zero. Thus we obtain the result that the work done by the impressed force during a complete period of a forced oscillation of an undamped system is equal to zero. The complete solution of Eq. (6.1) is given by

$$x = A \cos \omega_0 t + B \sin \omega_0 t + \frac{F_0 \sin \omega t}{m(\omega_0^2 - \omega^2)} \tag{6.7}$$

The constants of integration in Eq. (6.7) are determined by the initial values of x and dx/dt for $t = 0$. Putting $x = 0$ and $dx/dt = v_0$ for $t = 0$, the following expression is easily obtained:

$$x = v_0 \frac{\sin \omega_0 t}{\omega_0} + \frac{F_0}{m(\omega_0^2 - \omega^2)\omega_0} (\omega_0 \sin \omega t - \omega \sin \omega_0 t) \tag{6.8}$$

We shall investigate the case for which the values of ω and ω_0 are only slightly different. To discuss this case, we transform the expression (6.8) into

$$x = v_0 \frac{\sin \omega_0 t}{\omega_0} + \frac{F_0}{m(\omega_0^2 - \omega^2)\omega_0} \left[(\omega_0 - \omega) \cos \left(\frac{\omega - \omega_0}{2} t \right) \sin \left(\frac{\omega + \omega_0}{2} t \right) \right.$$
$$\left. + (\omega + \omega_0) \sin \left(\frac{\omega - \omega_0}{2} t \right) \cos \left(\frac{\omega + \omega_0}{2} t \right) \right]$$

or

$$x = v_0 \frac{\sin \omega_0 t}{\omega_0} + \frac{F_0}{m(\omega_0 + \omega)\omega_0} \cos \left(\frac{\omega - \omega_0}{2} t \right) \sin \left(\frac{\omega + \omega_0}{2} t \right)$$
$$+ \frac{F_0}{m(\omega_0 - \omega)\omega_0} \sin \left(\frac{\omega - \omega_0}{2} t \right) \cos \left(\frac{\omega + \omega_0}{2} t \right) \tag{6.9}$$

In Fig. 6.2 the displacement x is plotted for the ratio $\omega/\omega_0 = \frac{5}{4}$. For small values of $\omega - \omega_0$ the factor of the third term on the right side of (6.9) is large compared to the first two terms. The third term represents *beats*, *i.e.*, it can be interpreted as a harmonic

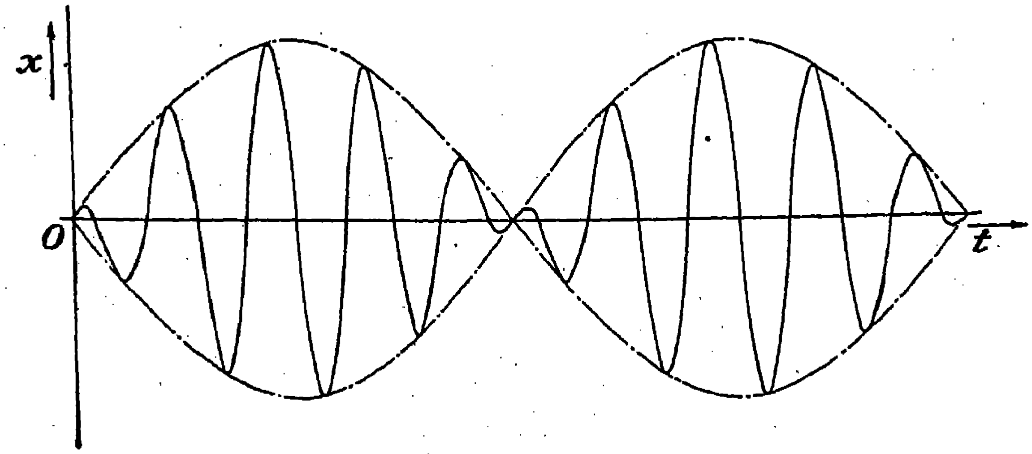

FIG. 6.2.—Beats. The frequency of the imposed periodic force differs slightly from the natural frequency of the oscillating mass.

motion with the angular frequency $\dfrac{\omega + \omega_0}{2} \cong \omega_0$ and with a periodically varying amplitude. The number of the beats in unit time is equal to $\left|\dfrac{\omega - \omega_0}{2\pi}\right|$, *i.e.*, it is equal to the difference between the frequency of the natural vibration and the frequency of the impressed force.

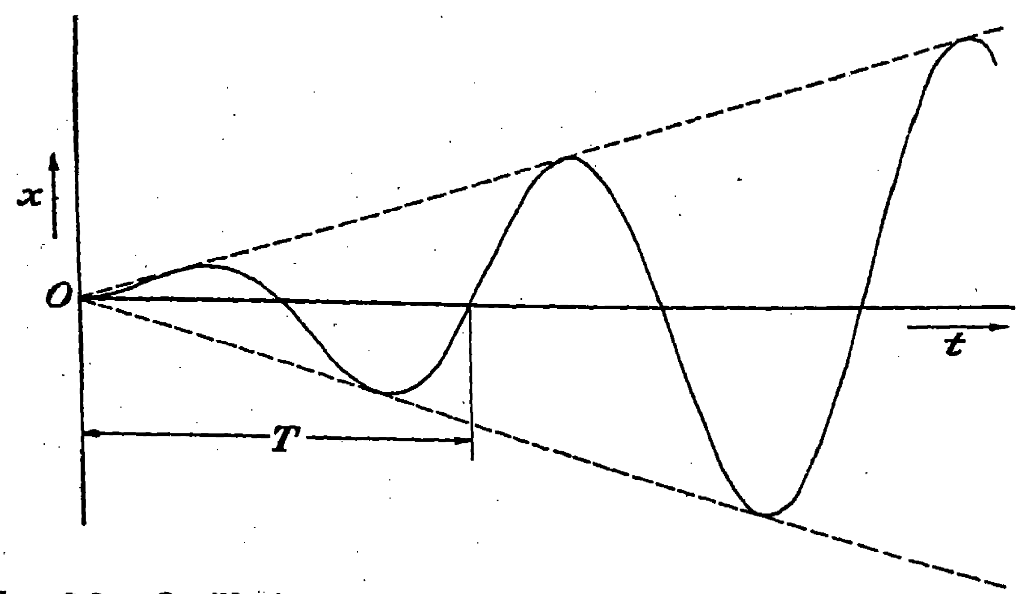

FIG. 6.3.—Oscillation with increasing amplitude in case of resonance.

The solution (6.7) fails for $\omega = \omega_0$ (resonance). In this case the particular solution (6.4) cannot be used and has to be replaced by a solution of the form $x = Ct \cos \omega_0 t$. Introducing this expression for x into (6.1), we have

$$-2Cm\omega_0 \sin \omega_0 t - Cmt\omega_0^2 \cos \omega_0 t + kCt \cos \omega_0 t = F_0 \sin \omega_0 t$$

Taking into account that $m\omega_0^2 = k$, we obtain

$$C = -\frac{F_0}{2m\omega_0}$$

and the complete solution of (6.1) becomes

$$x = A \cos \omega_0 t + B \sin \omega_0 t - \frac{F_0}{2m\omega_0} t \cos \omega_0 t \qquad (6.10)$$

The last term represents oscillations with indefinitely increasing amplitude (Fig. 6.3). Thus we obtain the result that if the frequency of the impressed force coincides with the natural frequency of an undamped system, the amplitude increases ad infinitum.

7. Forced Oscillation of a Mass with Damping.—The equation of the motion is, in this case,

$$m \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + kx = F_0 \sin \omega t \qquad (7.1)$$

A particular solution can be found by putting

$$x = a \sin \omega t + b \cos \omega t \qquad (7.2)$$

Introducing this expression into (7.1), the following relations for a and b are obtained:

$$\begin{aligned} (k - m\omega^2)a - \beta\omega b &= F_0 \\ (k - m\omega^2)b + \beta\omega a &= 0 \end{aligned}$$

Solving these equations for a and b and introducing their values into (7.2), we have

$$x = F_0 \frac{(k - m\omega^2) \sin \omega t - \beta\omega \cos \omega t}{(k - m\omega^2)^2 + \beta^2\omega^2} \qquad (7.3)$$

The complete solution is composed of Eq. (7.3) and the general solution (5.2) representing free damped oscillations or subsident motion. As (7.3) corresponds to a simple harmonic motion with constant amplitude while the motion represented by the terms of Eq. (5.2) dies out with time, it is clear that for sufficiently large values of the time the solution will be practically independent of the initial conditions and will be represented by Eq. (7.3) alone. We write (7.3) in the form:

$$x = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \beta^2\omega^2}} \sin(\omega t - \psi) \tag{7.4}$$

where $\tan\psi = \dfrac{\beta\omega}{k - m\omega^2}$. The angle ψ is called the *phase lag*. The lag is smaller than $\pi/2$ when $\omega < \sqrt{k/m}$; it is equal to $\pi/2$ for $\omega = \sqrt{k/m}$; and it is larger than $\pi/2$ when $\omega > \sqrt{k/m}$. For $\omega \to \infty$, $\psi \to \pi$. The amplitude of the harmonic oscillation represented by (7.4) is equal to

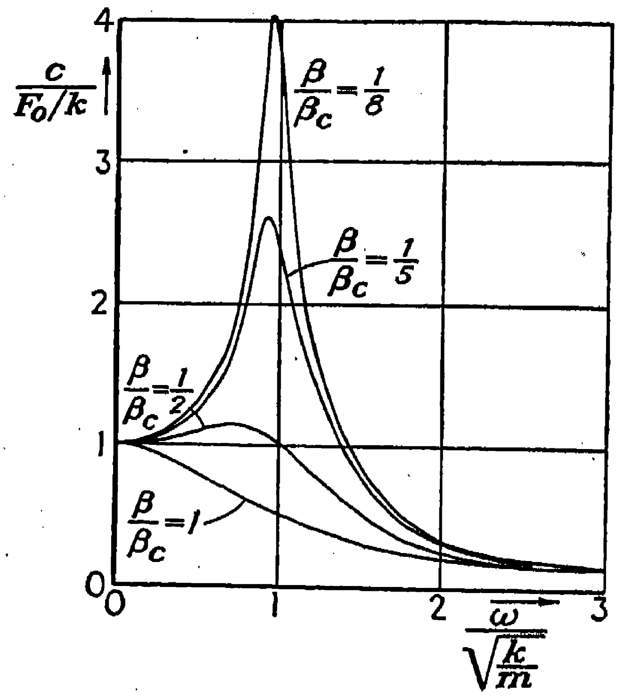

FIG. 7.1.—Amplitudes of forced oscillations as function of the frequency for various values of the damping factor β (resonance curve).

$$c = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \beta^2\omega^2}} \tag{7.5}$$

Let us determine the maximum value of c when β is kept constant and ω varies. We find the value of ω for which c reaches its maximum by differentiating the expression under the radical with respect to ω. We have

$$[-2m(k - m\omega^2) + \beta^2]\omega = 0 \tag{7.6}$$

The expression (7.6) vanishes either for $\omega = 0$ or for $\omega^2 = \dfrac{2km - \beta^2}{2m^2}$. The maximum of c occurs for $\omega = 0$ if $\beta^2 > 2km$, since in this case the second condition does not give real values for ω. If $\beta^2 < 2km$, the maximum of c occurs for $\omega^2 = \dfrac{2km - \beta^2}{2m^2}$ and is equal to

$$c_{\max} = F_0 \frac{2m}{\beta\sqrt{4km - \beta^2}} = \frac{F_0}{k} \frac{2km/\beta^2}{\sqrt{(4km/\beta^2) - 1}} \tag{7.7}$$

The amplitude ratio $c/(F_0/k)$ is plotted in Fig. 7.1 as function of the ratio $\omega/\sqrt{k/m}$ for different values of the ratio β/β_c, where $\beta_c = 2\sqrt{km}$ is the critical damping, *i.e.*, the value of the damping factor which separates the oscillatory and subsident types of motion in the case of free oscillations (cf. section 5).

For small values of β/β_c,

$$c_{max} \cong \frac{F_0}{\beta}\sqrt{\frac{m}{k}} \cong \frac{F_0}{\beta\omega_0} \tag{7.8}$$

In this case the phase lag ψ is very small when $\omega < \omega_0$, increases rapidly if we pass through $\omega = \omega_0$, and it is slightly less than π if $\omega > \omega_0$.

8. Motion of a Particle under Action of Gravity and Air Resistance (the Ballistic Problem).—Let us consider now the motion of a particle of mass m in a vertical plane under the action

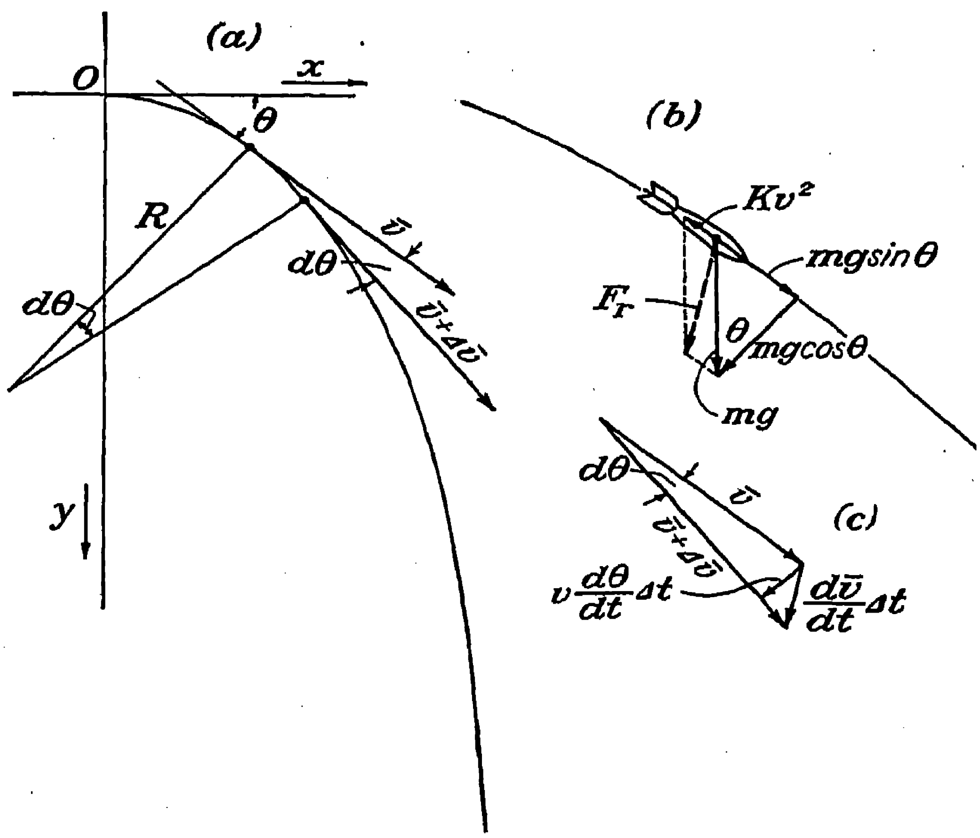

FIG. 8.1.—*a*. The trajectory of a bomb. *b*. The diagram of the forces acting on the bomb. *c*. Diagram of accelerations.

of gravity and air resistance. The action of gravity is represented by a force of magnitude mg acting in a vertical direction downward. The air resistance is represented by a force acting in opposite direction to the resultant velocity v of the mass point; its magnitude is kv^2. We first consider k as constant. This assumption is a fair approximation to the drag of a body of revolution if its Reynolds number* is not too small and the speed

* The Reynolds number is a dimensionless quantity defined as $R = vd\rho/\mu$ where v is the velocity of the body, d one of its characteristic linear dimensions, *e.g.*, its diameter, μ and ρ the viscosity and specific mass of the fluid, respectively.

v is smaller than about $\frac{8}{10}$ of the velocity of sound. The calculation with constant k gives, for example, a good approximation for the motion of bombs dropped from aircraft.

The equations of motion are given by Newton's laws (Chapter III, sections 1 and 3). It is convenient to consider the components of the forces and the accelerations in the direction of the tangent and of the normal to the path curve. Figure 8.1c shows that the component of the acceleration in the tangential direction is equal to dv/dt, and the acceleration in the normal direction is equal to $v\,\frac{d\theta}{dt}$, where θ is the angle between the tangent to the path curve and an arbitrary fixed direction. We choose as angle θ the inclination between the velocity vector $\bar{v}$ and the positive x-axis and count θ positive when $\bar{v}$ is directed downward. We notice that $v\,\frac{d\theta}{dt}$ can be written in the form v^2/R, where R is the radius of curvature of the path curve (Fig. 8.1a). In fact $1/R = d\theta/ds$, hence,

$$\frac{v^2}{R} = v^2\frac{d\theta}{ds} = v^2\frac{d\theta}{dt}\frac{dt}{ds}$$

Since $\frac{ds}{dt} = v$, we have $\frac{v^2}{R} = v\,\frac{d\theta}{dt}$.

We obtain the equations of motion by putting the product of the mass and the acceleration equal to the force in both the tangential and normal directions.

We obtain in this way (cf. Fig. 8.1b) the following two equations

$$\begin{aligned} m\frac{dv}{dt} &= mg\sin\theta - kv^2 \\ mv\frac{d\theta}{dt} &= mg\cos\theta \end{aligned} \tag{8.1}$$

From the second equation follows:

$$\frac{d\theta}{dt} = \frac{g}{v}\cos\theta \tag{8.2}$$

Noting that $\frac{dv}{dt} = \frac{dv}{d\theta}\frac{d\theta}{dt}$, and introducing this relation into the first equation (8.1), we obtain

$$m\frac{dv}{d\theta}\frac{g}{v}\cos\theta = mg\sin\theta - kv^2$$

or

$$\frac{dv}{d\theta}\cos\theta - v\sin\theta = -\frac{k}{mg}v^3 \tag{8.3}$$

Equation (8.3) is a first-order differential equation for v as function of θ. Taking the horizontal component of the velocity $v_x = v\cos\theta$ as the unknown variable, we write (8.3) in the form:

$$\frac{dv_x}{d\theta} = -\frac{k}{mg}\frac{v_x^3}{\cos^3\theta} \tag{8.4}$$

In Eq. (8.4) the variables can be separated:

$$\frac{dv_x}{v_x^3} = -\frac{k}{mg}\frac{d\theta}{\cos^3\theta} \tag{8.5}$$

Integrating (8.5), we obtain

$$\frac{1}{2v_x^2} = \frac{k}{mg}\int\frac{d\theta}{\cos^3\theta} + C \tag{8.6}$$

Let us assume now as initial conditions $v_x = v_0$ and $\theta = 0$ for $t = 0$. (This will be the case, for example, if a bomb is released from an airplane flying with the velocity v_0.) Then from Eq. (8.6) follows:

$$\frac{1}{v_x^2} = \frac{1}{v_0^2} + \frac{2k}{mg}\int^{\theta}\frac{d\theta}{\cos^3\theta} \tag{8.7}$$

Equation (8.7) determines the horizontal velocity v_x as function of the inclination θ. Carrying out the integration—using $\tan\theta = \tau$ as an auxiliary variable—we obtain

$$\frac{1}{v_x^2} = \frac{1}{v_0^2} + \frac{k}{mg}\left(\log\frac{1+\sin\theta}{\cos\theta} + \frac{\sin\theta}{\cos^2\theta}\right) \tag{8.8}$$

We write (8.8) in the form:

$$v_x = v_0 f(\theta) = v_0\frac{1}{\sqrt{1 + \frac{kv_0^2}{mg}\left(\log\frac{1+\sin\theta}{\cos\theta} + \frac{\sin\theta}{\cos^2\theta}\right)}} \tag{8.9}$$

Equation (8.9) may be considered as the equation for the so-called *hodograph* of the motion. Let us plot the velocity vector $\bar{v}$

in a v_xv_y plane (Fig. 8.2) from the origin O. The curve described by the end point of the $\bar{v}$ vector is called the *hodograph* of the motion. The equation of the hodograph in parametric form is

$$\begin{aligned} v_x &= v_0 f(\theta) \\ v_y &= v_0 f(\theta) \tan\theta \end{aligned} \tag{8.10}$$

From the hodograph equation we obtain the equations of the path curve by integration. First, from (8.2) it follows that:

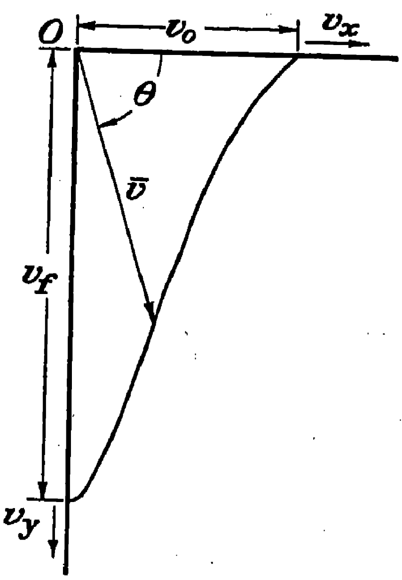

FIG. 8.2.—The hodograph of a bomb trajectory.

$$dt = \frac{v}{g}\frac{d\theta}{\cos\theta} = \frac{v_x\,d\theta}{g\cos^2\theta} \tag{8.11}$$

Then substituting v_x from Eq. (8.9) in Eq. (8.11) and integrating, we have

$$t = \frac{v_0}{g}\int_0^\theta \frac{f(\theta)\,d\theta}{\cos^2\theta} \tag{8.12}$$

Furthermore, from (8.9) and (8.10) follow:

$$\begin{aligned} x &= \int_0^t v_x\,dt = \frac{v_0^2}{g}\int_0^\theta \frac{[f(\theta)]^2\,d\theta}{\cos^2\theta} \\ y &= \int_0^t v_y\,dt = \frac{v_0^2}{g}\int_0^\theta \frac{[f(\theta)]^2 \sin\theta\,d\theta}{\cos^3\theta} \end{aligned} \tag{8.13}$$

For constant k, *i.e.*, for the case of the quadratic resistance law, the ballistic problem is completely solved by Eqs. (8.12) and (8.13). The corresponding values of t, x, and y can be plotted numerically from these equations. The function $f(\theta)$ contains one dimensionless parameter, *viz.*, $kv_0^2/mg = \lambda$. The parameter λ has a simple physical meaning. For the asymptotic value v_f of the velocity of a falling body, known as the final or *terminal velocity*, the air resistance must be equal to the weight. From $kv_f^2 = mg$ follows $v_f^2 = mg/k$, and therefore $\lambda = (v_0/v_f)^2$. Hence, the parameter λ is equal to the ratio between the squares of the initial and the terminal velocities. In most cases λ is small, and thus $f(\theta)$ can be developed in a series of increasing powers of the parameter λ.

In general, especially in the case of shells with high initial velocity, k is variable. If the variation of the density of the air with altitude can be neglected, k may be considered as a given function of the velocity v. Then from (8.3) follows:

$$\frac{dv}{d\theta} = v \tan \theta - \frac{k(v)}{mg} \frac{v^3}{\cos \theta} \tag{8.14}$$

Equation (8.14) may be considered again as the differential equation of the hodograph. However, in this general case the variables cannot be separated and the integration has to be carried out by some numerical step-by-step method or by development in series. A further complication results from the fact that the air resistance depends also on the density of the air, and often the density varies over a wide range along the trajectory. Hence, k is a function of the velocity v and also of the altitude, *i.e.*, of the coordinate y. In this case the system of equations (8.1) cannot be reduced to an equation of first order and must be integrated directly by approximate methods. As a matter of fact, the trajectories of shells are at present usually computed by means of the differential analyzer.

If we substitute $\lambda = 0$ in Eq. (8.9), we obtain $f(\theta) = 1$, and from Eqs. (8.13)

$$\begin{aligned} x &= \frac{v_0^2}{g} \int_0^\theta \frac{d\theta}{\cos^2 \theta} = \frac{v_0^2}{g} \tan \theta \\ y &= \frac{v_0^2}{g} \int_0^\theta \frac{\sin \theta \, d\theta}{\cos^3 \theta} = \frac{v_0^2}{2g} \tan^2 \theta \end{aligned} \tag{8.15}$$

These expressions give the flight path when air resistance is neglected. Hence, the difference between the respective values given by Eqs. (8.13) and Eqs. (8.15) may be considered as the correction due to air resistance.

9. Equations of Motion of an Airplane.—Let us consider an airplane as a rigid body, symmetric with respect to a vertical plane, and assume that its motion consists of a displacement of its center of gravity in this plane and a rotation around a horizontal axis perpendicular to the plane of symmetry. Then the equations of motion are given by the two statements: (1) the products of the mass and the components of the acceleration of the center of gravity are equal to the respective components of the resultant of the external forces and (2) the rate of change of the angular momentum is equal to the moment of the forces around the center of gravity (see Chapter III, section 6).

If we denote the velocity of the center of gravity by v, the slope of its path by θ (measured as positive downward) (Fig. 9.1),

then the two components of the acceleration—parallel and normal to the path—are equal to $\frac{dv}{dt}$ and $v\frac{d\theta}{dt}$. The latter expression is equivalent to the centripetal acceleration v^2/R, where R is the radius of curvature of the path defined by $1/R = d\theta/ds$ and s is the length of the arc measured along the path. Instead of $v\frac{d\theta}{dt}$, we can also write $v^2\frac{d\theta}{ds}$.

Denote the components of the resultant air force in the direction of the motion and normal to it by $-D$ and L, respectively, and their moment with respect to the center of gravity by M.

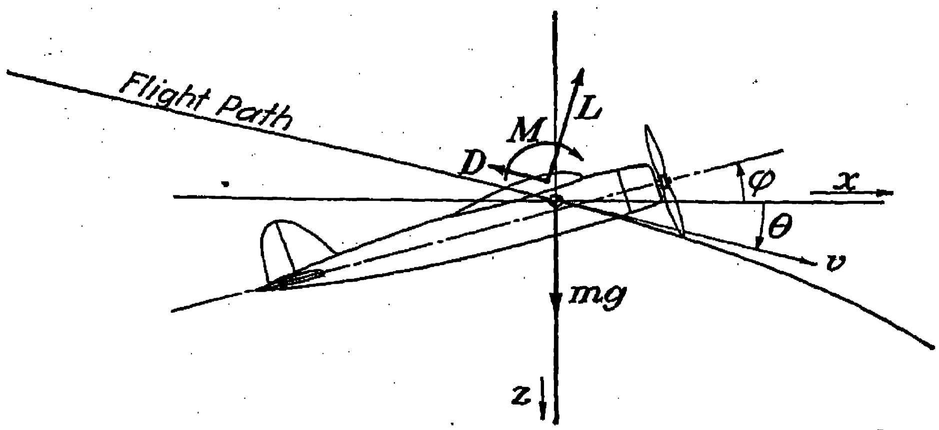

FIG. 9.1.—Diagram of the forces acting on an airplane.

We assume that the air forces include both the air forces proper acting on wings, fuselage, control surfaces, etc., and the forces produced by the propulsion. Thus D is the total drag minus the component of the propeller thrust in the direction of flight, and, correspondingly, any component of the propeller thrust normal to the flight direction is included in the lift L. The mass of the airplane is denoted by m, its moment of inertia with respect to an axis normal to the plane of symmetry intersecting this plane at the center of gravity by I. Finally, the inclination between the horizontal direction and an arbitrary axis of reference connected with the airplane, *e.g.*, the axis of the propeller shaft, is denoted by φ measured positive upward. Then the equations of motion are

$$m\frac{dv}{dt} = -D + mg\sin\theta \tag{9.1}$$

$$mv^2\frac{d\theta}{ds} = -L + mg\cos\theta \tag{9.2}$$

$$I\frac{d^2\varphi}{dt^2} = -M \tag{9.3}$$

The forces D and L and the moment M depend, in general, on the velocity v, the angle of attack of the plane as a whole, and on the local angle of attack of the control surfaces, flaps, etc.

10. The Flight Path of an Airplane with High Stability and Small Moment of Inertia (Phugoid Motion).—In this section the following idealized case will be treated:

a. It will be assumed that $D = 0$, *i.e.*, the engine is controlled in such a way that at any moment the propeller thrust is equal to the drag.

b. It will be assumed that the stabilizing moment M is so large and the moment of inertia of the airplane I is so small that any variation in the angle of attack is instantly corrected, and thus, the angle of attack remains practically constant during flight. The resulting motion is known as *phugoid motion.** Owing to assumption a, v is completely determined by Eq. (9.1). In fact, multiplying both sides by v, we obtain

$$\frac{d}{dt}\left(\frac{v^2}{2}\right) = gv \sin \theta$$

or, taking into account that $v \sin \theta$ is the vertical component of the velocity and is equal to dz/dt, where z is the vertical coordinate of the center of gravity measured positive downward,

$$v^2 = 2gz + \text{const.} \tag{10.1}$$

Eq. (10.1) states that the change of kinetic energy is equal to the work done by gravity; this is true in our pecial case since $D = 0$. Choosing as origin for z the height at which the velocity v vanishes, we obtain the simple relation

$$v = \sqrt{2gz} \tag{10.2}$$

Since a constant angle of attack is assumed, the lift L depends only on the velocity and can be put equal to Cv^2 where C is a constant. The constant C can be expressed conveniently by taking into account that Eqs. (9.1) and (9.2) include the case of uniform horizontal flight ($\theta = 0$). In this case, from Eq. (9.2), $L = mg$, and denoting the velocity of uniform level flight by v_0, obviously $mg = Cv_0^2$ and the general expression for L will be

$$L = mg\left(\frac{v}{v_0}\right)^2 \tag{10.3}$$

* See F. W. Lanchester, "Aerodonetics (Ref. 5)."

Introducing Eq. (10.3) into Eq. (9.2), we obtain

$$mv^2 \frac{d\theta}{ds} = -mg\frac{v^2}{v_0^2} + mg \cos \theta \tag{10.4}$$

Using (10.2) and taking into account that $\frac{d\theta}{ds} = \frac{d\theta}{dz}\frac{dz}{ds} = \frac{d\theta}{dz} \sin \theta$, the following differential equation results, containing as variables the height z and the slope of the flight path θ:

$$2z \sin \theta \frac{d\theta}{dz} = -\frac{2gz}{v_0^2} + \cos \theta \tag{10.5}$$

It is convenient to write $v_0^2 = 2gH$, where obviously H is equal to the *velocity height* corresponding to the velocity v_0 of the uniform horizontal flight. Furthermore, in order to use only dimensionless variables, we write $\zeta = z/H$. Then Eq. (10.5) becomes

$$2\zeta \sin \theta \frac{d\theta}{d\zeta} = -\zeta + \cos \theta \tag{10.6}$$

Let us first consider $\cos \theta$ as function of ζ. Since

$$\frac{d}{d\zeta}(\cos \theta) = -\sin \theta \frac{d\theta}{d\zeta}$$

Eq. (9.9) can be written in the form:

$$\frac{d}{d\zeta}(\cos \theta) + \frac{\cos \theta}{2\zeta} = \frac{1}{2} \tag{10.7}$$

It is easily seen that $\sqrt{\zeta}$ is an integrating factor (see Chapter I, section 4) of the left side. Multiplying both sides with this factor, we obtain

$$\frac{d}{d\zeta}(\sqrt{\zeta} \cos \theta) = \frac{\sqrt{\zeta}}{2} \tag{10.8}$$

Integrating (10.8) we have

$$\cos \theta = \frac{\zeta}{3} + \frac{A}{\sqrt{\zeta}} \tag{10.9}$$

Equation (10.9) determines a family of curves which includes all possible flight paths compatible with the condition of the same constant value of the total energy. It obviously includes the horizontal flight for which $\cos \theta = 1$ and, according to Eq. (10.7),

$\zeta = 1$. The corresponding value of A is, according to Eq. (10.9), $A = \frac{2}{3}$. To obtain the whole family of flight paths, we plot $\cos\theta$ as function of ζ. This is done in Fig. 10.1. We note that only the strip $\zeta > 0$ and $-1 < \cos\theta < 1$ has physical significance.

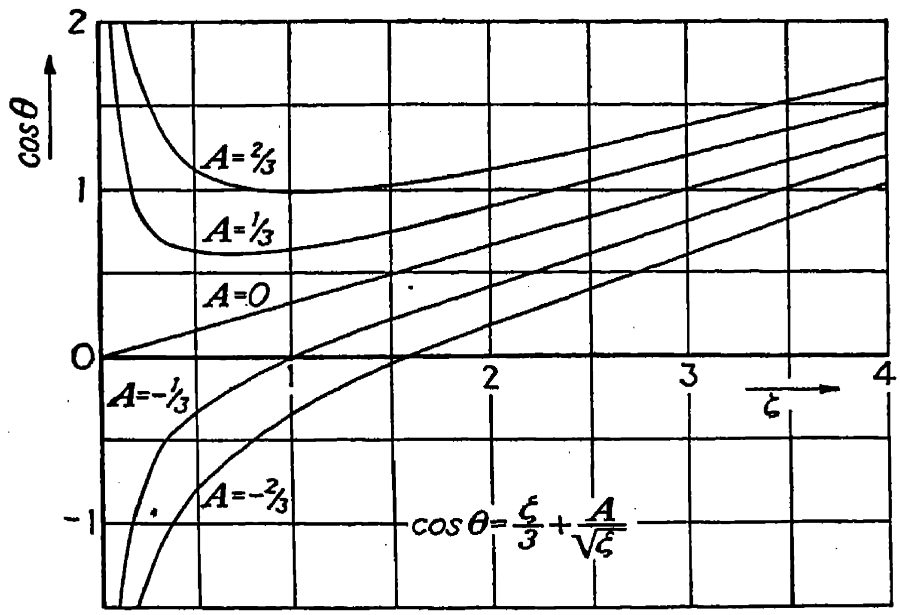

FIG. 10.1.—Phugoid motion. The cosine of the angle of inclination θ between the flight path and the horizontal as function of the vertical displacement ζ.

To calculate the flight paths, we introduce the horizontal coordinate $\xi = x/H$. Then we have $\tan\theta = dz/dx = d\zeta/d\xi$. This is a first-order differential equation for the flight path, since $\tan\theta$ is a given function of ζ through Eq. (10.9). This differential

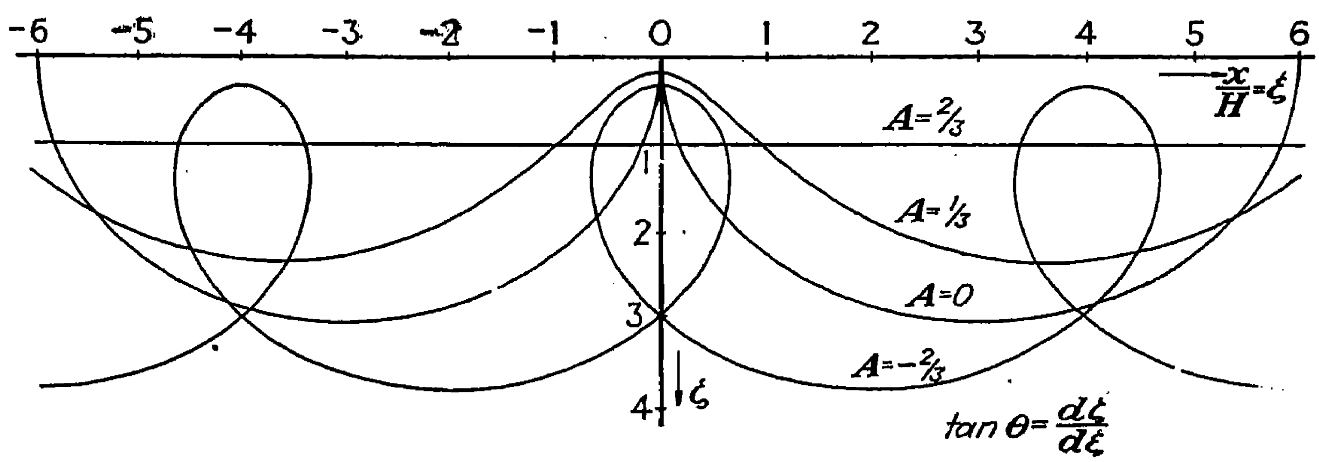

FIG. 10.2.—Different types of phugoid motion.

equation may be integrated graphically or by any of the numerical methods indicated in Chapter I. The flight paths obtained by integration are plotted in a dimensionless coordinate system ξ,ζ in Fig. 10.2. The integral curves of (10.9) show that the case

$A = 0$ is a limiting case between two different types of motion. It is seen from Fig. 10.1 that for $A > 0$, $\cos\theta$ is always positive and has a minimum value, say a; then θ will oscillate between $\pm\cos^{-1} a$. The corresponding flight path is a wavelike line without loops (Fig. 10.2). For $A < 0$, all values of $\cos\theta$ between $+1$ and -1 occur during the motion, and thus in this case the flight path has the shape of a "loop." The case $A = 0$ represents the limit between wavelike lines and loops. For $A = 0$,

$$\zeta = 3\cos\theta \tag{10.10}$$

or

$$z = 3H\cos\theta \tag{10.11}$$

Equation (10.11) is the equation of a half circle with the radius $3H$. The path curve degenerates in this case into a series of half circles; at the peak the airplane has zero velocity of flight and makes a 180° turn with infinite angular velocity.*

The point $\zeta = 0$, $\cos\theta = 0$ is a *singular point of the differential equation* (10.7). Writing (10.7) in the form:

$$\frac{d(\cos\theta)}{d\zeta} = \frac{\zeta - \cos\theta}{2\zeta} \tag{10.12}$$

it is seen that for $\zeta = 0$ and $\cos\theta = 0$ the value of $d(\cos 6)/d\zeta$ is not determined. This is seen geometrically from Fig. 10.1. Apparently $d(\cos\theta)/d\zeta$ is equal to the slope of the curves plotted in the figure, and this takes all possible values between $-\infty$ and ∞ in the immediate neighborhood of the point $\zeta = 0$, $\cos\theta = 0$. Equation (10.9) reveals that ∞, $\frac{1}{3}$, $-\infty$ are the values of the slope at the singular point itself. The same result can be obtained from the differential equation itself by putting $\cos\theta = \lambda\zeta$ + higher order terms in ζ and expanding the right side of Eq. (10.12). We obtain

$$\lambda = \frac{1 - \lambda}{2}$$

or $3\lambda = 1$. This gives us a straight line with the slope $\frac{1}{3}$. Furthermore, by changing the independent variable in Eq. (10.12), *i.e.*, considering ζ as a function of $\cos\theta$, we have

$$\frac{d\zeta}{d(\cos\theta)} = \frac{2\zeta}{\zeta - \cos\theta} \tag{10.13}$$

* This physically impossible condition is introduced into the problem by the idealized assumption of zero moment of inertia.

It is seen then that the straight line $\zeta = 0$ is also a solution of Eq. (10.13). The corresponding values of the slope $\dfrac{d(\cos\theta)}{d\zeta}$ are $+\infty$ and $-\infty$.

The differential equation (10.6) has a second singular point. In fact, Eq. (10.6) may be written in the form:

$$\frac{d\theta}{d\zeta} = \frac{\cos\theta - \zeta}{2\zeta\sin\theta} \tag{10.14}$$

Then it is apparent that $d\theta/d\zeta$ is undetermined, not only for $\zeta = 0$, $\cos\theta = 0$, the case just discussed, but also for $\zeta = 1$, $\theta = 0$. In order to study the solution in the neighborhood of this point, we expand $\sin\theta$ and $\cos\theta$ in power series and write $\zeta = 1 + \zeta'$. Then, neglecting terms of higher order than the first, we have

$$\frac{d\theta}{d\zeta} = -\frac{\zeta'}{2\theta} \tag{10.15}$$

Taking into account that $d\theta/d\zeta = d\theta/d\zeta'$, we have

$$2\theta\,d\theta + \zeta'\,d\zeta' = 0 \tag{10.16}$$

Integrating (10.16), we obtain

$$\theta^2 + \frac{\zeta'^2}{2} = \text{const.} \tag{10.17}$$

Equation (10.17) yields curves representing flight paths in the neighborhood of uniform level flight. The family of curves in the $\zeta\theta$ plane represented by Eq. (10.17) consists of ellipses having the same ratio between major and minor axis. The complete family of integral curves of Eq. (10.14) is represented in Fig. 10.3. Points B and C are the singular points. The center of the ellipses given by (10.17) is the point B. In section 11 it will be seen that B is a so-called *vortex point* and C a so-called *saddle point.*

For small values of θ, $\theta = dz/dx$ where x is the horizontal coordinate of the $C.G.$ Hence putting $\zeta' = \dfrac{z - H}{H}$, we obtain from (10.7) the following approximate differential equation for z:

$$\left(\frac{dz}{dx}\right)^2 + \frac{(z - H)^2}{2H^2} = \text{const.} \tag{10.18}$$

The general solution of this equation is

$$z = H + A \cos \frac{x}{H\sqrt{2}} + B \sin \frac{x}{H\sqrt{2}} \tag{10.19}$$

where A and B are two constants of integration. Equation (10.19) represents a sinusoidal flight path with the wave length $2\pi H\sqrt{2}$. The motion may be considered as a vertical oscillation superposed upon a horizontal uniform motion. The period of oscillation is approximately the wave length divided by the mean velocity v_0, *i.e.*, $T = 2\pi\sqrt{2}\,\frac{H}{v_0}$ or with $v_0 = \sqrt{2gH}$, $T = 2\pi\sqrt{H/g}$. By comparison with the formula $T = 2\pi\sqrt{l/g}$, which gives the

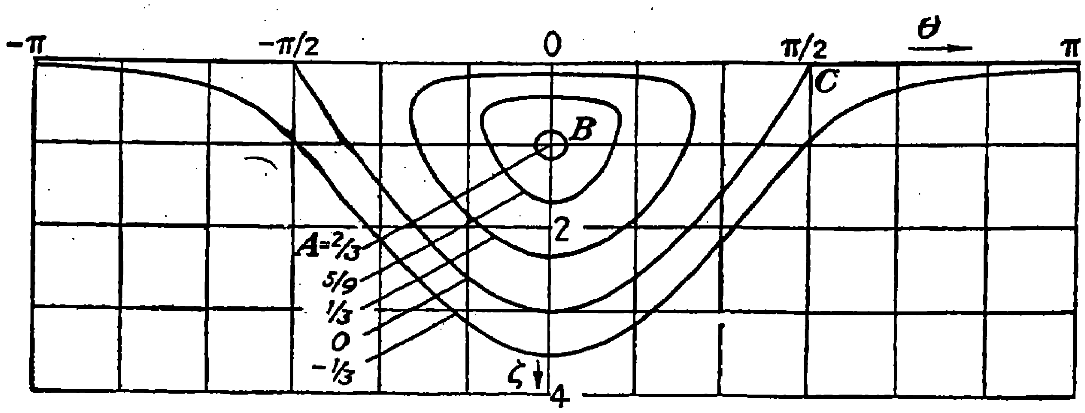

FIG. 10.3.—Phugoid motion. Plot of the inclination of the flight path vs. the vertical displacement ζ.

period of oscillation for a mathematical pendulum of the length l, it is seen that the frequency of the oscillatory motion of the airplane is equal to the frequency of a mathematical pendulum of the length $v_0^2/2g = H$.

Note that from Eq. (10.18) we obtain by differentiation

$$\frac{d^2z}{dx^2} + \frac{(z - H)}{2H^2} = 0 \tag{10.20}$$

Equation (10.20) is a linear differential equation of the second order of the type encountered in undamped oscillations. Equation (10.18) is a first integral of (10.20).

11. Singular Points of Differential Equations of the First Order.—We encountered in the last section two different types of singular points of first-order differential equations. In this section we give a systematic classification of such points.

Consider a differential equation of the form:

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} \tag{11.1}$$

and assume that for $x = y = 0$, $f(x,y) = g(x,y) = 0$, *i.e.*, the point $x = 0$, $y = 0$ is a singular point of the differential equation (11.1) in the sense explained in Chapter I, section 3. The assumption that the singular point is located at the point $x = y = 0$ does not restrict the general character of the following discussion because, if the coordinates of the singular point are $x = x_0$ and $y = y_0$, it can be transferred to the origin of the coordinate system by the transformation $\xi = x - x_0$, $\eta = y - y_0$. Then in the $\xi\eta$ plane the coordinates of the singular point will be $\xi = \eta = 0$.

Let us now assume that both $f(x,y)$ and $g(x,y)$ can be expanded in the neighborhood of the origin $x = y = 0$ in power series starting with the first powers of x and y. Then the differential equation (11.1) can be approximated by an equation of the form:

$$\frac{dy}{dx} = \frac{ax + by}{cx + dy} \tag{11.2}$$

We suppose that at least one of the coefficients is different from zero both in the numerator and in the denominator and investigate how the types of solution depend on the coefficients a, b, c, and d. Speaking more precisely, we shall find the different possible patterns of integral curves in the neighborhood of the singular point. Let us write Eq. (11.2) in the form:

$$\frac{dy}{dx} = \frac{a + b\dfrac{y}{x}}{c + d\dfrac{y}{x}} \tag{11.3}$$

The geometrical meaning of $dy/dx = \tan \alpha$ is the slope of the integral curve at a certain point x,y. The geometrical meaning of $\tan \beta = y/x$ is the slope of the radius vector connecting the origin with the point in question. Hence, Eq. (11.3) determines the relation between the slope of the integral curve and the slope of the radius vector. According to (11.3), the slope of the integral curves is the same at all points which lie on the same straight line $y = x \tan \beta$ and is given by

$$\tan \alpha = \frac{a + b \tan \beta}{c + d \tan \beta} \tag{11.4}$$

Before we discuss Eq. (11.2) or (11.4) in general, let us consider a few elementary cases:

a. First assume that $\alpha = \beta$, *i.e.*, $\tan \alpha = \tan \beta$. Then Eq. (11.2) reads

$$\frac{dy}{dx} = \frac{y}{x} \tag{11.5}$$

The general solution of (11.15) is

$$y = Cx \tag{11.6}$$

where C is an arbitrary constant.

Equation (11.6) represents a family of straight lines passing through the origin. Hence, in this case an infinite number of integral curves go through the singular point. A singular point with an infinite number of integral curves passing through it is called a *nodal point* (Fig. 11.1*a*).

If we generalize Eq. (11.5) by multiplying the right side by a positive numerical factor λ,

$$\frac{dy}{dx} = \lambda\frac{y}{x} \tag{11.7}$$

the equation of the integral curves will be

$$y = Cx^\lambda \tag{11.8}$$

In this case again an infinite number of integral curves pass through $x = y = 0$. The axes $x = 0$ and $y = 0$ always represent two integral curves; the remaining integral curves have the x-axis as their common tangent at $x = y = 0$, when $\lambda > 1$, and the y-axis, when $\lambda < 1$. For $\lambda = 1$ we obtain the solution (11.6).

b. Let us now assume that $\alpha = \frac{\pi}{2} + \beta$; then the differential equation becomes

$$\tan \alpha = -\frac{1}{\tan \beta}$$

or

$$\frac{dy}{dx} = -\frac{x}{y} \tag{11.9}$$

By separating the variables, we obtain

$$y\,dy = -x\,dx$$

and

$$x^2 + y^2 = C \tag{11.10}$$

where C is again an arbitrary constant. The integral curves are circles with the origin $x = y = 0$ as center. A singular point

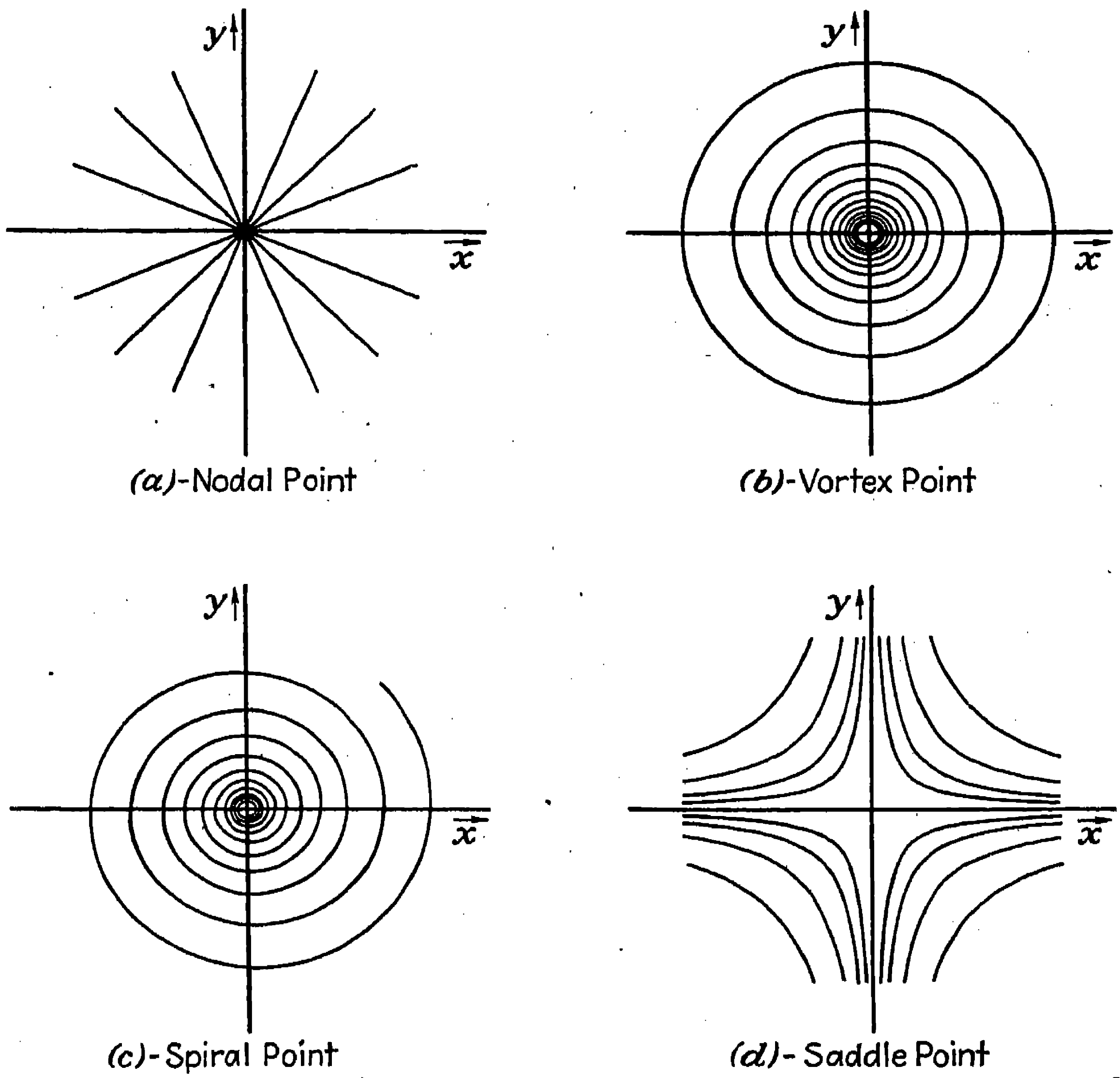

FIG. 11.1.—Various types of singular points of differential equations of first order.

surrounded by closed integral curves such that none of them goes through the point itself is called a *vortex point* (Fig. 11.1*b*).

If we write instead of (11.9)

$$\frac{dy}{dx} = -\lambda\frac{x}{y} \tag{11.11}$$

where λ is a positive numerical factor, the type of the singular

point remains the same; the integral curves are ellipses instead of circles.

c. Our next assumption is $\alpha = \gamma + \beta$, where γ is a given constant and $\gamma \neq 0$ and $\gamma \neq \pi/2$. Then:

$$\tan \alpha = \frac{\tan \gamma + \tan \beta}{1 - \tan \gamma \tan \beta} \tag{11.12}$$

or

$$\frac{dy}{dx} = \frac{x \tan \gamma + y}{x - y \tan \gamma} \tag{11.13}$$

The solutions are logarithmic spirals. This can be seen by using polar coordinates. We write $x = r \cos \beta$, $y = r \sin \beta$. Since $\alpha = \gamma + \beta$, an integral curve intersects all radius vectors by the same angle γ. The tangent of this angle is equal to $r\,d\beta/dr$. Hence,

$$\cot \gamma = \frac{dr}{r\,d\beta}$$

By integration we have

$$\log r = \beta \cot \gamma + \text{const.}$$

or

$$r = Ce^{\beta \cot \gamma} \tag{11.14}$$

The singular point in this case is called a *spiral point* (Fig. 11.1*c*).

d. Finally, assume that $\alpha + \beta = \pi$. Then $\tan \alpha = -\tan \beta$, and the differential equation has the form:

$$\frac{dy}{dx} = -\frac{y}{x} \tag{11.15}$$

The general solution of Eq. (11.15) is

$$xy = C \tag{11.16}$$

The integral curves are the x- and y-axes and hyperbolas having these axes as asymptotes. If a finite number of integral curves go through the singular point and the remaining integral curves pass by, we call the singular point a *saddle point* (Fig. 11.1*d*).

If we multiply the right side of (11.15) by a positive numerical factor λ, the solution will be

$$x^\lambda y = C \tag{11.17}$$

The type of the singular point remains the same.

Summarizing the examples presented under *a*, *b*, *c*, *d*, we find the following types of singular points (Fig. 11.1):

a. Nodal point: infinite number of integral curves passing through the singular point.

b. Vortex point: no integral curve through the singular point; closed integral curves around the singular point.

c. Spiral point: integral curves approaching the singular point asymptotically in spirals.

d. Saddle point: finite number of integral curves through the singular point, the other integral curves passing by.

It will be shown that this classification includes all possible cases for the differential equation (11.2). Consider the general case:

$$\frac{dy}{dx} = \frac{a + b\dfrac{y}{x}}{c + d\dfrac{y}{x}} \tag{11.18}$$

First we investigate the integral curves consisting of straight lines through the origin. We must have in this case $dy/dx = y/x = \tan \beta$. Hence, Eq. (11.18) yields the condition

$$\tan \beta = \frac{a + b \tan \beta}{c + d \tan \beta}$$

or

$$\tan^2 \beta + \frac{c - b}{d} \tan \beta - \frac{a}{d} = 0 \tag{11.19}$$

Equation (11.19) is a quadratic equation for $\tan \beta$. We have to distinguish between the following cases:

a. Every value of $\tan \beta$ is a solution of Eq. (11.19). This will be the case when $a = 0$, $d = 0$, and $c - b = 0$. Then obviously, $dy/dx = y/x$, and we obtain a *nodal point* with an infinite number of straight integral curves.

b. Equation (11.19) has two real roots. Denote the two real roots by $\tan \beta_1$ and $\tan \beta_2$. Then the straight lines $y = x \tan \beta_1$ and $y = x \tan \beta_2$ are integral curves. Now we can easily verify that substituting for the variables x,y in Eq. (11.18) new variables ξ,η by the linear substitution

$$\begin{aligned} x &= p\xi + q\eta \\ y &= r\xi + s\eta \end{aligned} \tag{11.20}$$

the type of differential equation (11.18) remains unchanged, *i.e.*, we obtain

$$\frac{d\eta}{d\xi} = \frac{a' + b'\frac{\eta}{\xi}}{c' + d'\frac{\eta}{\xi}} \tag{11.21}$$

The substitution (11.20) is a so-called *affine transformation* which changes the coordinate system x,y into a coordinate system ξ,η which, in general, is nonorthogonal. Let us choose (11.20) such that the ξ- and η-axes coincide with the integral curves $y = x \tan \beta_1$ and $y = x \tan \beta_2$. We obtain this result by writing

$$\begin{aligned} x &= \xi \cos \beta_1 + \eta \cos \beta_2 \\ y &= \xi \sin \beta_1 + \eta \sin \beta_2 \end{aligned} \tag{11.22}$$

Introducing (11.22) in (11.18), we obtain a differential equation of the form (11.21), but the values a', b', c', and d' must satisfy particular conditions, *viz.*, $\xi = 0$ and $\eta = 0$ must be solutions of the equation. Hence, $a' = 0$ and $d' = 0$, and Eq. (11.21) becomes

$$\frac{d\eta}{d\xi} = \frac{b'}{c'}\frac{\eta}{\xi} \tag{11.23}$$

The solution of Eq. (11.23) has the form:

$$\eta = K\xi^{\frac{b'}{c'}} \tag{11.24}$$

where K is an arbitrary constant.

If b'/c' is positive, we obtain an infinite number of integral curves through $\xi = \eta = 0$. They touch the ξ-axis at that point when $b'/c' > 1$ and the η-axis when $b'/c' < 1$. Hence, when $b'/c' > 0$, the singular point is a *nodal point.* When $b'/c' < 0$, no curves determined by (11.24) can go through $\xi = \eta = 0$ with the exception of the straight lines $\xi = 0$ and $\eta = 0$. Hence, for $b'/c' < 0$ we obtain a *saddle point.*

The difference between the general case and the examples under b or d (pages 152 and 154) is the different angle of intersection of the straight lines which represent integrals. They are normal to each other in the simple examples treated before and inclined at an arbitrary angle in the general case.

c. Equation (11.19) has two equal real roots. This real root may be denoted by $\tan \beta_1$. In this case we use the transformation (11.22) so that $\eta = 0$ represents $y/x = \tan \beta$, the value for β_2 is left arbitrary. Then the Eq. (11.21) obtained by this substitution must be such that a solution of the type $\eta/\xi = \lambda$ is only possible if $\lambda = 0$. Substituting $\eta = \xi\lambda$ in Eq. (11.21), we find $\lambda^2 + \dfrac{c' - b'}{d'}\lambda - \dfrac{a'}{d'} = 0$. This equation must reduce to $\lambda^2 = 0$ and, therefore, $a' = 0$ and $c' = b'$. Hence, Eq. (11.21) must have the form:

$$\frac{d\eta}{d\xi} = \frac{b'\frac{\eta}{\xi}}{b' + d'\frac{\eta}{\xi}} \tag{11.25}$$

or

$$\frac{d\xi}{d\eta} = \frac{d'}{b'} + \frac{\xi}{\eta} \tag{11.26}$$

If we introduce $\xi/\eta = \zeta$ as a new variable, then, substituting $\xi = \zeta\eta$ into (11.26), we obtain

$$\eta\frac{d\zeta}{d\eta} + \zeta = \frac{d'}{b'} + \zeta$$

or by integration

$$\zeta = \frac{d'}{b'}\log\eta + C\cdot$$

and finally,

$$\xi = \eta\frac{d'}{b'}\log\eta + C\eta \tag{11.27}$$

The solution (11.27) gives an infinite number of integral curves through $\xi = \eta = 0$.* The ξ-axis ($\eta = 0$) is their common tangent at this point. The ξ-axis itself is an integral corresponding to $C = \infty$. This singular point is also a *nodal point*.

d. Equation (11.19) has two conjugate complex roots. Proceeding as we did under (*b*), the differential equation can be reduced by the substitution of (11.20) to the form:

$$\frac{d\eta}{d\xi} = \frac{a' + \frac{\eta}{\xi}}{1 - a'\frac{\eta}{\xi}} \tag{11.28}$$

If we put $a' = \tan\gamma$, Eq. (11.28) becomes identical with Eq. (11.13) and leads to a *spiral point*, with the exception of $a' = 0$ and $a' = \infty$. In the case $a' = 0$, we obtain a *nodal point* with an infinite number of straight integral curves, whereas with $a' \to \infty$, the equation becomes

$$\frac{d\eta}{d\xi} = -\frac{\xi}{\eta}$$

and we obtain a *vortex point* according to Eq. (11.9).

Summarizing the result of this general discussion, we find that our classification yields four types of singular points, and it is complete as far as the differential equation (11.2) is concerned. However, if the expansions of $f(x,y)$ and $g(x,y)$ do not start with the first power of at least one of the

* Note that $\eta\log\eta \to 0$ for $\eta \to 0$.

variables, further investigation of the character of the singularity is necessary.

Problems

1. Calculate the deceleration of a boat if the propulsion is stopped at the instant when the velocity is v_0. The fluid resistance is given by the empirical law

$$f(v) = \alpha v + \beta v^2$$

where α and β are constants. The mass of the boat is M. Find the expression for the total distance that the boat will travel from the instant that the propulsion is stopped.

2. In a heavy circular disk of 20-in. diameter and uniform thickness, there is a circular hole of 4-in. diameter. The center of the hole is at a distance of 6 in. from the center of the disk. The disk is mounted so that it can rotate freely and without friction about a horizontal axis which passes through its center. Find the period of small oscillations.

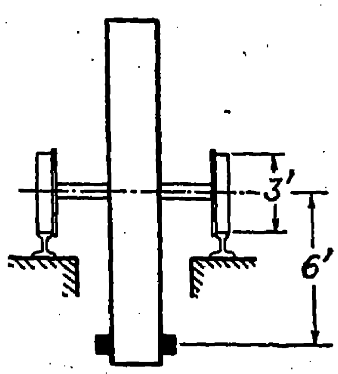

FIG. P.3.

3. The moment of inertia of a heavy spoked flywheel is to be determined experimentally by the setup shown in Fig. P.3. The flywheel is mounted on a two-wheel truck which rolls on a frictionless track. The mass of the truck can be neglected. The flywheel has a weight of 10 tons; it is balanced, but it carries a 100-lb. weight whose center of gravity is at a distance of 6 ft. from the center of the flywheel. The period of small oscillations was measured and found to be equal to 36 sec. Find the moment of inertia of the flywheel.

4. Find the weight per running foot of a thin-walled steel cylinder of elliptic cross section. The major and minor axes are 3 ft. and 2 ft., respectively, and the thickness of the wall is $\frac{1}{4}$ in.

5. Find the period of a compound pendulum if it is released when the center of gravity lies on a level with the point of suspension. Compare the exact value with that obtained by the approximate formula (3.10).

6. Describe the motion of a mass constrained to a circular frictionless path in a vertical plane and moving permanently in the same direction in such a way that the velocity at the highest point is half the velocity at the lowest point.

7. Reduce the integral

$$\int_0^{\psi} \frac{d\psi}{\sqrt{1 + c^2 \sin^2 \psi}}$$

to the standard form.

8. Calculate the definite integral

$$\int_2^3 \frac{dx}{\sqrt{(x-1)(x-2)(x-3)(x-4)}}$$

Hint: Use the transformation $x = z + 2.5$.

9. Calculate the definite integral

$$\int_1^2 \frac{dx}{\sqrt{(x-1)(x-2)(x-3)}}$$

Hint: Use the transformation $x = \dfrac{z}{1+z}$ with the limits of integration $z = -2$ and $z = -\infty$.

10. Calculate the definite integral

$$\int_0^{\pi/2} \frac{d\alpha}{\sqrt{2 - \sin \alpha}}$$

11. Find the period of small oscillations of a so-called *pendulum counterweight* which is used for the damping of torsional vibrations of the crankshaft in airplane engines (Fig. P.11). The counterweight can freely move in a segment of a circular groove of radius r cut in the counterbalance of the crankshaft; the center of curvature of the segment is displaced a distance e from the axis of rotation. The angular velocity of the crankshaft is equal to ω. Determine e so that the period of the small oscillations is equal to the period of revolution of the crankshaft.

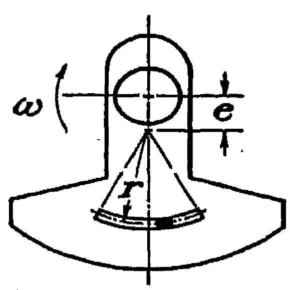

FIG. P.11.

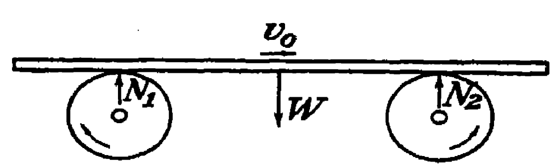

FIG. P.12.

12. Discuss the motion of a heavy bar of weight W which lies horizontally on two pulleys of equal radii rotating with the same angular velocity in opposite senses (Fig. P.12). The coefficient of friction between the pulleys and the rod is equal to f, so that the friction is equal to fN_1 and fN_2, respectively, where N_1 and N_2 are the normal loads on the pulleys. Assume that at $t = 0$ the center of gravity of the bar is at the mid-point of the two pulleys and the bar is set in motion with the initial velocity v_0. Show that the motion is periodic and that the frequency is proportional to $\sqrt{f}$.

13. A periodic force $F_0 \sin \omega t$ acts on a mass with elastic restraint and damping. Does an increase in damping produce an increase in energy dissipated?

Hint: The average energy dissipation per unit time is given by $\frac{1}{2}\beta v^2$ where β is the damping factor and v the velocity amplitude.

14. A periodic force $F_0 \sin \omega t$ starts acting on a mass with elastic restraint and damping at $t = 0$. The mass is initially at rest. Find the motion of the mass. Examine the case of resonance and what happens in this case when the damping tends to vanish.

15. Find the motion of a body falling vertically under the action of gravity and take into account the air resistance, given by the quadratic law $D = -kv^2$.

16. A bomb is released with a horizontal velocity of 250 miles/hr. at an altitude of 3,000 ft. What is the horizontal distance covered by the bomb when it reaches the ground, neglecting air resistance?

17. Calculate in Prob. 16 the correction of the horizontal distance due to air resistance. The weight of the bomb is 300 lb. The drag is given by the formula $D_{\text{lb.}} = C_D \rho d^2 v^2$ where C_D is the drag coefficient, ρ the average density of the air, d the diameter of the bomb, v the velocity. Take $C_D = 0.087$, $d = 1$ ft., $\rho = 0.00227$ slugs/cu. ft. and v in feet per second.

Hint: Calculate the parameter $\lambda = (v_0/v_f)^2$. Since λ is small, write for $[f(\theta)]^2$

$$[f(\theta)]^2 = 1 - \lambda\left(\log \frac{1 + \sin \theta}{\cos \theta} + \frac{\sin \theta}{\cos^2 \theta}\right)$$

and substitute this expression in Eq. (8.13).

Plot y as function of θ according to Eq. (8.13) and determine the value of θ corresponding to $y = 3{,}000$ ft. Then calculate the correction for the horizontal distance x using Eqs. (8.13) and (8.15).

18. Imagine a straight tunnel connecting Los Angeles with San Francisco. The length of the tunnel is 400 miles. Calculate the time in which a train would cover the distance under gravity alone neglecting friction and air resistance. Calculate the maximum speed which could be reached. The radius of the earth is slightly less than 4,000 miles.

19. Discuss the behavior of the integral curves of the differential equation

$$\frac{dy}{dx} = \frac{6y^2 - \sin x}{2x + y}$$

in the vicinity of the origin $x = 0$, $y = 0$.

20. Discuss the behavior of the integral curves of the differential equation

$$\frac{dy}{dx} = \frac{4 + y - 5x + x^2}{2 - 3y + y^2}$$

in the vicinity of the point $x = 3$, $y = 2$.

21. *A Problem in Hydraulics.*—The water level of a broad river is determined by the energy equation

$$\frac{d}{dx}\left(h + \frac{v^2}{2g}\right) - \alpha + c_f \frac{v^2}{2gh} = 0$$

and by the continuity equation in steady flow

$$hv = Q$$

In these equations h is the height of the water level over the bed, v the mean velocity through a cross section, α the slope of the bed, c_f the coefficient of

hydraulic friction, and Q the volume of the water flowing through a cross section per unit time. Q is considered as a given constant.

Find the differential equation for h by eliminating v. Discuss the integral curves and determine which curves can have physical significance as lines of free water surface.

Hint: Express the slope dh/dx as a function of h. The isoclines are straight lines parallel to the bed. Notice that the pattern of integral curves is different depending on whether $\alpha < \frac{1}{2}c_f$ or $\alpha > \frac{1}{2}c_f$. A stream of uniform depth in the case $\alpha < \frac{1}{2}c_f$ is called a *river*, and in the case $\alpha > \frac{1}{2}c_f$, it is called a *torrent*. In the first case the integral curves with no vertical tangent represent the free surface of a river flowing into a reservoir. In the case $\alpha > \frac{1}{2}c_f$ a physically possible solution is given by portions of two branches of an integral curve, the transition from one branch to the other occurs physically by a sudden change of height and velocity called a *hydraulic jump*.

References

General:

1. Reference 2, Chapter III.
2. Routh, E. J.: "A Treatise on Dynamics of a Particle," Cambridge University Press, London, 1898.
3. Den Hartog, J. P.: "Mechanical Vibrations," McGraw-Hill Book Company, Inc., New York, 1933.
4. Timoshenko, S.: "Vibration Problems in Engineering," D. Van Nostrand Company, Inc., New York, 1928.

Special problems:

5. Lanchester, F. W.: "Aerodonetics" (for section 10), Constable & Company, Ltd., London, 1908.
6. Hayes, T. J.: "Elements of Ordnance," Chaps. X, XI, XII (for section 8), John Wiley & Sons, Inc., New York, 1938.

Singular points of differential equations:

7. Ince, E. L.: "Ordinary Differential Equations," Longmans, Green & Company, New York, 1927.

Elliptic integrals and functions:

8. Goursat, E.: "A Course in Mathematical Analysis," translated by E. R. Hedrick, Ginn and Company, Boston, 1904.
9. Hancock, H.: "Elliptic Integrals," Mathematical Monographs, John Wiley & Sons, Inc., New York, 1917.
10. King, L. V.: "On the Direct Numerical Calculation of Elliptic Functions and Integrals," Cambridge University Press, London, 1924.
11. Reference 5, to Chapter II.

CHAPTER V

SMALL OSCILLATIONS OF CONSERVATIVE SYSTEMS

> The methods which I present here do not require either constructions or reasonings of geometrical or mechanical nature, but only algebraic operations proceeding after a regular and uniform plan. Those who love the Analysis, will see with pleasure Mechanics made a branch of it and will be grateful to me for having thus extended its domain.
>
> —J. L. LAGRANGE,
> "Mécanique Analytique" (1788).

Introduction.—In this chapter we take up the problem of small oscillations of conservative systems. The theory of small oscillations is, in general, an approximate theory of the motion of mechanical or other physical systems in the neighborhood of an equilibrium position or a state of uniform motion. In this chapter we shall restrict ourselves to oscillations of conservative systems in the neighborhood of a stable equilibrium position. We deal with the free oscillations of such systems and with their forced oscillations produced by periodic forces. Whereas in the last chapter the equations of motion were obtained by direct application of Newton's laws, in this chapter we shall use Lagrange's equations for generalized coordinates and forces.

1. Small Oscillations of Conservative Systems—General Remarks.—The definition of a conservative mechanical system was given in Chapter IV. A system is called conservative if the work done by the forces acting on the system depends only on the end positions or, more exactly, on the end-point coordinates. In this case the forces can be derived from a potential energy U, which is a function of the coordinates $q_1, q_2, \ldots, q_n$. However, U contains an arbitrary additive constant, which can be chosen arbitrarily.

We shall choose the coordinates in such a way that in the equilibrium condition, in the neighborhood of which the oscillation takes place, $q_1 = q_2 = \cdots = q_n = 0$. Furthermore, we shall choose the arbitrary constant in $U(q_1, q_2, \ldots, q_n)$—*i.e.*,

the "level" of the potential energy—so that in the equilibrium position $U = 0$. Then we expand the function U in a Taylor series in the neighborhood of $q_1 = q_2 = \cdots = q_n = 0$. For simplicity's sake we consider the case $n = 3$, *i.e.*, a system with 3 degrees of freedom, but the results can easily be generalized for arbitrary values of n. The Taylor expansion of U is given by

$$U = \frac{\partial U}{\partial q_1}q_1 + \frac{\partial U}{\partial q_2}q_2 + \frac{\partial U}{\partial q_3}q_3 + \frac{1}{2}\left(\frac{\partial^2 U}{\partial q_1^2}q_1^2 + \frac{\partial^2 U}{\partial q_2^2}q_2^2 + \frac{\partial^2 U}{\partial q_3^2}q_3^2 \right.$$
$$\left. + 2\frac{\partial^2 U}{\partial q_1\,\partial q_2}q_1q_2 + 2\frac{\partial^2 U}{\partial q_2\,\partial q_3}q_2q_3 + 2\frac{\partial^2 U}{\partial q_3\,\partial q_1}q_3q_1\right) + \cdots \tag{1.1}$$

where the derivatives are to be taken for $q_1 = q_2 = q_3 = 0$. In the theory of small oscillations we shall neglect all terms that are higher than the second order.

Now the equilibrium condition (cf. Chapter III, section 9) requires that $\delta U = 0$, and therefore the generalized forces $Q_1 = -\partial U/\partial q_1$, $Q_2 = -\partial U/\partial q_2$, $Q_3 = -\partial U/\partial q_3$ vanish for $q_1 = q_2 = q_3 = 0$. Hence, in the expansion of U only the quadratic terms remain, and we have as an approximate expression for U the following quadratic function, or *quadratic form:*

$$U = \frac{1}{2}\left(\frac{\partial^2 U}{\partial q_1^2}q_1^2 + \frac{\partial^2 U}{\partial q_2^2}q_2^2 + \frac{\partial^2 U}{\partial q_3^2}q_3^2 + 2\frac{\partial^2 U}{\partial q_1\,\partial q_2}q_1q_2 + 2\frac{\partial^2 U}{\partial q_2\,\partial q_3}q_2q_3 \right.$$
$$\left. + 2\frac{\partial^2 U}{\partial q_3\,\partial q_1}q_3q_1\right) \tag{1.2}$$

We shall obtain a more symmetric form of expression (1.2) if we remember that

$$\frac{\partial^2 U}{\partial q_i\,\partial q_j} = \frac{\partial^2 U}{\partial q_j\,\partial q_i}$$

and substitute, for example,

$$2\frac{\partial^2 U}{\partial q_1\,\partial q_2}q_1q_2 = \frac{\partial^2 U}{\partial q_1\,\partial q_2}q_1q_2 + \frac{\partial^2 U}{\partial q_2\,\partial q_1}q_2q_1$$

and write all terms containing products of coordinates in a similar way. Then Eq. (1.2) can be put in the following simple form:

$$U = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial^2 U}{\partial q_i\,\partial q_j}q_iq_j \tag{1.3}$$

An analogous form holds for arbitrary n. Since the derivatives are taken at the equilibrium position, they represent constant coefficients, and we write $\partial^2 U/\partial q_i \partial q_j = k_{ij}$. It is evident that $k_{ij} = k_{ji}$. Then the expression (1.3) becomes

$$U = \tfrac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} k_{ij} q_i q_j \tag{1.4}$$

We shall consider in this chapter stable systems only. In this case the potential energy has a minimum in the equilibrium position. Hence, U has a minimum for $q_1 = q_2 = q_3 = 0$. Since U itself is equal to zero at this point, it must be positive everywhere else. We call a quadratic function of n variables, which is never negative for arbitrary values of the variables, a *positive definite quadratic function* or a *positive definite quadratic form.* The potential energy of a stable system is represented by such a positive definite quadratic form of the coordinates.

The forces Q_1, Q_2, and Q_3 acting on the system, if it is in the arbitrary position q_1, q_2, q_3, are obtained from (1.4) by differentiation

$$\begin{aligned} -Q_1 &= \frac{\partial U}{\partial q_1} = k_{11}q_1 + k_{12}q_2 + k_{13}q_3 \\ -Q_2 &= \frac{\partial U}{\partial q_2} = k_{21}q_1 + k_{22}q_2 + k_{23}q_3 \\ -Q_3 &= \frac{\partial U}{\partial q_3} = k_{31}q_1 + k_{32}q_2 + k_{33}q_3 \end{aligned} \tag{1.5}$$

We multiply the Eqs. (1.5) by q_1, q_2, and q_3, respectively, and obtain by addition

$$-Q_1q_1 - Q_2q_2 - Q_3q_3 = \sum_{i=1}^{3} \frac{\partial U}{\partial q_i} q_i = \sum_{i=1}^{3}\sum_{j=1}^{3} k_{ij}q_iq_j = 2U \tag{1.6}$$

or

$$-\frac{1}{2}\sum_{i=1}^{3} Q_iq_i = \frac{1}{2}\sum_{i=1}^{3} \frac{\partial U}{\partial q_i} q_i = U \tag{1.7}$$

This relation has the following physical significance: The Q_i are the forces produced by a deflection q_1, q_2, and q_3; we call them in the case of a stable system the *restoring forces*, and the coefficients k_{ij}, the *spring constants*. Hence, the forces which, when applied

to the system from the outside, would produce the deflections q_1, q_2, and q_3 are equal to $-Q_1$, $-Q_2$, and $-Q_3$. The expression (1.7) states that if the system moves under the action of external forces from the equilibrium position to an arbitrary position q_1, q_2, q_3, its potential energy increases by an amount equal to the work of the forces which produce the displacement. Equation (1.7) holds also for an arbitrary number of degrees of freedom.

Let us consider now the kinetic energy of the system. The kinetic energy is a homogeneous quadratic function of the generalized velocities, where in general the coefficients are functions of the coordinates. Hence, we can write the kinetic energy T in the form:

$$T = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} m_{ij}\dot{q}_i\dot{q}_j \tag{1.8}$$

where $m_{ij} = m_{ji}$.

We shall neglect terms of higher than second order in the q's and $\dot{q}$'s. Then we have to replace the functions m_{ij} by their values for $q_1 = q_2 = q_3 = 0$, *i.e.*, by the first terms of their expansions in the neighborhood of the equilibrium position. All other terms of the expansions contribute to the kinetic energy T terms at least of the third order. We call the constants m_{ij} (0, 0, 0) (denoted by m_{ij}) the *inertia parameters* of the system. The kinetic energy cannot be negative by its definition. Hence, T is a positive definite quadratic form of the velocities.

Since T does not depend on the coordinates, we have $\partial T/\partial q_i = 0$. Therefore, in the case of small oscillations, Lagrange's equations appear in the following simple form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = -\frac{\partial U}{\partial q_i} \tag{1.9}$$

We shall write these equations explicitly for $n = 3$. Substituting (1.5) and using (1.8), we obtain

$$\begin{aligned} m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + m_{13}\ddot{q}_3 &= -(k_{11}q_1 + k_{12}q_2 + k_{13}q_3) \\ m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + m_{23}\ddot{q}_3 &= -(k_{21}q_1 + k_{22}q_2 + k_{23}q_3) \\ m_{31}\ddot{q}_1 + m_{32}\ddot{q}_2 + m_{33}\ddot{q}_3 &= -(k_{31}q_1 + k_{32}q_2 + k_{33}q_3) \end{aligned} \tag{1.10}$$

The terms containing $k_{12} = k_{21}$, $k_{23} = k_{32}$, and $k_{31} = k_{13}$ are called the *static coupling terms*, whereas those containing

$m_{12} = m_{21}$, $m_{23} = m_{32}$, and $m_{31} = m_{13}$ are called the *dynamic coupling terms*. The physical significance of the static coupling term $-k_{ij}q_j$ is evident from Eq. (1.5): it represents the contribution of the deflection q_j to the force Q_i. The physical meaning of the dynamic coupling terms is seen if we consider the generalized momentum components [cf. Chapter III, section 11]; $m_{ij}\dot{q}_j$ is the contribution of the velocity component $\dot{q}_j$ to the ith component of the generalized momentum.

In the next sections we shall consider systems with static coupling terms only. In the last two sections we shall deal with systems with dynamic coupling and show how this case can be reduced to that of static coupling.

2. Linear Oscillation of Two Coupled Masses.—Let us assume that a system consists of two masses, m_1 and m_2. Each mass is restrained by an elastic spring to a fixed base, and a third elastic spring connects the two masses. We assume that in the equilibrium position all three springs are unstressed.

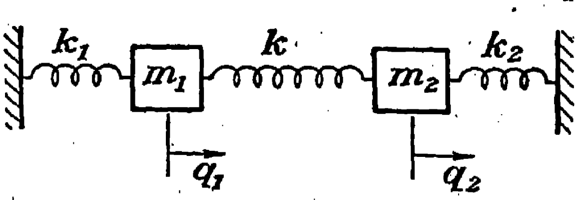

FIG. 2.1.—A coupled system with two degrees of freedom.

We choose as coordinates q_1 and q_2 the displacements of the masses in the positive x-direction (Fig. 2.1). The elastic restraints produced by the two springs acting between the masses and the fixed bases are equal to $-k_1q_1$ and $-k_2q_2$, respectively. The elastic force of the spring acting between the masses is equal to the product of a coefficient k and the change of the distance between the two masses. If we consider all forces positive in the direction of the positive x-axis, the force exerted by this spring on m_1 will be equal to $-k(q_1 - q_2)$, and the force exerted on m_2 is equal to $-k(q_2 - q_1)$, so that the total force acting on m_1 is equal to $Q_1 = -k_1q_1 - k(q_1 - q_2)$, the total force acting on m_2 is equal to $Q_2 = -k_2q_2 - k(q_2 - q_1)$.

The potential energy is given by [cf. Eq. (1.7)]

$$U = \tfrac{1}{2}\,[k_1q_1^2 + k_2q_2^2 + k(q_1 - q_2)^2] \tag{2.1}$$

It is seen that $Q_1 = -\partial U/\partial q_1$ and that $Q_2 = -\partial U/\partial q_2$.

The kinetic energy is equal to

$$T = \tfrac{1}{2}(m_1\dot{q}_1^2 + m_2\dot{q}_2^2) \tag{2.2}$$

We derive from Eqs. (2.1) and (2.2) the equations of motion

$$\begin{aligned} m_1\ddot{q}_1 &= -(k_1 + k)q_1 + kq_2 \\ m_2\ddot{q}_2 &= kq_1 - (k_2 + k)q_2 \end{aligned} \tag{2.3}$$

The method leading to the solution of such systems of linear differential equations was discussed in Chapter I. We put $q_1 = A_1e^{\lambda t}$, $q_2 = A_2e^{\lambda t}$—where λ, A_1, and A_2 may be real or complex—and obtain for A_1 and A_2 the two linear equations

$$\begin{aligned} A_1(m_1\lambda^2 + k_1 + k) - A_2k &= 0 \\ -A_1k + A_2(m_2\lambda^2 + k_2 + k) &= 0 \end{aligned} \tag{2.4}$$

The system of linear equations has solutions different from $A_1 = A_2 = 0$, if

$$(m_1\lambda^2 + k_1 + k)(m_2\lambda^2 + k_2 + k) - k^2 = 0 \tag{2.5}$$

This equation is the *characteristic equation* of the system (2.3). We write (2.5) in the form:

$$\left(\lambda^2 + \frac{k_1}{m_1} + \frac{k}{m_1}\right)\left(\lambda^2 + \frac{k_2}{m_2} + \frac{k}{m_2}\right) - \frac{k^2}{m_1m_2} = 0 \tag{2.6}$$

and solve Eq. (2.6) for λ^2. We can write λ^2 in the two following forms:

$$\lambda^2 = -\frac{1}{2}\left(\frac{k_1 + k}{m_1} + \frac{k_2 + k}{m_2}\right) \pm \frac{1}{2}\sqrt{\left(\frac{k_1 + k}{m_1} + \frac{k_2 + k}{m_2}\right)^2 - 4\frac{k_1k_2 + k(k_1 + k_2)}{m_1m_2}} \tag{2.7}$$

$$\lambda^2 = -\frac{1}{2}\left(\frac{k_1 + k}{m_1} + \frac{k_2 + k}{m_2}\right) \pm \frac{1}{2}\sqrt{\left(\frac{k_1 + k}{m_1} - \frac{k_2 + k}{m_2}\right)^2 + \frac{4k^2}{m_1m_2}} \tag{2.8}$$

The expression under the radical sign in Eq. (2.8) is always positive; hence, λ^2 is real. In Eq. (2.7) the absolute value of the second term is smaller than that of the first term since k, k_1, and k_2 are positive. Hence λ^2 is negative, and we can write $\lambda = \pm i\omega$, *i.e.*, we obtain undamped harmonic oscillations with the frequency ω.

Let us use the notations $\omega_{11}^2 = \dfrac{k_1 + k}{m_1}$, $\omega_{22}^2 = \dfrac{k_2 + k}{m_2}$, and $\omega_{12}^2 = k/\sqrt{m_1m_2}$. The quantities ω_{11} and ω_{22} have simple phys-

ical meanings. Let us assume that the mass m_2 is held fixed, *i.e.*, $q_2 = 0$. Then the equation of motion for m_1 will be

$$m_1\ddot{q}_1 = -(k_1 + k)q_1$$

Hence, the angular frequency of the mass m_1 is in this case equal to ω_{11}. Correspondingly, ω_{22} is equal to the angular frequency of m_2 if the mass m_1 is held fixed.

With these notations and with $\lambda^2 = -\omega^2$, we obtain from Eq. (2.8)

$$\omega^2 = \tfrac{1}{2}(\omega_{11}^2 + \omega_{22}^2) \pm \tfrac{1}{2}\sqrt{(\omega_{11}^2 - \omega_{22}^2)^2 + 4\omega_{12}^4} \tag{2.9}$$

Equation (2.9) gives two values for ω^2, say ω_1^2 and ω_2^2; we call ω_1 and ω_2 the *natural frequencies of the coupled system.*

There are four values of λ, *viz.*, $i\omega_1$, $-i\omega_1$, $i\omega_2$, and $-i\omega_2$. If we substitute any of these values into Eq. (2.4), the two equations become identical, and we obtain

$$\frac{A_2}{A_1} = \frac{m_1\lambda^2 + k_1 + k}{k} = \frac{k}{m_2\lambda^2 + k_2 + k} \tag{2.10}$$

The two expressions in (2.10) are equal as a result of (2.5).

The general solution of the system of equations (2.3) will be

$$\begin{aligned} q_1 &= A_1^{(1)}e^{i\omega_1 t} + \bar{A}_1^{(1)}e^{-i\omega_1 t} + A_1^{(2)}e^{i\omega_2 t} + \bar{A}_1^{(2)}e^{-i\omega_2 t} \\ q_2 &= A_2^{(1)}e^{i\omega_1 t} + \bar{A}_2^{(1)}e^{-i\omega_1 t} + A_2^{(2)}e^{i\omega_2 t} + \bar{A}_2^{(2)}e^{-i\omega_2 t} \end{aligned} \tag{2.11}$$

The constants in Eqs. (2.11) are not arbitrary. Since q_1 and q_2 are real, the coefficients of the type $A_1^{(1)}$ and $\bar{A}_1^{(1)}$ must be complex conjugate. Hence, putting

$$A_1^{(1)} = \frac{C_1^{(1)}}{2}e^{i\psi_1}, \qquad \bar{A}_1^{(1)} = \frac{C_1^{(1)}}{2}e^{-i\psi_1} \tag{2.12}$$

and so on, we have

$$q_1 = \tfrac{1}{2}C_1^{(1)}(e^{i(\omega_1 t+\psi_1)} + e^{-i(\omega_1 t+\psi_1)}) + \tfrac{1}{2}C_1^{(2)}(e^{i(\omega_2 t+\psi_2)} + e^{-i(\omega_2 t+\psi_2)})$$

or

$$q_1 = C_1^{(1)} \cos(\omega_1 t + \psi_1) + C_1^{(2)} \cos(\omega_2 t + \psi_2) \tag{2.13}$$

and similarly, with $A_2^{(1)} = \dfrac{C_2^{(1)}}{2}e^{i\psi_1}$ and $\bar{A}_2^{(1)} = \dfrac{C_2^{(1)}}{2}e^{-i\psi_1}$ we have

$$q_2 = C_2^{(1)} \cos(\omega_1 t + \psi_1) + C_2^{(2)} \cos(\omega_2 t + \psi_2) \tag{2.14}$$

We see that $C_2^{(1)}/C_1^{(1)} = A_2^{(1)}/A_1^{(1)}$, and $C_2^{(2)}/C_1^{(2)} = A_2^{(2)}/A_1^{(2)}$, so

that the amplitude ratios C_2/C_1 are directly determined by Eq. (2.10).

Equations (2.13) and (2.14) contain four arbitrary real constants: $C_1^{(1)}$, $C_1^{(2)}$, ψ_1 and ψ_2; these are determined by the initial positions and velocities of the two masses. The equations show that the most general motion of the system is made up of the superposition of two pure harmonic oscillations:

$$\begin{aligned} q_1 &= C_1^{(1)} \cos (\omega_1 t + \psi_1) \\ q_2 &= C_2^{(1)} \cos (\omega_1 t + \psi_1) \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} q_1 &= C_1^{(2)} \cos (\omega_2 t + \psi_2) \\ q_2 &= C_2^{(2)} \cos (\omega_2 t + \psi_2) \end{aligned} \tag{2.16}$$

In each of these oscillations the two masses oscillate with the same frequency and in the same phase. Their amplitudes are in a definite ratio which is given by Eq. (2.10). The pure harmonic oscillations given by (2.15) and (2.16) are called the *principal oscillations* or the *principal modes of oscillation* of the system.

Let us apply these general results to a case of two equal masses, restrained by identical springs to the fixed bases. Then with $m_1 = m_2 = m$, $k_1 = k_2 = K$, we have $\omega_{11}^2 = \omega_{22}^2 = \dfrac{K + k}{m}$, $\omega_{12}^2 = k/m$, and from Eq. (2.9)

$$\omega^2 = \omega_{11}^2 \pm \omega_{12}^2 \tag{2.17}$$

The equations for the two frequencies are

$$\omega_1^2 = \frac{K}{m}, \qquad \omega_2^2 = \frac{K + 2k}{m}$$

The amplitude ratios are given by

$$\frac{C_2}{C_1} = \frac{-m\omega^2 + K + k}{k} \tag{2.18}$$

If we substitute $\omega^2 = \omega_1^2$, we obtain $C_2^{(1)}/C_1^{(1)} = 1$; for $\omega^2 = \omega_2^2$ we have $C_2^{(2)}/C_1^{(2)} = -1$.

The first ratio corresponds to the lower frequency called the *fundamental mode of oscillation*. In this mode of oscillation the two masses oscillate with the same amplitude and in the same direction. In the second mode of oscillation, which corresponds

to the higher frequency, the two masses oscillate with the same amplitude, but in opposite directions. These two principal oscillations are not coupled; each of them can occur independently. For example, if at $t = 0$ the two masses are set in motion with equal velocities in the same direction, their motion will be identical all the time. If their initial velocities are equal and opposite their velocities will remain equal and opposite. In any other case, for example, if at $t = 0$ the mass m_1 has a certain initial velocity, while the initial velocity of the mass m_2 is equal to zero, both principal oscillations are excited simultaneously. If $k \ll K$, the two frequencies ω_1 and ω_2 are only slightly different; then the superposition of the two principal oscillations will bring about the well-known phenomenon of *beats*. The reader will easily verify (using the method of Chapter IV, section 6) that the amplitude of the mass m_1 will decrease and that of m_2 increase until the mass m_1 comes to rest and the amplitude of m_2 is the same as that of m_1 at the beginning of the motion. Then the amplitude of m_2 decreases again, and that of m_1 increases. This process is sometimes called the *wandering of the energy* between the degrees of freedom. The energy of the principal oscillations is constant; there is no transfer of energy between them. The notion of the principal oscillations will be expanded in more detail in the next sections.

3. Conservative System with Static Coupling. General Theory.—Let us assume that the potential energy of the oscillating system is given by

$$U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}q_iq_j \tag{3.1}$$

where $q_1, q_2, \ldots, q_n$ are generalized coordinates. We assume that $k_{ij} = k_{ji}$. We also assume that U has a minimum for $q_1 = q_2 = \cdots = q_n = 0$, *i.e.*, the equilibrium position is stable. Therefore, U is positive for any other arbitrary values of the q_i's. Since there is no dynamic coupling, the kinetic energy is given by

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i\dot{q}_i^2 \tag{3.2}$$

Lagrange's equations in this case become

$$\begin{aligned} -m_1\ddot{q}_1 &= k_{11}q_1 + k_{12}q_2 + \cdots + k_{1n}q_n \\ -m_2\ddot{q}_2 &= k_{21}q_1 + k_{22}q_2 + \cdots + k_{2n}q_n \\ &\cdots\cdots\cdots\cdots\cdots \\ -m_n\ddot{q}_n &= k_{n1}q_1 + k_{n2}q_2 + \cdots + k_{nn}q_n \end{aligned} \tag{3.3}$$

We shall consider solutions corresponding to pure harmonic motion*

$$q_i = A_i \sin(\omega t + \psi) \tag{3.4}$$

where ω is the angular frequency of the oscillation and ψ an arbitrary phase angle. Then $\ddot{q}_i = -\omega^2 q_i$, and substituting this expression in (3.3), we obtain n homogeneous linear equations for $q_1, q_2, \ldots, q_n$, *viz.*,

$$\begin{aligned} k_{11}q_1 + k_{12}q_2 + \cdots + k_{1n}q_n &= m_1\omega^2 q_1 \\ k_{21}q_1 + k_{22}q_2 + \cdots + k_{2n}q_n &= m_2\omega^2 q_2 \\ \cdots\cdots\cdots\cdots\cdots & \\ k_{n1}q_1 + k_{n2}q_2 + \cdots + k_{nn}q_n &= m_n\omega^2 q_n \end{aligned} \tag{3.5}$$

The system in (3.5) has solutions different from zero if the determinant of the system vanishes, *i.e.*,

$$\begin{vmatrix} k_{11} - m_1\omega^2 & k_{12} & \cdots & k_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} - m_n\omega^2 \end{vmatrix} = 0 \tag{3.6}$$

Equation (3.6) is an algebraic equation of the degree n for ω^2 and is called the *frequency equation*. It can be proved that, if U is positive definite (stable equilibrium), the roots ω^2 are all positive. The cases in which (3.6) has a zero root or two equal roots will be excluded, and the n roots will be denoted by $\omega_1^2, \omega_2^2, \ldots, \omega_n^2$. Then substituting the roots successively into (3.5), we obtain n sets of expressions for the q's; each set corresponds to one of the frequencies $\omega_1, \omega_2, \ldots, \omega_n$. The general solution is, therefore,

$$q_i = \sum_{r=1}^{n} A_i^{(r)} \sin(\omega_r t + \psi_r) \tag{3.7}$$

* By substituting $q_i = A_i e^{\lambda t}$, we would obtain complex values for A_i and would have to eliminate the complex quantities as we did in the last section. Hence, it is simpler to assume directly a solution in the real form. Whether we use sine or cosine is arbitrary.

where $A_1^{(1)}, A_2^{(1)}, \ldots, A_n^{(1)}$ correspond to the frequency ω_1; $A_1^{(2)}, A_2^{(2)}, \ldots, A_n^{(2)}$ to ω_2; and so on. As the $A_i^{(r)}$'s satisfy the same system of homogeneous equations (3.5) as the q_i's, they are determined only as far as their ratios are concerned. Every set contains an undetermined multiplicative factor.

We call the oscillation determined by one set of the $A_i^{(r)}$'s, for example, the oscillation given by $q_1 = A_1^{(1)} \sin(\omega_1 t + \psi_1)$, $q_2 = A_2^{(1)} \sin(\omega_1 t + \psi_1)$, and so on, a *principal oscillation* or a *principal mode of oscillation* of the system. The number of principal oscillations is equal to the number of degrees of freedom.

Every principal oscillation is a pure harmonic motion. The most general form of oscillation of a system with n degrees of freedom consists, therefore, of the superposition of n pure harmonic motions. If, for instance, the system is composed of n masses, and we assume that the coefficients $A_i^{(r)}$ of only one principal oscillation are different from zero, the n masses oscillate with the same frequency and in the same phase. Then we say that the system oscillates in a principal mode.

The frequencies of the principal oscillations are also called the *natural frequencies* of the system. The lowest frequency is called the *fundamental frequency*. The principal oscillations are often numbered as *first, second, and so on, principal oscillations* in order of ascending frequencies.

4. Orthogonality of the Principal Oscillations.—We shall prove in this section a fundamental property of the principal oscillations that is important for the practical calculations of the natural frequencies and also for the treatment of forced vibrations.

Let us consider two principal oscillations, *e.g.*, those corresponding to ω_r^2 and ω_s^2. The coefficients of these two modes of vibration $A_1^{(r)}, A_2^{(r)}, \ldots, A_n^{(r)}$ and $A_1^{(s)}, A_2^{(s)}, \ldots, A_n^{(s)}$ satisfy the following equations [cf. Eq. (3.5)]:

$$k_{i1}A_1^{(r)} + k_{i2}A_2^{(r)} + \cdots + k_{in}A_n^{(r)} = m_i\omega_r^2 A_i^{(r)} \tag{4.1}$$

for $i = 1, 2, \ldots, n$, and

$$k_{i1}A_1^{(s)} + k_{i2}A_2^{(s)} + \cdots + k_{in}A_n^{(s)} = m_i\omega_s^2 A_i^{(s)} \tag{4.2}$$

also for $i = 1, 2, \ldots, n$.

We multiply Eq. (4.1) by $A_i^{(s)}$, and Eq. (4.2) by $A_i^{(r)}$. Then we obtain

$$k_{i1}A_1^{(r)}A_i^{(s)} + k_{i2}A_2^{(r)}A_i^{(s)} + \cdots + k_{in}A_n^{(r)}A_i^{(s)} = m_i\omega_r^2 A_i^{(r)}A_i^{(s)} \tag{4.3}$$

and

$$k_{i1}A_1^{(s)}A_i^{(r)} + k_{i2}A_2^{(s)}A_i^{(r)} + \cdots + k_{in}A_n^{(s)}A_i^{(r)} = m_i\omega_s^2 A_i^{(s)}A_i^{(r)} \tag{4.4}$$

If we now add the n equations of the form (4.3) obtained by substituting $i = 1, 2, \ldots, n$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}A_j^{(r)}A_i^{(s)} = \omega_r^2\sum_{i=1}^{n} m_iA_i^{(r)}A_i^{(s)} \tag{4.5}$$

Adding the n equations of the form (4.4) obtained by substituting $i = 1, 2, \ldots, n$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}A_j^{(s)}A_i^{(r)} = \omega_s^2\sum_{i=1}^{n} m_iA_i^{(s)}A_i^{(r)} \tag{4.6}$$

The left sides of Eqs. (4.5) and (4.6) are identical. To show this, let us interchange in the summation on the left side of Eq. (4.5) the subscripts i and j. Then we obtain $\sum_{j=1}^{n}\sum_{i=1}^{n} k_{ji}A_i^{(r)}A_j^{(s)}$. Since $k_{ij} = k_{ji}$, this sum is identical with the left side of (4.6). Therefore, the right sides must also be equal, *i.e.*,

$$(\omega_r^2 - \omega_s^2)\sum_{i=1}^{n} m_iA_i^{(r)}A_i^{(s)} = 0 \tag{4.7}$$

Since ω_r^2 and ω_s^2 are different, it follows that

$$\sum_{i=1}^{n} m_iA_i^{(r)}A_i^{(s)} = 0 \tag{4.8}$$

The relation (4.8) is known as the *orthogonality relation for the principal modes of oscillation.* Sometimes we also say that *the principal modes are orthogonal.*

This terminology will be justified by the illustrative example given below. We notice that our proof is based on the condition that for all i's and j's, $k_{ij} = k_{ji}$. We remember that this is true only for a system of forces which can be derived from a potential energy. Hence, the orthogonality of the modes of

vibration is a property of systems with a potential energy. It does not hold, for example, for nonconservative systems.

An Illustrative Example.—Let us assume that a particle of unit mass can oscillate in all directions and is restrained to its equilibrium position $x_1 = x_2 = x_3 = 0$ by forces which are linear functions of the three rectangular components of its displacements. We assume that the rectangular components of the force which produces a displacement x_1, x_2, x_3 are given by

$$\begin{aligned} X_1 &= k_{11}x_1 + k_{12}x_2 + k_{13}x_3 \\ X_2 &= k_{21}x_1 + k_{22}x_2 + k_{23}x_3 \\ X_3 &= k_{31}x_1 + k_{32}x_2 + k_{33}x_3 \end{aligned} \tag{4.9}$$

The system of forces is supposed to be conservative and, therefore, $k_{12} = k_{21}$, $k_{23} = k_{32}$, $k_{31} = k_{13}$. The potential energy is given by

$$U = \tfrac{1}{2}(X_1x_1 + X_2x_2 + X_3x_3) = \tfrac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} k_{ij}x_ix_j \tag{4.10}$$

The force system (4.9) can be realized, for example, by the following arrangement (Fig. 4.1): the mass is suspended at the point D between three arbitrary fixed points A, B, C by three springs whose spring constants are k_a, k_b, and k_c. If the displacements of the mass are small compared to the lengths of the springs, the spring forces will act approximately along their respective axes corresponding to their equilibrium position. If the components of the displacement of the mass in the direction of the three springs are equal to a, b, and c, respectively, the potential energy of the system is given by

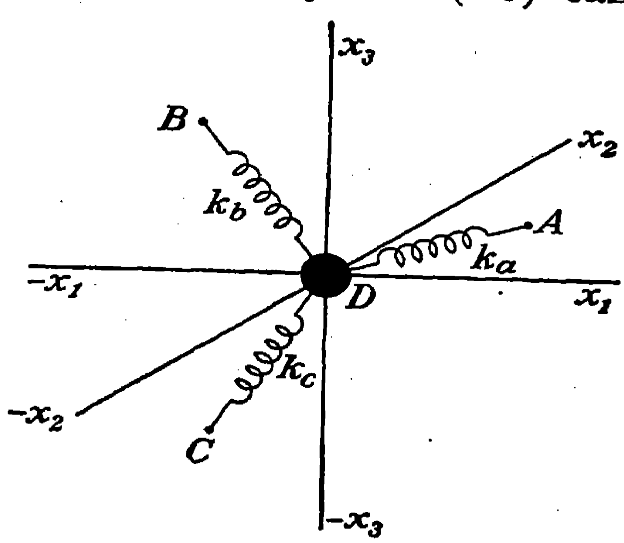

FIG. 4.1.—A mass elastically restrained in space illustrating an oscillating system with three degrees of freedom.

$$U = \tfrac{1}{2}(k_aa^2 + k_bb^2 + k_cc^2)$$

Then if we express a, b, and c in terms of x_1, x_2, and x_3, we obtain the general form (4.10).

The kinetic energy of the mass is equal to

$$T = \tfrac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) \tag{4.11}$$

The equations of motion are

$$\begin{aligned} -\ddot{x}_1 &= k_{11}x_1 + k_{12}x_2 + k_{13}x_3 \\ -\ddot{x}_2 &= k_{21}x_1 + k_{22}x_2 + k_{23}x_3 \\ -\ddot{x}_3 &= k_{31}x_1 + k_{32}x_2 + k_{33}x_3 \end{aligned} \tag{4.12}$$

The frequency equation is given, according to (3.6), by the determinant

$$\begin{vmatrix} k_{11} - \omega^2 & k_{12} & k_{13} \\ k_{21} & k_{22} - \omega^2 & k_{23} \\ k_{31} & k_{32} & k_{33} - \omega^2 \end{vmatrix} = 0 \tag{4.13}$$

Denoting the roots of (4.13) by ω_1^2, ω_2^2, and ω_3^2, we obtain the general solution of the problem of free oscillations [cf. Eq. (3.7)]:

$$\begin{aligned} x_1 &= A_1^{(1)} \sin(\omega_1 t + \psi_1) + A_1^{(2)} \sin(\omega_2 t + \psi_2) \\ &\qquad + A_1^{(3)} \sin(\omega_3 t + \psi_3) \\ x_2 &= A_2^{(1)} \sin(\omega_1 t + \psi_1) + A_2^{(2)} \sin(\omega_2 t + \psi_2) \\ &\qquad + A_2^{(3)} \sin(\omega_3 t + \psi_3) \\ x_3 &= A_3^{(1)} \sin(\omega_1 t + \psi_1) + A_3^{(2)} \sin(\omega_2 t + \psi_2) \\ &\qquad + A_3^{(3)} \sin(\omega_3 t + \psi_3) \end{aligned} \tag{4.14}$$

Consider, for example, the first mode of oscillation. The equations

$$x_1 = A_1^{(1)} \sin(\omega_1 t + \psi_1), \qquad x_2 = A_2^{(1)} \sin(\omega_1 t + \psi_1), \qquad x_3 = A_3^{(1)} \sin(\omega_1 t + \psi_1) \tag{4.15}$$

are equations for a segment of a straight line in the parametric form with t as parameter. If we denote the direction cosines of this straight line by $\varphi_1^{(1)}$, $\varphi_2^{(1)}$, and $\varphi_3^{(1)}$, we can write

$$A_1^{(1)} = c_1\varphi_1^{(1)}, \qquad A_2^{(1)} = c_1\varphi_2^{(1)}, \qquad A_3^{(1)} = c_1\varphi_3^{(1)} \tag{4.16}$$

where c_1 is an arbitrary constant. Then (4.15) becomes

$$\begin{aligned} x_1 &= c_1\varphi_1^{(1)} \sin(\omega_1 t + \psi_1) \\ x_2 &= c_1\varphi_2^{(1)} \sin(\omega_1 t + \psi_1) \\ x_3 &= c_1\varphi_3^{(1)} \sin(\omega_1 t + \psi_1) \end{aligned} \tag{4.17}$$

Obviously c_1 is the amplitude of the oscillation, since

$$\varphi_1^{(1)^2} + \varphi_2^{(1)^2} + \varphi_3^{(1)^2} = 1 \tag{4.18}$$

and, therefore, the displacement of the mass point at the time t is equal to

$$\sqrt{x_1^2 + x_2^2 + x_3^2} = c_1 \sin (\omega_1 t + \psi_1)$$

Let us now consider the second mode of oscillation which corresponds to ω_2^2. We have

$$\begin{aligned} x_1 &= A_1^{(2)} \sin (\omega_2 t + \psi_2) \\ x_2 &= A_2^{(2)} \sin (\omega_2 t + \psi_2) \\ x_3 &= A_3^{(2)} \sin (\omega_2 t + \psi_2) \end{aligned} \tag{4.19}$$

We now write

$$A_1^{(2)} = c_2\varphi_1^{(2)}, \qquad A_2^{(2)} = c_2\varphi_2^{(2)}, \qquad A_3^{(2)} = c_2\varphi_3^{(2)} \tag{4.20}$$

where c_2 is the amplitude of the oscillation and $\varphi_1^{(2)}$, $\varphi_2^{(2)}$, and $\varphi_3^{(2)}$ are the direction cosines of the straight line along which the mass oscillates.

If we substitute the coefficients $A_i^{(1)}$ and $A_i^{(2)}$ from (4.16) and (4.20) in Eq. (4.8), we obtain

$$c_1c_2(\varphi_1^{(1)}\varphi_1^{(2)} + \varphi_2^{(1)}\varphi_2^{(2)} + \varphi_3^{(1)}\varphi_3^{(2)}) = 0 \tag{4.21}$$

According to a well-known relation of analytical geometry, the expression in the parentheses is equal to the cosine of the angle between the two directions defined by the direction cosines $\varphi_1^{(1)}$, $\varphi_2^{(1)}$, $\varphi_3^{(1)}$ and $\varphi_1^{(2)}$, $\varphi_2^{(2)}$, $\varphi_3^{(2)}$. Hence, in this case Eq. (4.8), which we called the *orthogonality relation*, has the simple geometrical meaning that the two principal oscillations take place along straight lines which are perpendicular to each other. The same is true for any two principal oscillations. And any oscillation of the mass in space—no matter how complicated it may look—is made up by superposition of three linear oscillations in three perpendicular directions.

Normal Modes of Oscillation.—Let us return to the first principal oscillation. If we put $c_1 = 1$, we obtain a mode of oscillation *with unit amplitude* given by the equations

$$\begin{aligned} x_1 &= \varphi_1^{(1)} \sin (\omega_1 t + \psi_1) \\ x_2 &= \varphi_2^{(1)} \sin (\omega_1 t + \psi_1) \\ x_3 &= \varphi_3^{(1)} \sin (\omega_1 t + \psi_1) \end{aligned} \tag{4.22}$$

We call (4.22) a *normal mode* of oscillation. Since the sum of the squares of $\varphi_1^{(1)}$, $\varphi_2^{(1)}$ and $\varphi_3^{(1)}$ is equal to unity, we obtain the normal mode of oscillation from the expression (4.15) by putting

$$A_1^{(1)^2} + A_2^{(1)^2} + A_3^{(1)^2} = 1 \tag{4.23}$$

In other words we use the term, *principal oscillation,* for an oscillation of undetermined amplitude; the term, *normal mode,* is used for a principal oscillation normalized by the *normalizing condition* (4.23).

The normalization of the principal oscillations is useful also in the case when the kinetic energy appears in the more general form (3.2), *i.e.*, the inertia coefficients m_i are not equal to unity, as they were assumed in this simple example. In the general case it is convenient to use as normalizing conditions n relations of the following form (for $r = 1, 2, \ldots n,$):

$$\sum_{i=1}^{n} m_i A_i^{(r)^2} = M \tag{4.24}$$

where M is a constant of the dimension of the inertia coefficients m_i.* We choose $M = \frac{1}{n}\sum_{i=1}^{n} m_i$. Then in the special case $m_1 = m_2 = \cdots = m_n = 1$, the condition (4.24) is identical with (4.23).

We write now by analogy with (4.16) and (4.20)

$$A_i^{(r)} = c_r \varphi_i^{(r)} \tag{4.25}$$

and call the common factor c_r the *amplitude* of the rth principal oscillation. Then the expressions for the rth *normal mode* of oscillation become with $c_r = 1$:

$$q_i = \varphi_i^{(r)} \sin (\omega_r t + \psi_r) \tag{4.26}$$

where $i = 1, 2, \cdots, n$. To be sure, the n^2 coefficients $\varphi_i^{(r)}$, which we call the *coefficients of the normal modes*, no longer have the significance of direction cosines. They are numerical constants which satisfy the systems of equations ($i = 1, 2, \ldots, n$):

$$k_{i1}\varphi_1^{(r)} + k_{i2}\varphi_2^{(r)} + \cdots + k_{in}\varphi_n^{(r)} = m_i\omega_r^2\varphi_i^{(r)} \tag{4.27}$$

* We assume that the coordinates $q_1, q_2, \ldots, q_n$ have the same physical dimension; for example, they are all lengths or angles, etc. This can always be accomplished by multiplication by some constant parameter of the physical setup or by a combination of some of them. The expressions become more symmetrical in this way than in coordinates of different dimensions.

and the normalizing condition

$$\sum_{i=1}^{n} m_i \varphi_i^{(r)^2} = M \tag{4.28}$$

Through these equations they are determined entirely by the parameters (spring constants and inertia factors) of the system. They also satisfy the orthogonality condition

$$\sum_{i=1}^{n} m_i \varphi_i^{(r)} \varphi_i^{(s)} = 0 \tag{4.29}$$

provided $r \neq s$.

5. Example of a System with Three Degrees of Freedom.—A shaft with a uniform torsional stiffness C carries three disks of equal shape and size at three equidistant points (Fig. 5.1*a*). The shaft is fixed at $x = 0$; the distance of the first disk from the fixed end and the distance between the disks is equal to l. At $x = 3l$ the shaft has a free end. Then denoting the inertia moment of each disk by I, the angular deflection of the three disks by q_1, q_2, and q_3, the kinetic energy of the system is given by

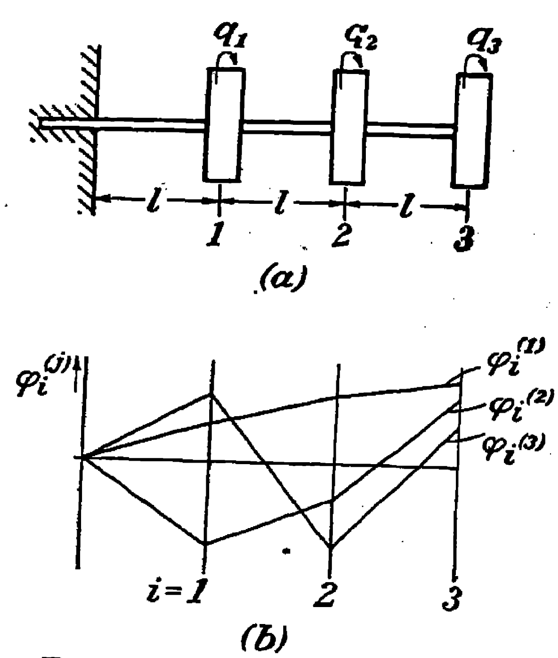

FIG. 5.1.—Normal modes of torsional oscillation of a shaft carrying three disks.

$$T = \tfrac{1}{2}\sum_{i=1}^{3} I\dot{q}_i^2 \tag{5.1}$$

The potential energy of each section of the shaft is equal to $\frac{1}{2}c(\Delta q)^2$ where Δq is the relative angular displacement of two adjacent disks and $c = \dfrac{C}{l}$. Hence, the potential energy is given by the expression

$$U = \frac{c}{2}\left[q_1^2 + (q_2 - q_1)^2 + (q_3 - q_2)^2\right] \tag{5.2}$$

or

$$U = \frac{c}{2}(2q_1^2 + 2q_2^2 + q_3^2 - 2q_1q_2 - 2q_2q_3)$$

The equations of motion are

$$\begin{aligned} -I\ddot{q}_1 &= c(\ 2q_1 - \ q_2 \qquad) \\ -I\ddot{q}_2 &= c(-q_1 + 2q_2 - q_3) \\ -I\ddot{q}_3 &= c(\qquad - \ q_2 + q_3) \end{aligned} \tag{5.3}$$

Hence, the frequency equation is

$$\begin{vmatrix} 2 - \dfrac{I\omega^2}{c} & -1 & 0 \\ -1 & 2 - \dfrac{I\omega^2}{c} & -1 \\ 0 & -1 & 1 - \dfrac{I\omega^2}{c} \end{vmatrix} = 0 \tag{5.4}$$

or with $\lambda = I\omega^2/c$

$$\lambda^3 - 5\lambda^2 + 6\lambda - 1 = 0 \tag{5.5}$$

The three roots of Eq. (5.5) are approximately $\lambda_1 = 0.1981$, $\lambda_2 = 1.555$, and $\lambda_3 = 3.247$. (For the calculation of these roots the reader is referred to section 8 of this chapter.)

The normalizing condition (4.28) for the normal modes becomes

$$I\varphi_1^{(r)2} + I\varphi_2^{(r)2} + I\varphi_3^{(r)2} = M = \frac{I + I + I}{3} = I$$

Hence the coefficients of the normal modes of oscillations satisfy the equations

$$\begin{aligned} (2 - \lambda_r)\varphi_1^{(r)} - \varphi_2^{(r)} &= 0 \\ -\varphi_1^{(r)} + (2 - \lambda_r)\varphi_2^{(r)} - \varphi_3^{(r)} &= 0 \\ -\varphi_2^{(r)} + (1 - \lambda_r)\varphi_3^{(r)} &= 0 \end{aligned} \tag{5.6}$$

and

$$\varphi_1^{(r)2} + \varphi_2^{(r)2} + \varphi_3^{(r)2} = 1 \tag{5.7}$$

These equations give the following values:
for $r = 1$,

$\varphi_1^{(1)} = 0.328, \qquad \varphi_2^{(1)} = 0.591, \qquad \varphi_3^{(1)} = 0.736,$

$$\omega_1 = 0.445\sqrt{\frac{c}{I}}$$

for $r = 2$,

$\varphi_1^{(2)} = -0.736, \qquad \varphi_2^{(2)} = -0.328, \qquad \varphi_3^{(2)} = 0.591,$

$$\omega_2 = 1.247\sqrt{\frac{c}{I}}$$

for $r = 3$,

$$\varphi_1^{(3)} = 0.591, \qquad \varphi_2^{(3)} = -0.736, \qquad \varphi_3^{(3)} = 0.328,$$

$$\omega_3 = 1.801\sqrt{\frac{C}{I}}$$

The amplitudes of the normal modes are shown in Fig. 5.1*b*. The sign chosen for any particular mode is arbitrary. We may verify the orthogonality relations. For example:

$$\varphi_1^{(1)}\varphi_1^{(2)} + \varphi_2^{(1)}\varphi_2^{(2)} + \varphi_3^{(1)}\varphi_3^{(2)} = (0.328)(-0.736) + (0.591)(-0.328) + (0.736)(0.591) = 0$$

6. The Kinetic and Potential Energies in Terms of the Principal Oscillations.—We shall now express the kinetic and potential energies of a conservative system with static coupling in terms of the amplitudes of the principal oscillations. We found in section 3 that the most general form of oscillation of such a system is represented by superposition of n pure harmonic oscillations [cf. Eq. (3.7)]

$$q_i = \sum_{r=1}^{n} A_i^{(r)} \sin (\omega_r t + \psi_r) \tag{6.1}$$

Introducing the amplitudes of the principal oscillations by putting $A_i^{(r)} = \varphi_i^{(r)} c_r$ [Eq. (4.25)], we have

$$q_i = \sum_{r=1}^{n} \varphi_i^{(r)} c_r \sin (\omega_r t + \psi_r) \tag{6.2}$$

Let us calculate first the kinetic energy of the system, which is given by Eq. (3.2). Differentiating (6.2), we obtain for the generalized velocity component $\dot{q}_i$ the expression

$$\dot{q}_i = \sum_{r=1}^{n} \varphi_i^{(r)} c_r \omega_r \cos (\omega_r t + \psi_r) \tag{6.3}$$

If we substitute this expression in the formula for T, we obtain terms that contain the squares of the amplitudes and terms that contain products of the form $c_r c_s$. We show that the terms containing products vanish as a result of the orthogonality of the principal oscillations. In fact, the coefficient of $c_r c_s$ in the expression

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i[\varphi_i^{(1)}\omega_1 c_1 \cos(\omega_1 t + \psi_1) + \cdots + \varphi_i^{(n)}\omega_n c_n \cos(\omega_n t + \psi_n)]^2 \tag{6.4}$$

becomes

$$\omega_r\omega_s \cos(\omega_r t + \psi_r)\cos(\omega_s t + \psi_s)\sum_{i=1}^{n} m_i\varphi_i^{(r)}\varphi_i^{(s)} \tag{6.5}$$

This expression vanishes according to Eq. (4.29) when $r \neq s$. The coefficient of the square c_r^2 in Eq. (6.4) is

$$\frac{1}{2}\omega_r^2 \cos^2(\omega_r t + \psi_r)\sum_{i=1}^{n} m_i\varphi_i^{(r)2} \tag{6.6}$$

According to Eq. (4.28) (normalizing condition)

$$\sum_{i=1}^{n} m_i\varphi_i^{(r)2} = M$$

where $M = (1/n)\sum_{i=1}^{n} m_i$, *i.e.*, M is equal to the mean value of the inertia coefficients in the original expression for T. Hence, (6.4) is reduced to the form

$$T = \frac{1}{2}\sum_{r=1}^{n} M\omega_r^2 c_r^2 \cos^2(\omega_r t + \psi_r) \tag{6.7}$$

Let us assume, for example, that $n = 3$. Then Eq. (6.7) would represent the kinetic energy of a single mass M which oscillates in three perpendicular directions with the frequencies ω_1, ω_2, and ω_3 with the amplitudes c_1, c_2 and c_3 and with the phase angles ψ_1, ψ_2, and ψ_3, respectively.

Let us now calculate the potential energy. We shall use the expression (1.7) for the potential energy

$$U = -\frac{1}{2}\sum_{i=1}^{n} Q_i q_i \tag{6.8}$$

where Q_i is the generalized force corresponding to the coordinate q_i

$$Q_i = -\sum_{j=1}^{n} k_{ij}q_j \tag{6.9}$$

If we introduce for the q's the expression (6.2), we obtain again terms containing the squares of the amplitudes c_r and terms containing products of the form $c_r c_s$.

We shall prove that the coefficients of the products vanish. It is sufficient to consider a motion consisting of two arbitrary principal oscillations, assuming, for example, that only the rth and the sth mode of vibrations are present. Then the coordinate q_i becomes

$$q_i = \varphi_i^{(r)} c_r \sin(\omega_r t + \psi_r) + \varphi_i^{(s)} c_s \sin(\omega_s t + \psi_s) \tag{6.10}$$

and

$$\begin{aligned} -Q_i = {} & c_r(k_{i1}\varphi_1^{(r)} + k_{i2}\varphi_2^{(r)} + \cdots + k_{in}\varphi_n^{(r)}) \sin(\omega_r t + \psi_r) \\ & + c_s(k_{i1}\varphi_1^{(s)} + k_{i2}\varphi_2^{(s)} + \cdots + k_{in}\varphi_n^{(s)}) \sin(\omega_s t + \psi_s) \end{aligned} \tag{6.11}$$

The coefficient in parentheses in the first term is, according to Eq. (4.27), equal to $m_i\omega_r^2\varphi_i^{(r)}$; the coefficient in parentheses in the second term is equal to $m_i\omega_s^2\varphi_i^{(s)}$. As a matter of fact, this means that $q_i = \varphi_i^{(r)} \sin(\omega_r t + \psi_r)$ and $q_i = \varphi_i^{(s)} \sin(\omega_s t + \psi_s)$ satisfy the equations of motion, *i.e.*, represent principal modes of oscillation. Hence, we have

$$-Q_i = m_i\omega_r^2\varphi_i^{(r)} c_r \sin(\omega_r t + \psi_r) + m_i\omega_s^2\varphi_i^{(s)} c_s \sin(\omega_s t + \psi_s) \tag{6.12}$$

If we substitute (6.10) and (6.12) into the expression (6.8) for the potential energy, we see that the coefficient of $c_r c_s$ vanishes owing to the orthogonality relation (4.29) and that the coefficient of c_r^2 becomes

$$\tfrac{1}{2}\omega_r^2 \sum_{i=1}^{n} m_i \varphi_i^{(r)2} \sin^2(\omega_r t + \psi_r) = \tfrac{1}{2} M\omega_r^2 \sin^2(\omega_r t + \psi_r) \tag{6.13}$$

Thus, we obtain the following important expressions for the kinetic and potential energies of the system:

$$\begin{aligned} T &= \tfrac{1}{2} M \sum_{r=1}^{n} c_r^2\omega_r^2 \cos^2(\omega_r t + \psi_r) \\ U &= \tfrac{1}{2} M \sum_{r=1}^{n} c_r^2\omega_r^2 \sin^2(\omega_r t + \psi_r) \end{aligned} \tag{6.14}$$

Each expression consists of a sum of n terms, each representing the energy of a single mass M oscillating with the amplitude c_r and the frequency ω_r and with the phase angle ψ_r. If we call

$$\xi_r = c_r \sin(\omega_r t + \psi_r)$$

the *displacement* of the rth normal mode, we see that

$$T = \tfrac{1}{2}M\sum_{r=1}^{n}\dot{\xi}_r^2$$

and (6.15)

$$U = \tfrac{1}{2}M\sum_{r=1}^{n}\omega_r^2\xi_r^2$$

Each term of the kinetic energy has the form of the kinetic energy of a single mass M whose displacement is ξ_r. The corresponding term of the potential energy is equal to the potential energy of a mass point restrained to its equilibrium position by a spring of the spring constant $K_r = M\omega_r^2$. We remember that in the case of a single mass with elastic restraint the frequency is equal to $\omega = \sqrt{K/M}$; hence, the spring constant is given by $K = M\omega^2$.

It is seen that the total kinetic and potential energies of a conservative system can be expressed as sums of the energies of the normal modes, where the kinetic and the potential energies of each normal mode can be calculated as the kinetic and the potential energies of a single mass.

The expressions (6.15) show that if we use the displacements of the normal modes as coordinates of the system, the oscillation problem is greatly simplified. The equations of motion become simply

$$\ddot{\xi}_r = -\omega_r^2\xi_r \tag{6.16}$$

where $r = 1, 2, \ldots, n$.

In the case of our illustrative example treated in section 4, this amounts to the use of a rectangular coordinate system ξ_1, ξ_2, ξ_3, whose axes coincide with the directions of the principal oscillations of the mass point, instead of the arbitrary coordinate system x_1, x_2, x_3.

We see from Eq. (6.2) that the coordinates q_i and the displacements of the normal modes ξ_r, which we also might call the *normal coordinates* of the system, are connected by the relations

$$q_i = \sum_{r=1}^{n}\varphi_i^{(r)}\xi_r \tag{6.17}$$

Hence, the coordinates $q_i (i = 1, 2, \ldots, n)$ are linear combinations of the normal coordinates ξ_r $(i = 1, 2, \ldots, n)$, and vice versa. Therefore, the oscillation problem of a conservative system can be formulated in the following way: to find linear combinations of the coordinates q_i such that the kinetic and the potential energies expressed in the new coordinates appear in the form (6.15), *i.e.*, the mixed products are zero. The solution of this problem is equivalent to the solution of the oscillation problem given in section 3 and, in general, requires the same calculations. However, in some cases it may be simpler to find directly the *transformation* between the original and the normal coordinates than to calculate the frequencies and normal modes.

Let us consider, for example, the linear oscillation of two masses treated already in section 2. Assume that the two masses are equal: $m_1 = m_2 = m$. Then the kinetic energy is equal to

$$T = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2) \tag{6.18}$$

and the potential energy:

$$U = \tfrac{1}{2}(k_{11}q_1^2 + 2k_{12}q_1q_2 + k_{22}q_2^2) \tag{6.19}$$

where $k_{11} = k_1 + k$, $k_{22} = k_2 + k$ and $k_{12} = -k$. We introduce as new coordinates linear combinations of q_1 and q_2 such that the kinetic energy keeps the form (6.18), and the potential energy becomes a sum of squares multiplied by certain coefficients. The easiest way to find such a transformation is to consider q_1 and q_2 as rectangular coordinates in a plane. Then the equation

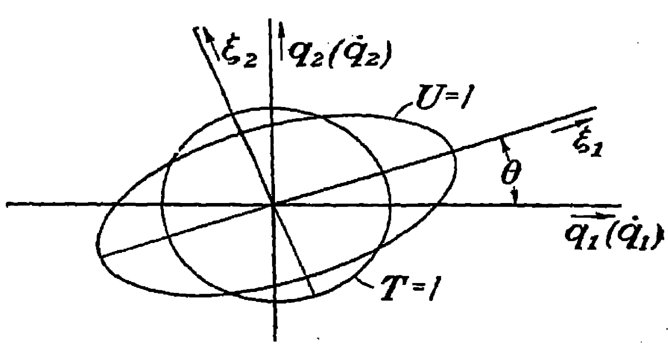

FIG. 6.1.—Graphical representation of the potential and kinetic energies of a coupled system.

$$\tfrac{1}{2}(k_{11}q_1^2 + 2k_{12}q_1q_2 + k_{22}q_2^2) = 1 \tag{6.20}$$

represents an ellipse (Fig. 6.1). [The case in which Eq. (6.20) defines a hyperbola contradicts the assumption of stability. The positions q_1,q_2, corresponding to a given U, must be restricted to a finite neighborhood of the equilibrium position, whereas a hyperbola has points at infinity.] Now it is evident that if

we turn the coordinate system so that the new axes ξ_1 and ξ_2 coincide with the principal axes of the ellipse, the equation of the ellipse becomes

$$\frac{\xi_1^2}{a^2} + \frac{\xi_2^2}{b^2} = 1$$

where a and b are its semiaxes.

Let us turn the coordinate system by the angle θ; then

$$\begin{aligned} q_1 &= \xi_1 \cos\theta - \xi_2 \sin\theta \\ q_2 &= \xi_1 \sin\theta + \xi_2 \cos\theta \end{aligned} \tag{6.21}$$

If we substitute (6.21) in Eq. (6.20), the coefficient of $\xi_1\xi_2$ becomes

$$(k_{22} - k_{11}) \sin\theta \cos\theta + k_{12}(\cos^2\theta - \sin^2\theta)$$

This coefficient vanishes if

$$\tan 2\theta = \frac{2k_{12}}{k_{11} - k_{22}} \tag{6.22}$$

Equation (6.22) yields two values for θ which differ by $\pi/2$, corresponding to the cases in which we turn the q_1-axis into the direction of the major or of the minor axis. The reader will have no difficulty in verifying—substituting (6.21) and (6.22) into (6.19)—that the coefficients of ξ_1^2 and ξ_2^2 in the expression for U become equal to $\frac{1}{2}m\omega_1^2$ and $\frac{1}{2}m\omega_2^2$, where ω_1^2 and ω_2^2 are given by Eq. (2.9). Hence,

$$U = \frac{m}{2}(\omega_1^2\xi_1^2 + \omega_2^2\xi_2)$$

Substituting

$$\begin{aligned} \dot{q}_1 &= \dot{\xi}_1 \cos\theta - \dot{\xi}_2 \sin\theta \\ \dot{q}_2 &= \dot{\xi}_1 \sin\theta + \dot{\xi}_2 \cos\theta \end{aligned}$$

into the expression (6.18) for the kinetic energy, we see immediately that

$$T = \frac{m}{2}(\dot{\xi}_1^2 + \dot{\xi}_2^2)$$

If $k_{11} = k_{22}$, according to (6.22), $\tan 2\theta = \infty$ and $\theta = \pi/4$ or $\theta = 3\pi/4$. Then from (6.21), taking $\theta = \pi/4$,

$$q_2 = \frac{1}{\sqrt{2}}(\xi_1 + \xi_2), \qquad q_1 = \frac{1}{\sqrt{2}}(\xi_1 - \xi_2)$$

or

$$\xi_1 = \frac{q_1 + q_2}{\sqrt{2}}, \qquad \xi_2 = \frac{q_2 - q_1}{\sqrt{2}} \tag{6.23}$$

In this case the normal coordinates have a simple physical meaning: ξ_1 is proportional to the displacement of the center of gravity of the two masses and ξ_2 is proportional to the relative displacement of the two masses.

In Fig. (6.1) the curve $T = 1$ is also plotted, using $\dot{q}_1$ and $\dot{q}_2$ as coordinates. Since $m_1 = m_2$, the curve $T = 1$ is a circle, which remains a circle when the coordinate system is rotated. If $m_1 \neq m_2$, $T = 1$ would represent an ellipse; in this case we might first use an affine transformation, introducing $q_1/\sqrt{m_1}$ and $q_2/\sqrt{m_2}$ as intermediary coordinates. With this transformation $T = 1$ again becomes a circle, and we can continue the calculation as we did before.

In the case of three degrees of freedom the problem of finding normal coordinates is equivalent to the problem of finding the directions of the principal axes of an ellipsoid $U = 1$, where we consider q_1, q_2, and q_3 as rectangular coordinates in space. The mathematician would consider that even for the case $n > 3$ the equation $U(q_1, q_2, \ldots, q_n) = 1$ represents a quadratic surface in n-dimensional space and would speak of the principal axes of such a surface. However, we shall refrain from expanding this idea further.

7. Forced Oscillations.—We consider in this section the motion of a statically coupled conservative system under the action of periodic external forces. We assume that the equations of motion are given in the following form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial U}{\partial q_i} = F_i \sin (\omega t + \psi) \tag{7.1}$$

where F_i is the amplitude of the ith component of the generalized external force. Substituting the expressions (3.1) and (3.2) for U and for T in (7.1), the equations of motion become $(i = 1, 2, \ldots, n)$:

$$m_i\ddot{q}_i + k_{i1}q_1 + k_{i2}q_2 + \cdots + k_{in}q_n = F_i \sin (\omega t + \psi) \tag{7.2}$$

These equations constitute a system of nonhomogeneous differential equations for the q_i's. The general solution of the system

is obtained by superposition of the general solution of the associated homogeneous system (substituting zero on the right side) and a particular solution of the nonhomogeneous system. The solution of the homogeneous system represents free oscillations where the amplitudes of the various modes of oscillation are undetermined. In this section we are especially concerned with the methods of obtaining a particular solution of the nonhomogeneous equations. We shall show that if the homogeneous problem is solved, *i.e.*, the frequencies and the modes of free oscillation are known, there is a relatively simple way of obtaining such a particular solution.

We have seen in the last section that the law of motion for each normal mode of oscillation is the same as the law of motion for a single mass. We have treated the problem of forced oscillation of a particle in section 6 of the preceding chapter. We have found especially that the amplitude c produced by a periodic force $F_0 \sin \omega t$ is equal to [cf. Chapter IV, Eq. (6.3)]

$$c = \frac{F_0}{m} \frac{1}{\omega_0^2 - \omega^2}$$

where ω_0 is the frequency of the free oscillation. Let us consider this relation from another point of view. Assume that we want to maintain an oscillation of the amplitude c with a frequency ω different from the natural frequency ω_0. Then we see that a periodic force of the amplitude

$$F_0 = mc(\omega_0^2 - \omega^2) \tag{7.3}$$

has to be applied to the mass. We shall transfer this viewpoint to the case of a statically coupled system. We consider an oscillation that is similar to the rth normal mode of oscillation to the extent that the displacements $q_1, q_2, \ldots, q_r$ are in the same ratio, however, its frequency is equal to ω, *i.e.*, equal to the frequency of the external force. Therefore, we put

$$q_i = \varphi_i^{(r)} c_r \sin (\omega t + \psi) \tag{7.4}$$

Substituting (7.4) in Eq. (7.2), we obtain

$$-m_i \varphi_i^{(r)} c_r \omega^2 + c_r \sum_{j=1}^{n} k_{ij} \varphi_j^{(r)} = F_i \tag{7.5}$$

or, taking into account that the $\varphi_i^{(r)}$'s satisfy Eq. (4.27),

$$(-m_i\varphi_i^{(r)}\omega^2 + m_i\varphi_i^{(r)}\omega_r^2)c_r = F_i$$

Hence,

$$F_i = m_i\varphi_i^{(r)}c_r(\omega_r^2 - \omega^2) \tag{7.6}$$

This equation is obviously analogous to (7.3); it states that the components of the force required to maintain a forced oscillation of the type (7.4) are proportional to the inertia factors m_i, to the coefficients of the normal mode considered, and to the difference $(\omega_r^2 - \omega^2)$. Using (7.6), we are able to solve the general problem of forced oscillations, provided we can expand the given forces $F_1, F_2, \ldots, F_n$ in sums of the form:

$$F_i = \sum_{r=1}^{n} f_r m_i \varphi_i^{(r)} \tag{7.7}$$

In fact, if the coefficients f_r are known, it follows from (7.6) and (7.7) that

$$c_r = \frac{f_r}{\omega_r^2 - \omega^2} = \frac{f_r}{\omega_r^2}\frac{1}{1 - \dfrac{\omega^2}{\omega_r^2}} \tag{7.8}$$

and, therefore, the solution of the problem is given by

$$q_i = \sum_{r=1}^{n} \varphi_i^{(r)}c_r \sin(\omega t + \psi) = \sum_{r=1}^{n} \frac{f_r\varphi_i^{(r)}}{\omega_r^2}\frac{\sin(\omega t + \psi)}{1 - \dfrac{\omega^2}{\omega_r^2}} \tag{7.9}$$

The determination of the coefficients f_r is relatively easy if we take into account the orthogonality properties of the $\varphi_i^{(r)}$'s. Let us write (7.7) explicitly, thus,

$$F_j = f_1 m_j \varphi_j^{(1)} + \cdots + f_r m_j \varphi_j^{(r)} + \cdots + f_n m_j \varphi_j^{(n)} \tag{7.10}$$

where $j = 1, 2, \ldots, n$.

If we multiply the first equation ($j = 1$) by $\varphi_1^{(r)}$, the second ($j = 2$) by $\varphi_2^{(r)}$, and so on, and add the n equations, we obtain

$$\sum_{j=1}^{n} F_j\varphi_j^{(r)} = f_r\sum_{j=1}^{n} m_j\varphi_j^{(r)2} = f_r M \tag{7.11}$$

since on the right side, owing to the orthogonality relations, all

columns vanish with the exception of the rth column. We remember that $\sum_{j=1}^{n} m_j \varphi_j^{(r)2} = M = \frac{1}{n}\sum_{j=1}^{n} m_j$ [*cf.* Eq. (4.28)].

We now solve Eq. (7.11) for f_r:

$$f_r = \frac{1}{M}\sum_{j=1}^{n} F_j \varphi_j^{(r)} \tag{7.12}$$

and call (7.12) the *coefficient formula* for the expansion of the forces F_j into *normal components*. The justification for this terminology is the following: If we consider the quantity,

$$\Phi_r = Mf_r = \sum_{j=1}^{n} F_j \varphi_j^{(r)} \tag{7.13}$$

as a force, we can show that the work done by the external forces F_i in the displacements q_i is equal to the work done by the forces Φ_r in the displacements of the normal mode ξ_r. To show this, we multiply Eq. (7.13) by ξ_r and add the n equations corresponding to $r = 1, 2, \ldots, n$. We obtain

$$\sum_{r=1}^{n} \Phi_r \xi_r = \sum_{j=1}^{n} F_j \sum_{r=1}^{n} \varphi_j^{(r)} \xi_r$$

We notice that in the case of our illustrative example (section 4), Eq. (7.13) simply means that Φ_1, Φ_2, and Φ_3 are the components of the vector F_1, F_2, F_3 in the direction of the axes ξ_1, ξ_2, and ξ_3, since the $\varphi_i^{(r)}$ are the direction cosines of the axes ξ_1, ξ_2, and ξ_3.

Remembering Eq. (6.17), we obtain

$$\sum_{r=1}^{n} \Phi_r \xi_r = \sum_{j=1}^{n} F_j q_j \qquad \text{Q.E.D.} \tag{7.14}$$

We now substitute the coefficients f_r from (7.12) in Eq. (7.9) and obtain

$$q_i = \frac{1}{M}\sum_{r=1}^{n}\sum_{j=1}^{n} \frac{F_j \varphi_i^{(r)} \varphi_j^{(r)}}{\omega_r^2} \frac{\sin(\omega t + \psi)}{1 - \frac{\omega^2}{\omega_r^2}} \tag{7.15}$$

or

$$q_i = \sum_{r=1}^{n} \frac{\Phi_r \varphi_i^{(r)}}{M\omega_r^2} \frac{\sin(\omega t + \psi)}{1 - \frac{\omega^2}{\omega_r^2}} \tag{7.16}$$

Equation (7.16) can be given a simple physical interpretation. We have seen that for the free oscillations the quantities $M\omega_r^2$ play the role of spring constants. As Φ_r is the normal component of the force system, $\Phi_r/M\omega_r^2$ is the *static deflection of the normal mode* due to the normal component Φ_r of the force system. Now we know that in the case of the linear oscillation of a single mass the static deflection is increased by the *resonance factor* $\frac{1}{1 - (\omega^2/\omega_r^2)}$. In the same way, the amplitude of the forced vibration of the type of the rth normal mode is equal to $\frac{\Phi_r}{M\omega_r^2} \frac{1}{1 - (\omega^2/\omega_r^2)}$. Multiplying the amplitudes by the coefficients of the normal modes $\varphi_i^{(r)}$ and summing over all normal modes, we transform the amplitudes of the normal modes into the amplitudes of the coordinates q_i (Eq. 7.16).

Resonance occurs if $\omega^2 \rightarrow \omega_r^2$. Near resonance the amplitudes of all normal modes are small compared to the rth mode. Hence, if resonance occurs near the natural frequency ω_r, the amplitude ratios $q_1 : q_2 : \ldots : q_n$ of the forced vibration are approximately equal to the amplitude ratios of the normal mode corresponding to ω_r.

As an interesting result of this theory we notice that the so-called *influence coefficients* of an elastic system can be expressed in terms of the parameters of the free oscillations. If we put $F_j = 1$ and all other F's equal to zero, we have from Eq. (7.15):

$$q_i = \sum_{r=1}^{n} \frac{\varphi_i^{(r)} \varphi_j^{(r)}}{\omega_r^2} \frac{\sin(\omega t + \psi)}{1 - (\omega^2/\omega_r^2)} \tag{7.17}$$

If $\omega/\omega_r \ll 1$, the external and the elastic forces are practically in equilibrium. Therefore, $q_i = \sum_{r=1}^{n} \frac{\varphi_i^{(r)} \varphi_j^{(r)}}{\omega_r^2}$ is equal to the static deflection under action of a unit force F_j. This quantity is

called the *influence coefficient* of the force F_j on the deflection q_i. We denote this coefficient by g_{ij}. The formula

$$g_{ij} = \sum_{r=1}^{n} \frac{\varphi_i^{(r)}\varphi_j^{(r)}}{\omega_r^2} \tag{7.18}$$

shows that $g_{ij} = g_{ji}$ in accordance with the well-known *reciprocity theorem* of elastic structures, which is also known as *Maxwell's theorem*.

Example.—We consider the same example which we treated in section 5 and assume that the shaft with the three disks is subjected to an alternating torque $T_0 \sin \omega t$, which is imposed on the second disk. Then $F_1 = 0$, $F_3 = 0$, and $F_2 = T_0$; $\psi = 0$. We remember that $M = I$.

Thus from (7.15) we obtain the angular displacements of the three disks, using the numerical values of the $\varphi_i^{(r)}$'s obtained in section 5,

$$q_1 = \left[\frac{(0.328)(0.591)}{\omega_1^2 - \omega^2} + \frac{(-0.736)(-0.328)}{\omega_2^2 - \omega^2} + \frac{(0.591)(-0.736)}{\omega_3^2 - \omega^2}\right]\frac{T_0}{I}\sin \omega t$$

$$q_2 = \left[\frac{(0.591)^2}{\omega_1^2 - \omega^2} + \frac{(-0.328)^2}{\omega_2^2 - \omega^2} + \frac{(-0.736)^2}{\omega_3^2 - \omega^2}\right]\frac{T_0}{I}\sin \omega t$$

$$q_3 = \left[\frac{(0.591)(0.736)}{\omega_1^2 - \omega^2} + \frac{(-0.328)(0.591)}{\omega_2^2 - \omega^2} + \frac{(-0.736)(0.328)}{\omega_3^2 - \omega^2}\right]\frac{T_0}{I}\sin \omega t$$

8. Solution of Algebraic Equations with Real Roots.—Most vibration problems involve the calculation of the roots of algebraic equations. Since the frequency equation (3.6) of section 3 has only real roots, we restrict ourselves in this chapter to the computation of real roots.

Let us assume that $f(x)$ is a polynomial in x of degree n with real coefficients. Then if we plot the curve $y = f(x)$ for real values of x, the real roots of the equation $f(x) = 0$ will be represented by the intersections of the curve $y = f(x)$ with the x-axis. If the number of such intersections is less than n,* say equal to m, there are $(n - m)$ complex roots. If $\alpha + i\beta$ is a root of $f(x) = 0$, $\alpha - i\beta$ is also a root, and therefore $n - m$ must be an even number.

Since $f(x)$ is a continuous function, if $f(a)$ and $f(b)$ have opposite signs, there is at least one real root between $x = a$ and

* We shall exclude the case in which the curve $y = f(x)$ is tangent to the axis for $x = \alpha$, *i.e.*, $x = \alpha$ is a multiple root.

$x = b$. The methods of determining the exact number of real roots without plotting $y = f(x)$ will not be treated here because of their restricted practical value. However, a simple rule will be mentioned that gives an upper limit for the number of real roots between $x = 0$ and $x = \infty$; in other words, for the number of positive roots. This rule, given by Descartes, states that the number of positive roots of $f(x) = 0$ is equal to or less than the number of changes of sign between successive coefficients of the polynomial $f(x)$. Descartes has also shown that if the number of positive roots is smaller than the number of changes of sign, the difference between these two numbers must be even. If we now investigate the equation $f(-x) = 0$, the positive real roots of this equation correspond to the negative roots of the original equation. Hence by means of the rule of Descartes we obtain an upper limit for the number of the real roots and a lower limit for that of the complex roots.

For example, the equation $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$ cannot have positive roots, because all coefficients are positive. Replacing x by $-x$, we obtain $-x^5 + x^4 - x^3 + x^2 - x + 1 = 0$. The coefficients show five changes of sign. Hence, the number of negative roots can be 1, 3, or 5. In fact, there is one negative root, $x = -1$; the other roots are complex.

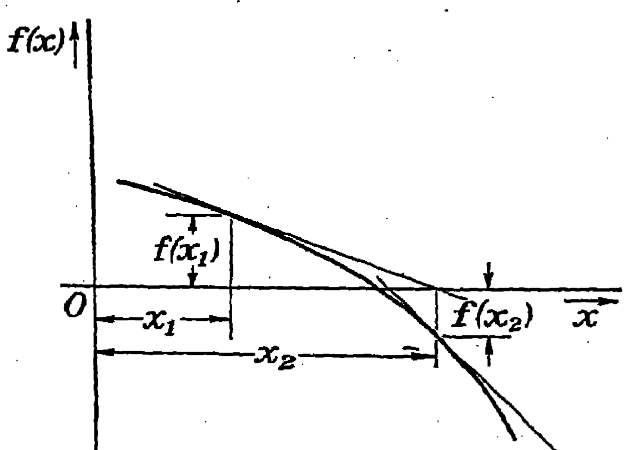

FIG. 8.1.—Newton's method for finding the roots of an algebraic equation.

Methods for Computing Real Roots. a. Newton's Method.—Assume that $x = x_1$ is a first approximation for a real root. Then replacing the curve $f(x)$ (Fig. 8.1) between $x = x_1$ and the unknown root by a straight line, we write

$$f(x) = f(x_1) + f'(x_1)(x - x_1)$$

From $f(x) = 0$ it follows that

$$x = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{8.1}$$

We denote the value of x corresponding to (8.1) by x_2 and repeat

the procedure by writing

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \tag{8.2}$$

and so on.

Let us apply this method to the Eq. (5.5), writing x instead of λ:

$$f(x) = x^3 - 5x^2 + 6x - 1 = 0 \tag{8.3}$$

The values of $f(x)$ for $x = 0, 1, 2, 3$ are $f(0) = -1$, $f(1) = 1$, $f(2) = -1$, $f(3) = -1$. For very large values of x, $f(x) > 0$. Consequently, there is one root between 0 and 1, one between 1 and 2, and one between 3 and ∞. Let us calculate the smallest one. If x is between 0 and 1, $x^3 < x^2 < x$; hence, we first neglect the terms with x^2 and x^3 and start with $x_1 = \frac{1}{6}$. Now $f(\frac{1}{6}) = -\frac{29}{216}$; the derivative of $f(x)$ is $f'(x) = 3x^2 - 10x + 6$ and $f'(\frac{1}{6}) = \frac{53}{12}$. The second approximation according to (8.1) is equal to $x_2 = \dfrac{1}{6} + \dfrac{29}{53 \times 18} = 0.1971$. Using x_2 to get the third approximation we obtain from Eq. (8.2), $x_3 = 0.1971 + \dfrac{0.0042}{4.146} = 0.1981$. The value of x_4 is found to be the same as that of x_3; therefore, the root calculated is accurate to the fourth decimal place.

b. Iteration Method.—Let us write

$$f(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

in the form:

$$x = -\frac{1}{a_{n-1}}(x^n + a_1x^{n-1} + \cdots + a_{n-2}x^2 + a_n) = g(x)$$

If $x = x_1$ is the exact value of a root, the two sides of the equation are exactly equal. If $x = x_1$ is a first approximate value of the root, we can obtain a second approximation by substituting $x = x_1$ on the right side and putting $x_2 = g(x_1)$.

Obviously, instead of the linear term, we can choose any of the terms containing an arbitrary power of x for solving formally the equation $f(x) = 0$. For example, to calculate the largest root, it will be more practical, especially if a_1 is not large in comparison with unity, to write

$$x = \sqrt[n]{-(a_1x^{n-1} + \cdots + a_n)}$$

and substitute $x = x_1$ in the radical on the right side.

Take the same example, $x^3 - 5x^2 + 6x - 1 = 0$. We calculate the root between 0 and 1 by the iteration method. Writing $x = \frac{1 + 5x^2 - x^3}{6}$ and substituting $x_1 = \frac{1}{6}$, we obtain

$$x_2 = \tfrac{245}{1296} = 0.1890$$

repeating the process,

$$x_3 = 0.1953, \qquad x_4 = 0.1972, \qquad x_5 = 0.1979, \qquad x_6 = 0.1981.$$

However, in many cases the method will converge very slowly.

c. Squaring the Roots (Graeffe's Method).—If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation $f(x) = 0$, the polynomial $f(x)$ can be written in the form $f(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$. Then obviously, $f(-x) = (-1)^n(x + \alpha_1)(x + \alpha_2) \cdots (x + \alpha_n)$. We multiply the two expressions and find

$$f(x)f(-x) = (-1)^n(x^2 - \alpha_1^2)(x^2 - \alpha_2^2) \cdots (x^2 - \alpha_n^2)$$

In other words, multiplying the two expressions $f(x)$ and $f(-x)$, we obtain a new algebraic equation of the order n, whose roots are $\alpha_1^2, \alpha_2^2, \ldots, \alpha_n^2$, *i.e.*, the squares of the roots of $f(x)$. Let us repeat this procedure ν times; then the roots will be $\alpha_1^{2\nu}$, $\alpha_2^{2\nu}, \ldots, \alpha_n^{2\nu}$. Let us assume that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real and are arranged so that $|\alpha_{k+1}| < |\alpha_k|$ (the case in which the absolute values of two or more roots are equal is excluded for the time being). Then in the sequence $\alpha_1^{2\nu}, \alpha_2^{2\nu}, \ldots, \alpha_n^{2\nu}$ each term will be much larger than all the following terms provided that ν is sufficiently large, *i.e.*, provided we carry out the squaring a sufficient number of times.

Hence, by the repeated multiplication of $f(x)$ and $f(-x)$ we obtain an algebraic equation whose roots are all of different orders of magnitude. Let us assume that the equation obtained by means of the procedure repeated ν times is

$$(x^{2\nu})^n + b_1(x^{2\nu})^{n-1} + \cdots + b_n = 0$$

or with $y = x^{2\nu}$

$$y^n + b_1y^{n-1} + \cdots + b_n = 0 \tag{8.4}$$

Then we can show that, owing to the different order of magnitude of the roots, the formulas

$$y_1 \cong -b_1, \qquad y_2 \cong -\frac{b_2}{b_1}, \qquad y_n \cong -\frac{b_n}{b_{n-1}} \tag{8.5}$$

e approximate values for the roots, arranged in order of decreas-
; magnitude. The coefficients of (8.4) are given by

$$\begin{aligned} b_1 &= -(y_1 + y_2 + \cdots + y_n) \\ b_2 &= (y_1y_2 + y_1y_3 + \cdots + y_{n-1}y_n) \\ &\cdots\cdots\cdots\cdots\cdots \\ b_n &= (-1)^n y_1y_2 \cdots y_n \end{aligned} \tag{8.6}$$

the roots are all very different and are arranged so that
$> y_2 > \cdots > y_n$, it is evident that in each of the expres-
ns (8.6) the first term is large compared to the rest of the terms.
nce, we have

$$\cong -y_1, \quad b_2 \cong y_1y_2, \quad \cdots, \quad b_{n-1} \cong (-1)^{n-1} y_1y_2 \cdots y_{n-1}, \qquad b_n \cong (-1)^n y_1y_2 \cdots y_n \tag{8.7}$$

om (8.7) the Eqs. (8.5) follow immediately.
This means that Eq. (8.4) is approximately equivalent to the
torized equation

$$(y + b_1)(y + b_2/b_1) \cdots (y + b_n/b_{n-1}) = 0 \tag{8.8}$$

We return to the original variable by substituting $y = x^{2^\nu}$.
en denoting the roots of the original equation by $\alpha_1, \alpha_2,$
$\cdot$, α_n, we have

$$= \sqrt[2^\nu]{-b_1}, \quad \alpha_2 = \sqrt[2^\nu]{-\frac{b_2}{b_1}}, \quad \cdots, \quad \alpha_n = \sqrt[2^\nu]{-\frac{b_n}{b_{n-1}}} \tag{8.9}$$

This method is especially successful if all roots are real.
mplex conjugate roots having equal absolute magnitudes can-
t be separated by the procedure of squaring, and, therefore,
e above conclusions do not apply without modification.
It is seen that the formulas (8.9) leave the signs of the real
ts $\alpha_1, \ldots, \alpha_n$ undetermined; the signs must be decided by
tting $f(x) = 0$ or by actually substituting the α's into the
ation $f(x) = 0$.
Take again, as an example, the equation

$$x^3 - 5x^2 + 6x - 1 = 0$$

e first squaring leads to

$$(x^3 - 5x^2 + 6x - 1)(x^3 + 5x^2 + 6x + 1) = -x^6 + 13x^4 - 26x^2 + 1 = 0.$$

The first approximations are

$$\alpha_1 = \sqrt{13} = 3.6056, \quad \alpha_2 = \sqrt{\tfrac{26}{13}} = 1.414, \quad \alpha_3 = \sqrt{\tfrac{1}{26}} = 0.196$$

The second squaring yields

$$-(x^6 - 13x^4 + 26x^2 - 1)(x^6 + 13x^4 + 26x^2 + 1) = -x^{12} + 117x^8 - 650x^4 + 1 = 0$$

Then our second approximations are

$$\alpha_1 = \sqrt[4]{117} = 3.289, \quad \alpha_2 = \sqrt[4]{\tfrac{650}{117}} = 1.535, \quad \alpha_3 = \sqrt[4]{\tfrac{1}{650}} = 0.1981.$$

The third approximations are

$$\alpha_1 = 3.248, \qquad \alpha_2 = 1.554, \qquad \alpha_3 = 0.1981,$$

and the fourth approximations,

$$\alpha_1 = 3.247, \qquad \alpha_2 = 1.555, \qquad \alpha_3 = 0.1981.$$

9. Solution of the Frequency Equation and Calculation of the Normal Modes by Use of Matrices.—The methods presented in the last section presume that the coefficients of the frequency equation are numerically known. Now the left side of the frequency equation appears in general in the form of a determinant. Hence, in many cases the numerical work will be reduced if we can avoid the calculation of the coefficients of the algebraic equation and use the coefficients of the linear equations immediately for the calculation of the frequencies and the normal modes.

The system of equations to be solved was given by Eq. (3.3):

$$m_i\ddot{q}_i + \sum_{j=1}^{n} k_{ij}q_j = 0 \tag{9.1}$$

Putting $q_i = A_i e^{i\omega t}$, we obtained n linear equations for the coefficients A_i in the form

$$m_iA_i\omega^2 = \sum_{j=1}^{n} k_{ij}A_j \tag{9.2}$$

Let us divide both sides by m_i; then we have

$$A_i\omega^2 = \sum_{j=1}^{n} \frac{k_{ij}}{m_i}A_j \tag{9.3}$$

These equations have solutions when ω^2 is equal to one of the roots of the frequency equation. Then the A_i's represent the principal mode of oscillation corresponding to this particular frequency. The solution of the oscillation problem consists of calculating the numerical values for ω^2 and the A_i's.

We shall use the iteration method for the solution of Eqs. (9.3). Denote the right side of Eqs. (9.3) by N_i, so that

$$\sum_{j=1}^{n} \frac{k_{ij}}{m_i} A_j = N_i \tag{9.4}$$

and Eqs. (9.3) become

$$N_i = A_i \omega^2 \tag{9.5}$$

We start with an arbitrary set $A_1, A_2, \ldots, A_n$; for convenience only we choose $A_n = 1$. Then from Eq. (9.4) we calculate $N_1, N_2, \ldots, N_n$. If $A_1, A_2, \ldots, A_n$ represented an exact solution of Eqs. (9.3), we should have

$$N_1 : N_2 : \cdots : N_{n-1} : N_n = A_1 : A_2 : \cdots : A_{n-1} : 1 \tag{9.6}$$

and

$$\frac{N_1}{A_1} = \frac{N_2}{A_2} = \cdots = \frac{N_n}{1} = \omega^2 \tag{9.7}$$

Hence, we use $N_1/N_n, N_2/N_n, \ldots, N_{n-1}/N_n, 1$, as new values for the A_i's for a new trial and calculate $N_1, N_2, N_3, \ldots, N_n$ again.

It is seen that if the procedure converges, the values N_1/N_n, $N_2/N_n, \ldots, N_n/N_n$ converge to definite values $A_1, A_2, \ldots, A_n$ which represent solutions of Eqs. (9.5), *i.e.*, coefficients of the principal modes, and $N_1/A_1, N_2/A_2, \ldots, N_n/A_n$ converge to a fixed value ω^2. It can be shown that the value ω^2 which we obtain by this iteration method is always the largest root of the frequency equation, and the values $N_1/N_n, N_2/N_n, \ldots, N_n/N_n$ are the coefficients of the principal mode corresponding to the highest frequency.

Consider for simplicity's sake a system with three degrees of freedom. Suppose that the exact values of the coefficients of the normal modes are $\varphi_1^{(1)}, \varphi_2^{(1)}, \varphi_3^{(1)}; \varphi_1^{(2)} \ldots$ Then the arbitrarily chosen initial values ${}_0A_1$, ${}_0A_2$, and ${}_0A_3$ can be written in the form:

$$\begin{aligned} {}_0A_1 &= \varphi_1^{(1)}\xi_1 + \varphi_1^{(2)}\xi_2 + \varphi_1^{(3)}\xi_3 \\ {}_0A_2 &= \varphi_2^{(1)}\xi_1 + \varphi_2^{(2)}\xi_2 + \varphi_2^{(3)}\xi_3 \\ {}_0A_3 &= \varphi_3^{(1)}\xi_1 + \varphi_3^{(2)}\xi_2 + \varphi_3^{(3)}\xi_3 \end{aligned} \tag{9.8}$$

where ξ_1, ξ_2, and ξ_3 are constant coefficients. Since every set of the $\varphi_i^{(r)}$'s satisfies individually Eqs. (9.3), *e.g.*,

$$\sum_{j=1}^{3} \frac{k_{ij}}{m_i}\varphi_j^{(r)} = \varphi_i^{(r)}\omega_r^2 \tag{9.9}$$

we obtain, by substituting (9.8) in Eqs. (9.4) and using (9.9),

$$\begin{aligned} {}_1N_1 &= \omega_1^2\varphi_1^{(1)}\xi_1 + \omega_2^2\varphi_1^{(2)}\xi_2 + \omega_3^2\varphi_1^{(3)}\xi_3 \\ {}_1N_2 &= \omega_1^2\varphi_2^{(1)}\xi_1 + \omega_2^2\varphi_2^{(2)}\xi_2 + \omega_3^2\varphi_2^{(3)}\xi_3 \\ {}_1N_3 &= \omega_1^2\varphi_3^{(1)}\xi_1 + \omega_2^2\varphi_3^{(2)}\xi_2 + \omega_3^2\varphi_3^{(3)}\xi_3 \end{aligned} \tag{9.10}$$

We now compute the new set of A_i's: ${}_1A_1 = {}_1N_1/{}_1N_3$, ${}_1A_2 = {}_1N_2/{}_1N_3$, and ${}_1A_3 = 1$. Substituting the new values ${}_1A_1$, ${}_1A_2$, and ${}_1A_3$ into (9.4), we obtain

$$\begin{aligned} {}_2N_1 &= \frac{1}{{}_1N_3}(\omega_1^4\varphi_1^{(1)}\xi_1 + \omega_2^4\varphi_1^{(2)}\xi_2 + \omega_3^4\varphi_1^{(3)}\xi_3) \\ {}_2N_2 &= \frac{1}{{}_1N_3}(\omega_1^4\varphi_2^{(1)}\xi_1 + \omega_2^4\varphi_2^{(2)}\xi_2 + \omega_3^4\varphi_2^{(3)}\xi_3) \\ {}_2N_3 &= \frac{1}{{}_1N_3}(\omega_1^4\varphi_3^{(1)}\xi_1 + \omega_2^4\varphi_3^{(2)}\xi_2 + \omega_3^4\varphi_3^{(3)}\xi_3) \end{aligned} \tag{9.11}$$

Continuing this process, it is seen that if, for example, ω_3 is the largest of the three frequencies, the influence of the terms containing ω_1 and ω_2 becomes decreasingly small compared with the terms containing ω_3. Hence,

$$\begin{aligned} \lim_{\nu\to\infty} {}_\nu A_1 &= \lim_{\nu\to\infty} \frac{{}_\nu N_1}{{}_\nu N_3} = \frac{\varphi_1^{(3)}}{\varphi_3^{(3)}} \\ \lim_{\nu\to\infty} {}_\nu A_2 &= \lim_{\nu\to\infty} \frac{{}_\nu N_2}{{}_\nu N_3} = \frac{\varphi_2^{(3)}}{\varphi_3^{(3)}} \end{aligned} \tag{9.12}$$

or in the limit

$$A_1 : A_2 : A_3 = \varphi_1^{(3)} : \varphi_2^{(3)} : \varphi_3^{(3)} \qquad \text{Q.E.D.} \tag{9.13}$$

We shall now show that, if one of the frequencies and the coefficients of one normalized mode are known, we can reduce the number of degrees of freedom by one.

Let us call the largest root ω_n^2 and assume that ω_n^2 and the corresponding set of values $A_1^{(n)}, A_2^{(n)}, \ldots, A_n^{(n)}$ are determined with sufficient accuracy. Then we proceed to the calculation of the next root by using the orthogonality relation proved in section 4. It was found that if $A_1^{(r)}, \ldots, A_n^{(r)}$ and $A_1^{(s)}, \ldots,$

$A_n^{(s)}$ are the coefficients of two normal modes of oscillations, the relation $\sum_{i=1}^{n} m_i A_i^{(r)} A_i^{(s)} = 0$ holds, provided $r \neq s$. If we now assume that a new set of values $A_1, A_2, \ldots, A_n$ represents a solution of the oscillation problem in which the principal mode corresponding to ω_n^2 is not present, it follows that

$$m_1 A_1^{(n)} A_1 + m_2 A_2^{(n)} A_2 + \cdots + m_n A_n^{(n)} A_n = 0 \qquad (9.14)$$

Using (9.14) we eliminate one of the coefficients, *e.g.*, A_n, and drop the corresponding last equation ($i = n$) in the system (9.3). The rest of the equations—after eliminating A_n—determine the oscillations of a mechanical system with $n - 1$ degrees of freedom, whose frequencies and normal modes are identical with $n - 1$ frequencies and normal modes of the original system. Applying the same method as above, we obtain the next largest root ω_{n-1}^2 and the corresponding set of values $A_1^{(n-1)}$, $A_2^{(n-1)}, \ldots, A_n^{(n-1)}$.

It is seen that the $A_i^{(r)}$ values obtained by this method differ by only a numerical factor from the coefficients $\varphi_i^{(r)}$, which we called the *coefficients of the normal mode*. As a matter of fact, the whole difference is that the A's are standardized in such a manner that $A_n = 1$, whereas the φ_i's satisfy the relation

$$\sum_{i=1}^{n} m_i \varphi_i^{(r)2} = M = \frac{1}{n} \sum_{i=1}^{n} m_i \qquad (9.15)$$

The practical application of the method explained in this section is facilitated if we use the quadratic scheme of coefficients $K_{ij} = k_{ij}/m_i$, which we call a *square matrix* $[K]$. (We note that $K_{ij} \neq K_{ji}$, hence, $[K]$ is, in general, not symmetric.) The values $A_1, \ldots, A_n$ constitute a *column matrix* $[A]$. Then we write symbolically Eq. (9.4) in the following way, placing the two matrices side by side,

$$\begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \\ \cdot \\ \cdot \\ \cdot \\ N_n \end{bmatrix} \qquad (9.16)$$

It is seen that the operation indicated by the left side of Eqs. (9.4) amounts to a multiplication of the elements of each row of the square matrix $[K]$ by the corresponding elements of the column matrix $[A]$, and a summation of the products. We start with an arbitrary ${}_0[A]$, and obtain ${}_1[N]$. Then dividing the elements of the latter by the last element ${}_1N_n$, we obtain a new matrix ${}_1[A]$ and repeat the operation. The operation indicated by $[K]\,[A]$ is called a *multiplication of matrices.*

Example.—Let us apply this scheme to the example treated in section 5, *i.e.*, to the torsional oscillation of three masses connected by an elastic shaft. Eqs. (9.3) for this system become

$$
\begin{aligned}
\frac{c}{I}(\ 2A_1 - \ A_2) \qquad\quad &= A_1\omega^2 \\
\frac{c}{I}(-A_1 + 2A_2 - A_3) &= A_2\omega^2 \\
\frac{c}{I}(\qquad\ - \ A_2 + A_3) &= A_3\omega^2
\end{aligned}
\tag{9.17}
$$

For convenience we multiply both sides by I/c. We start with the arbitrary values ${}_0A_1 = 1$, ${}_0A_2 = -1$, and ${}_0A_3 = 1$; we choose alternate signs for the A's because we know that for the higher normal modes of oscillation the masses oscillate with approximately opposite phases. Then the scheme of matrices according to (9.16) will be

$$
\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}
\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}
=
\begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}
= 2
\begin{bmatrix} \frac{3}{2} \\ -2 \\ 1 \end{bmatrix}
$$

The next steps are:

$$
\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}
\begin{bmatrix} \frac{3}{2} \\ -2 \\ 1 \end{bmatrix}
=
\begin{bmatrix} 5 \\ -\frac{13}{2} \\ 3 \end{bmatrix}
= 3
\begin{bmatrix} \frac{5}{3} \\ -\frac{13}{6} \\ 1 \end{bmatrix}
$$

$$
\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}
\begin{bmatrix} \frac{5}{3} \\ -\frac{13}{6} \\ 1 \end{bmatrix}
=
\begin{bmatrix} \frac{11}{2} \\ -7 \\ \frac{19}{6} \end{bmatrix}
= \frac{19}{6}
\begin{bmatrix} \frac{33}{19} \\ -\frac{42}{19} \\ 1 \end{bmatrix}
$$

$$
\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}
\begin{bmatrix} \frac{33}{19} \\ -\frac{42}{19} \\ 1 \end{bmatrix}
=
\begin{bmatrix} \frac{108}{19} \\ -\frac{136}{19} \\ \frac{61}{19} \end{bmatrix}
= \frac{61}{19}
\begin{bmatrix} \frac{108}{61} \\ -\frac{136}{61} \\ 1 \end{bmatrix}
$$

$$
\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}
\begin{bmatrix} \frac{108}{61} \\ -\frac{136}{61} \\ 1 \end{bmatrix}
=
\begin{bmatrix} \frac{352}{61} \\ -\frac{441}{61} \\ \frac{197}{61} \end{bmatrix}
= \frac{197}{61}
\begin{bmatrix} \frac{352}{197} \\ -\frac{441}{197} \\ 1 \end{bmatrix}
$$

The factors 2, 3, $\frac{19}{6}$, $\frac{61}{19}$, $\frac{197}{61}$ constitute a sequence of approximate values of $I\omega^2/c$. Using the last column on the right side, we easily calculate a further approximation which is equal to $\frac{638}{197}$. The sequence of approximate values consists of 2, 3, 3.167, 3.211, 3.230, 3.238. The ratios between the coeffi-

cients of the corresponding normal modes of oscillation are 352: −441:197 or 1.787: −2.238:1.

It is seen that even in the case in which the first arbitrary set of values is relatively far away from the correct values, after a few steps reasonable accuracy is reached. For instance, if we should continue the process in the above example, the next step would change the results by less than 1 per cent; the new values would be 1.795: −2.243:1. A great advantage of the method is that incidental numerical errors only slow down the convergence of the approximation and do not influence the final result.

In many practical applications the lowest frequency is of paramount interest. Hence, we should like to compute the smallest root of the frequency equation directly without starting with the largest root. This can be accomplished easily if the set of equations (9.3) is somewhat modified. We remember that in Eqs. (9.1) the generalized forces are given by $F_i = -\sum_{j=1}^{n} k_{ij}q_j$. From these relations we can express the coordinates q_i as linear functions of the forces F_i, say, in the form:

$$q_i = -\sum_{j=1}^{n} g_{ij}F_j \tag{9.18}$$

The coefficients g_{ij} are analogous to the influence numbers used in statics; for instance, in the case of an elastic system they represent the deflection due to a unit force.

Now the equations of motion (9.1) can be written in the form $F_j = m_j\ddot{q}_j$. Hence, we obtain by substitution of F_j in Eqs. (9.18)

$$q_i = -\sum_{j=1}^{n} m_j g_{ij}\ddot{q}_j \tag{9.19}$$

Putting $q_i = A_i e^{i\omega t}$, $m_j g_{ij} = \gamma_{ij}$ (where, in general, $\gamma_{ij} \neq \gamma_{ji}$):

$$A_i = \omega^2 \sum_{j=1}^{n} \gamma_{ij}A_j$$

We now divide both sides by ω^2 and write

$$\frac{1}{\omega^2}A_i = \sum_{j=1}^{n} \gamma_{ij}A_j \tag{9.20}$$

It is seen that the set of equations (9.20) is analogous to (9.3) or (9.16) with the difference that the matrix $[K]$ is replaced by

the matrix $[\Gamma]$ of the coefficients γ_{ij} and the factor of A_i is $1/\omega^2$ instead of ω^2. Hence, if we apply, as before, the method of multiplication of matrices, we obtain the largest value of $1/\omega^2$, *i.e.*, the smallest root ω^2 and the corresponding normal mode of oscillation.

Example.—Taking again the same example, we first write

$$\begin{aligned} -F_1 &= c(\ 2q_1 - \ \ q_2) \\ -F_2 &= c(-q_1 + 2q_2 - q_3) \\ -F_3 &= c(\ \ \ \ \ \ \ \ - \ \ q_2 + q_3) \end{aligned} \tag{9.21}$$

Solving these equations for q_1, q_2, q_3, we obtain

$$\begin{aligned} -q_1 &= \frac{1}{c}(F_1 + \ F_2 + \ F_3) \\ -q_2 &= \frac{1}{c}(F_1 + 2F_2 + 2F_3) \\ -q_3 &= \frac{1}{c}(F_1 + 2F_2 + 3F_3) \end{aligned}$$

Hence, substituting $F_i = I\ddot{q}_i$ and introducing $q_i = A_i e^{i\omega t}$, we obtain the following system of equations:

$$\begin{aligned} \frac{c}{I\omega^2}A_1 &= A_1 + A_2 + A_3 \\ \frac{c}{I\omega^2}A_2 &= A_1 + 2A_2 + 2A_3 \\ \frac{c}{I\omega^2}A_3 &= A_1 + 2A_2 + 3A_3 \end{aligned}$$

We choose as a first approximation $A_1 = A_2 = A_3 = 1$. Then the scheme of matrices leads to

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} \frac{1}{2} \\ \frac{5}{6} \\ 1 \end{bmatrix}$$

The subsequent steps are:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{5}{6} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{14}{6} \\ \frac{25}{6} \\ \frac{31}{6} \end{bmatrix} = \frac{31}{6} \begin{bmatrix} \frac{14}{31} \\ \frac{25}{31} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{14}{31} \\ \frac{25}{31} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{70}{31} \\ \frac{126}{31} \\ \frac{157}{31} \end{bmatrix} = \frac{157}{31} \begin{bmatrix} \frac{70}{157} \\ \frac{126}{157} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{70}{157} \\ \frac{126}{157} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{353}{157} \\ \frac{636}{157} \\ \frac{793}{157} \end{bmatrix} = \frac{793}{157} \begin{bmatrix} \frac{353}{793} \\ \frac{636}{793} \\ 1 \end{bmatrix}$$

The successive approximations of $I\omega^2/c$ are $\frac{1}{6}$, $\frac{6}{31}$, $\frac{31}{157}$, and $\frac{157}{793}$ or 0.1667, 0.1935, 0.1975, and 0.1980. The values of the ratios $A_1 : A_2 : A_3$ are, according to the last approximation calculated, 0.445:0.802:1.

The Case of Zero Frequency.—If $\omega^2 = 0$ is a root of the frequency equation (3.6), the system has at least one degree of unrestrained freedom of motion. If we put in this case

$$F_i = \sum_{j=1}^{n} k_{ij} q_j = 0$$

where $i = 1, 2, \cdots, n$, the equilibrium position

$$q_1 = q_2 = \cdots = q_n = 0$$

is not the unique solution of this homogeneous system of equations because the determinant of the coefficients k_{ij} is equal to zero. This means that the equations of motion are not independent, *i.e.*, there are certain linear relations between the displacements. In general, it is quite easy to find these relations by application of the general theorems given in Chapter III for arbitrary systems of mass points. Consider, for example, a shaft which carries n disks of the moment of inertia $I_1, I_2, \ldots, I_n$ and can rotate freely. It is known that the total moment of momentum must be constant unless external forces are acting on the system. This must hold for every mode of oscillation; therefore, if $q_1 = A_1 \sin \omega t$, $q_2 = A_2 \sin \omega t$, $\cdots$, $q_n = A_n \sin \omega t$,

$$(I_1 A_1 + I_2 A_2 + \cdots + I_n A_n)\omega = 0 \tag{9.22}$$

By means of this condition we can eliminate one coordinate, and we are left with a system of $n - 1$ degrees of freedom, where $\omega^2 = 0$ is no longer a root of the frequency equation.

It is interesting to notice that Eq. (9.22) may be considered also as the orthogonality condition between an arbitrary principal mode $A_1, A_2, \ldots, A_n$ and the principal mode corresponding to the displacement of the system as a rigid body, which is evidently given by $q_1 = q_2 = \cdots = q_n = C$, where C is an arbitrary constant. However, the unrestrained motion is not always that of a rigid body (*e.g.*, in the case of shafts connected by gears), and then the orthogonality relation may be most useful to eliminate the free motion.

Example.—Consider a shaft with four identical disks, and assume that the kinetic and the potential energies of the system are given by the following expressions:

$$T = \frac{I}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + \dot{q}_4^2) \tag{9.23}$$

$$U = \frac{c}{2}[(q_1 - q_2)^2 + (q_2 - q_3)^2 + (q_3 - q_4)^2] \tag{9.24}$$

Then the equations for the q's are, according to Lagrange's method,

$$\begin{aligned} \frac{I\omega^2}{c}q_1 &= q_1 - q_2 \\ \frac{I\omega^2}{c}q_2 &= -q_1 + 2q_2 - q_3 \\ \frac{I\omega^2}{c}q_3 &= -q_2 + 2q_3 - q_4 \\ \frac{I\omega^2}{c}q_4 &= -q_3 + q_4 \end{aligned} \tag{9.25}$$

The q's must satisfy Eq. (9.22), hence,

$$q_1 + q_2 + q_3 + q_4 = 0 \tag{9.26}$$

This also follows from addition of the four equations (9.25).

We can now take, for example, the first three equations (9.25) and eliminate q_4 by the use of (9.26). In this way we obtain

$$\begin{aligned} \frac{I\omega^2}{c}q_1 &= q_1 - q_2 \\ \frac{I\omega^2}{c}q_2 &= -q_1 + 2q_2 - q_3 \\ \frac{I\omega^2}{c}q_3 &= q_1 + 3q_3 \end{aligned} \tag{9.27}$$

The three equations (9.27) are now independent, and with $\lambda = I\omega^2/c$, we obtain the characteristic equation

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 4 - 10\lambda + 6\lambda^2 - \lambda^3 = 0 \tag{9.28}$$

The problem can be completed by use of any of the methods given in this and the preceding section.

10. Example of a Conservative System with Dynamic Coupling—Double Pendulum.—Up to this point we have assumed that the expression for the potential energy contains coupling

terms, *i.e.*, terms with products of the coordinates, but that the kinetic energy appears as a sum of squares multiplied with certain inertia coefficients. We shall now drop this last restriction and assume that the kinetic energy contains also terms with the products of the velocities. We call such terms *dynamic coupling terms* and say that there is *dynamic coupling* between the degrees of freedom that are defined by the generalized coordinates. An appropriate example of such a system is the *double pendulum*.

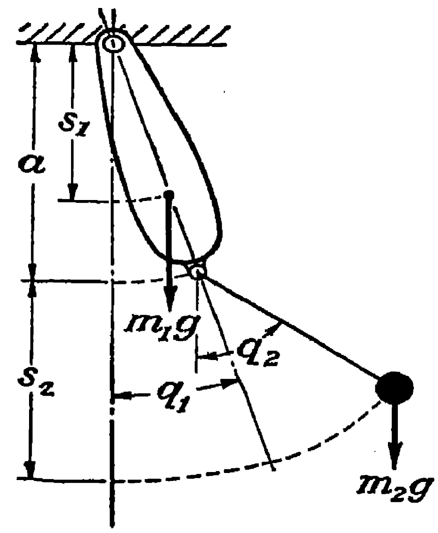

FIG. 10.1.—A double pendulum.

Let us assume that a double pendulum consists of a compound pendulum rotating around a fixed axis, called the axis of suspension, and a mathematical pendulum rotating around an axis fixed to the first pendulum, called the hinge axis (Fig. 10.1). We use the following notations:

m_1, m_2 for the masses of the two pendulums.
$I_1 = m_1 i_1^2$ for the moment of inertia of the compound pendulum.
a for the distance between the axes of suspension.
s_1 for the distance of the center of gravity of m_1 from the axis of suspension.
s_2 for the distance of the center of gravity of m_2 from the hinge axis.

We choose as coordinates the angular displacement q_1 of the first pendulum and the angular displacement q_2 of the second pendulum relative to the vertical direction. Both q_1 and q_2 are assumed to be small.

The kinetic energy of the first pendulum is equal to $\frac{1}{2}I_1\dot{q}_1^2$, while the kinetic energy of the simple pendulum is

$$\tfrac{1}{2}m_2(a\dot{q}_1 + s_2\dot{q}_2)^2$$

Hence, the kinetic energy of the system is given by

$$T = \tfrac{1}{2}\left[(I_1 + m_2a^2)\dot{q}_1^2 + 2m_2as_2\dot{q}_1\dot{q}_2 + m_2s_2^2\dot{q}_2^2\right] \qquad (10.1)$$

The potential energy of the system is given by the following expression:

$$U = -m_1gs_1 \cos q_1 - m_2g(a \cos q_1 + s_2 \cos q_2) + \text{const.} \qquad (10.2)$$

We shall assume that $U = 0$ in the equilibrium position. Expanding the trigonometric functions and retaining only terms of lowest order, we obtain

$$U = \frac{g}{2}[(m_1s_1 + m_2a)q_1^2 + m_2s_2q_2^2] \tag{10.3}$$

The equations of motion are easily obtained by Lagrange's method as follows:

$$\begin{aligned} (I_1 + m_2a^2)\ddot{q}_1 + m_2as_2\ddot{q}_2 &= -g(m_1s_1 + m_2a)q_1 \\ m_2as_2\ddot{q}_1 + m_2s_2^2\ddot{q}_2 &= -gm_2s_2q_2 \end{aligned} \tag{10.4}$$

If we substitute $q_1 = A_1 \sin(\omega t + \psi)$ and $q_2 = A_2 \sin(\omega t + \psi)$, we obtain the frequency equation, which is an equation of second degree in ω^2 and gives the frequencies of the two normal modes of oscillation.

A church bell and its clapper can be considered as a double pendulum; hence, Eqs. (10.4) can be used for the explanation of the curious phenomenon that some church bells remain silent when put into oscillation. Obviously this can occur if one of the normal modes does not involve relative motion between the bell and the clapper.

Let us investigate under what conditions $q_1 - q_2 = 0$ can represent a principal mode of oscillation. Substituting

$$q_1 = q_2 = q$$

into Eqs. (10.4), we obtain

$$\begin{aligned} (I_1 + m_2a^2 + m_2as_2)\ddot{q} &= -g(m_1s_1 + m_2a)q \\ (m_2as_2 + m_2s_2^2)\ddot{q} &= -gm_2s_2q \end{aligned} \tag{10.5}$$

The two equations are compatible if

$$\frac{I_1 + m_2a^2 + m_2as_2}{m_2as_2 + m_2s_2^2} = \frac{m_1s_1 + m_2a}{m_2s_2}$$

or

$$\begin{aligned} \frac{I_1}{m_2s_2(a + s_2)} + \frac{a}{s_2} &= \frac{m_1s_1}{m_2s_2} + \frac{a}{s_2} \\ \frac{I_1}{m_1s_1} &= a + s_2 \end{aligned} \tag{10.6}$$

Equation (10.6) can be given a simple physical interpretation. The length $l = I_1/m_1s_1$ is the length of a mathematical pendulum

whose frequency is the same as the frequency of the compound pendulum considered. In other words, we might concentrate the whole mass of the bell at a point P at a distance l from the axis of suspension, and the frequency of oscillation would not change. Obviously, to replace the compound pendulum the point P must be on the line connecting the axis of suspension and the center of gravity. This point is called the *center of percussion* of the compound pendulum.

Hence, Eq. (10.6) states that the bell may be silent if the center of gravity of the clapper coincides with the center of percussion of the bell. This conclusion has been confirmed by observations of the giant bell of the famous Cologne Cathedral in Germany.

Returning to the general problem of the double pendulum, we notice that whether the system is dynamically or statically coupled actually depends on the choice of the coordinates. As we defined the degrees of freedom by the coordinates q_1 and q_2 used above, we found dynamic coupling but no static coupling. However, if we should use, for example, the following linear combinations as coordinates:

$$\xi_1 = q_1$$
$$\xi_2 = q_2 + \frac{a}{s_2}q_1$$

we would obtain

$$T = \tfrac{1}{2}(I_1\dot{\xi}_1^2 + m_2s_2^2\dot{\xi}_2^2) \tag{10.7}$$

and

$$U = \frac{g}{2}\left[\left(m_1s_1 + m_2a + m_2\frac{a^2}{s_2}\right)\xi_1^2 - 2m_2a\xi_1\xi_2 + m_2s_2\xi_2^2\right] \tag{10.8}$$

Hence, if we use the coördinates ξ_1 and ξ_2, there is static coupling but no dynamic coupling between the degrees of freedom. In the next section we shall show that the most general case of static and dynamic coupling can be reduced to the case of static coupling only.

11. General Remarks on Systems with Dynamic Coupling.— Let us assume that the kinetic energy of the system is given in the general form:

$$T = \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j \tag{11.1}$$

Then the equations of motion appear in the form given in section 1 (Eq. 1.10). Putting $q_i = A_i \sin(\omega t + \psi)$, we obtain the following system of linear equations for the A_i's:

$$\omega^2(m_{i1}A_1 + m_{i2}A_2 + \cdots + m_{in}A_n) = k_{i1}A_1 + k_{i2}A_2 + \cdots + k_{in}A_n \qquad (11.2)$$

The frequency equation appears now in the form:

$$\begin{vmatrix} \omega^2 m_{11} - k_{11} & \cdots & \omega^2 m_{1n} - k_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \omega^2 m_{n1} - k_{n1} & \cdots & \omega^2 m_{nn} - k_{nn} \end{vmatrix} = 0 \qquad (11.3)$$

We again obtain n values for ω^2, say $\omega_1^2, \ldots, \omega_r^2, \ldots, \omega_n^2$ and n sets of values $A_1^{(r)}, A_2^{(r)}, \ldots, A_n^{(r)}$ for the coefficients of the n principal modes.

Of course it is always possible to evaluate the coefficients of the frequency equation by expanding the determinant in Eq. (11.3). However, in general, it will be more practical to use the matrix method in the following way: with the notations $N_i = A_i\omega^2$, the Eqs. (11.2) represent linear relations between the N_i's and the A_i's:

$$m_{i1}N_1 + m_{i2}N_2 + \cdots + m_{in}N_n = k_{i1}A_1 + k_{i2}A_2 + \cdots + k_{in}A_n \qquad (11.4)$$

where $i = 1, 2, \ldots, n$.

Let us solve the system for $N_1, N_2, \ldots, N_n$. Assume that we obtain

$$N_i = K_{i1}A_1 + K_{i2}A_2 + \cdots + K_{in}A_n \qquad (11.5)$$

Then the system of equations (11.5) can be written in matrix form:

$$\begin{bmatrix} N_1 \\ \cdot \\ \cdot \\ N_n \end{bmatrix} = \begin{bmatrix} K_{11} & \cdots & K_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ K_{n1} & \cdots & K_{nn} \end{bmatrix} \begin{bmatrix} A_1 \\ \cdot \\ \cdot \\ A_n \end{bmatrix} \qquad (11.6)$$

We now can start with an arbitrary set of values for $A_1, A_2, \ldots, A_n$ and apply the iteration method explained in section 9. We obtain in this way the highest frequency and the mode of oscillation corresponding to this frequency.

On the other hand, we can solve the system of linear equations (11.4) also for $A_1, A_2, \ldots, A_n$. Assume that we have

$$A_i = \gamma_{i1}N_1 + \gamma_{i2}N_2 + \cdots + \gamma_{in}N_n \tag{11.7}$$

where $i = 1, 2, \cdots, n$.

Now, remembering that $N_i = A_i\omega^2$ or $A_i = N_i/\omega^2$, we obtain from Eq. (11.7)

$$\frac{N_i}{\omega^2} = \gamma_{i1}N_1 + \gamma_{i2}N_2 + \cdots + \gamma_{in}N_n \tag{11.8}$$

where again $i = 1, 2, \cdots, n$. In matrix form:

$$\frac{1}{\omega^2}\begin{bmatrix} N_1 \\ \cdot \\ \cdot \\ N_n \end{bmatrix} = \begin{bmatrix} A_1 \\ \cdot \\ \cdot \\ A_n \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{bmatrix} \begin{bmatrix} N_1 \\ \cdot \\ \cdot \\ N_n \end{bmatrix} \tag{11.9}$$

Solving Eq. (11.9) by iteration, we obtain the lowest value of ω^2 and the corresponding mode of oscillation. It is seen that Eqs. (11.2) are reduced by (11.5) and (11.8) to the types of problems we solved in section 9.

The orthogonality relation for the principal modes in case of dynamic coupling has the form:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} A_i^{(r)} A_j^{(s)} = 0 \tag{11.10}$$

where $r \neq s$. It is seen that (11.10) is a sum of n^2 terms. Also the normalizing condition must be modified. We put

$$\sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} A_i^{(r)} A_j^{(r)} = M \tag{11.11}$$

where M is a suitably chosen constant of the dimension of the inertia coefficients m_{ij}. The values $A_i^{(r)}$ which satisfy Eq. (11.11) can be considered as the coefficients $\varphi_i^{(r)}$ of the rth normal mode.

If the expression of the kinetic energy (11.1) is given, it is always possible to find a linear combination of the coordinates $q_1, \ldots, q_n$ which transforms the kinetic energy into a sum of

squares of the velocities and thus reduces the dynamically coupled system to one which has static coupling only. In some cases the transformation of coordinates will actually be the shortest way of solving the oscillation problem. In other cases it will be simpler to solve the Eqs. (11.6) or (11.9).

Problems

1. A mass under the action of gravity is restrained to move without friction on the paraboloid

$$z = 2x^2 + 2xy + y^2$$

The z-axis is directed upward. Calculate the small oscillations of the mass near the origin.

2. What is the motion near the origin if the paraboloid in Prob. 1 is replaced by the quadric

$$z = 2x^2 + 2xy - y^2$$

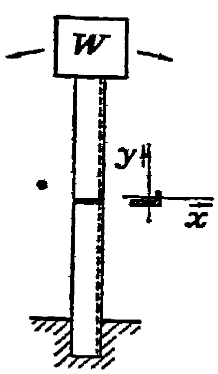

Fig. P.3.

3. An angle with unequal legs is clamped to a fixed horizontal base in a vertical position carrying at its free end a weight of 1 lb. The cross section of the angle is shown in Fig. P.3. A force of 1 lb. applied to the free end of the beam in the x-direction produces a deflection equal to 0.04 in. in the x-direction and 0.05 in. in the y-direction. A force of 1 lb. applied in the y-direction produces a deflection 0.15 in. in the y-direction and 0.05 in. in the x-direction. Find the directions and frequencies of the principal oscillations (neglecting the mass of the beam and the influence of the bending moment produced by the weight).

4. Find the natural frequencies of oscillation of a solid cube of concrete resting on springs at its four lower corners. The cube is 3 ft. in size. The springs are such that they deflect $\frac{1}{2}$ in. vertically under the weight of the cube and $\frac{1}{16}$ in. laterally under a horizontal force equal to one-third of its weight and applied to the base of the cube. Find the principal modes of vibration.

5. A small rotary pump is driven by an electric motor. The whole unit is mounted on a board supported by four identical helical springs whose

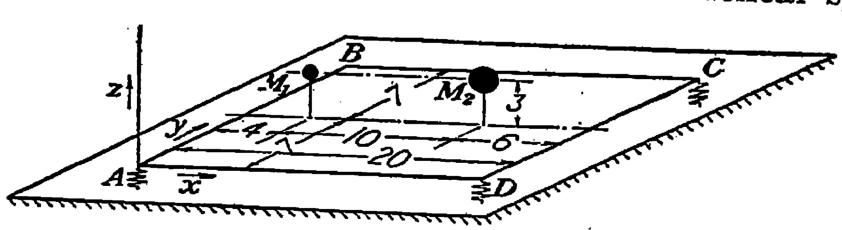

Fig. P.5.

spring constants k in the vertical and horizontal directions are equal. The number of revolutions of the motor is 1,400 per minute, that of the pump

600. The weight of the pump is 21 lb., that of the motor 31.5 lb. The location of the springs and the approximate mass distribution are shown in Fig. P.5. The masses of the pump and the motor are concentrated at M_1 and M_2, respectively; all other masses are neglected.

a. Determine the spring constant k so that no frequency is between 500 and 700 or above 1,100 per minute.

b. Determine the amplitudes of the displacements of the points of suspension caused by the eccentricity of the pump rotor. The axis of the rotor is parallel to the y-axis and passes through the point M_1; the eccentric rotating mass weighs 0.5 lb. and its eccentricity amounts to $\frac{1}{2}$ in.; $k = 70$ lb./in.

Hint: The unbalance moves in the vertical plane of symmetry passing through M_1M_2 and, therefore, excites only the three modes in that plane.

The system of the masses M_1,M_2, and the board, like any rigid system, has six degrees of freedom. We note that one principal oscillation, *viz.*, the rotation around the axis M_1M_2, has an infinite frequency. This is due to our assumption that the masses are concentrated at M_1 and M_2.

6. The hoist shown schematically in Fig. P.6 carries a weight W of 500 lb. The deflections of the springs S_1 and S_2 are equal to 1 in. per 1-ton load. The truck is prevented from rolling.

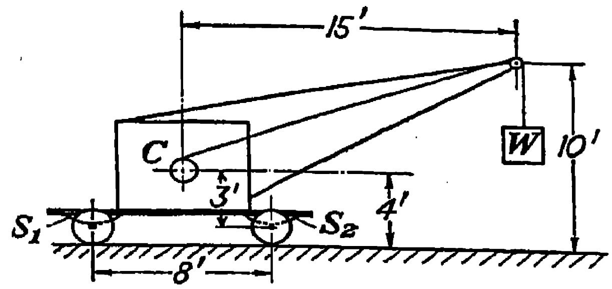

FIG. P.6.

a. Find the principal modes of oscillation of the system assuming that all masses move in a vertical plane and that the mass of the structure and the hoisting mechanism is replaced by a concentrated mass of 5 tons at the point C. The structure itself is considered rigid. Neglect the horizontal motion of W.

b. Assume that the weight W is hoisted up. A nonuniform speed $v_0 \sin \frac{2\pi t}{T}$ where $v_0 = 1$ ft./sec. and $T = 0.55$ sec. is superposed on the mean motion. Find the amplitudes of oscillation of the springs S_1 and S_2, neglecting the horizontal displacement of the weight W.

Hint: We may neglect the influence of gravity on the oscillation. If q_1 and q_2 are the deflections of the springs, the potential energy is

$$U = \tfrac{1}{2}k(q_1^2 + q_2^2)$$

The kinetic energy may be expressed in terms of $\dot{q}_1$ and $\dot{q}_2$. For example, since we neglect the horizontal motion of W, its kinetic energy is $\frac{1}{2}\frac{W}{g}\left[\frac{1}{2}(\dot{q}_1 + \dot{q}_2) + \frac{15}{8}(\dot{q}_2 - \dot{q}_1)\right]^2$. We can express the kinetic energy of the mass located at C in the same way.

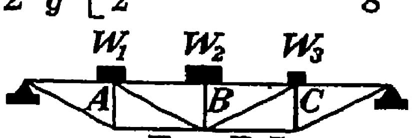

FIG. P.7.

7. The truss structure shown in Fig. P.7 carries three weights, $W_1 = 10$ tons, $W_2 = 15$ tons, and $W_3 = 5$ tons at A, B, C. The influence coefficients of the structure are given by the following table:

Measured at the point	Deflections in inches per 1-ton load applied—		
	At A	At B	At C
A	0.10	0.10	0.08
B	0.10	0.12	0.10
C	0.08	0.10	0.10

The mass of the structure is neglected.

a. Find the natural frequencies of the system and the modes of oscillation.

b. Assume that an engine with a vertical cylinder is running at the point B and its piston of 20 lb. weight is unbalanced. The length of the piston stroke is equal to 6 in. The weight of the engine is included in $W_2 = 15$ tons. Find the amplitudes of oscillation at the points A, B, and C as functions of the number of revolutions per minute.

Hint: The effect of the engine at B is the same as that of an external force equal to the inertia force. This alternating inertia force is in pounds, $\frac{20}{32.2} \cdot \frac{1}{4} \cdot \omega^2 \sin \omega t$ where ω is the angular velocity of the engine crank.

8. A radial airplane engine is supported by an elastic mounting at four points, A, B, C, D, equally spaced on a circular ring of radius r (Fig. P.8). The elastic properties of the mounting are chosen in such a way that a displacement τ parallel to the mount circle produces only a restoring force T in the same direction; however, a displacement ξ in the x-direction produces a force X in the x-direction and a radial force R; similarly, a displacement ρ in the radial direction produces a restoring force in the radial direction and in the x-direction. The equations for this are

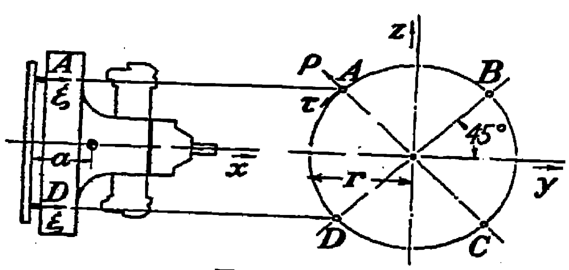

FIG. P.8.

$$T = k_{11}\tau$$
$$X = k_{22}\xi + k_{23}\rho$$
$$R = k_{23}\xi + k_{33}\rho$$

The engine block has axial symmetry, and its center of gravity is located on the x-axis a distance a ahead of the plane of the mounting.

a. What relation must prevail between a, r, and the spring constants to make the translatory oscillations of the block in the x-, y-, and z-directions and the rotational oscillations about the same axes uncoupled?

b. Show that if this condition is fulfilled, the resultant of the forces T, X, and R produced by a displacement ζ in the z-direction passes through the center of gravity.

Hint: It is seen from reasons of symmetry that the translation in the z-direction is coupled only with the rotation around the y-axis. To find the condition for elimination of this coupling, give to the engine an arbitrary displacement ζ parallel to the z-axis; calculate the corresponding values of

ρ at the four points of suspension and the forces X and R acting at the same four points. Express the condition for the vanishing of the moment of these forces with respect to the center of gravity.

9. A man traveling in a special train observes that the oscillation of the water in a bath tub is in resonance sidewise with the oscillation of the car. The bath tub is 2 ft. wide and the water stands 10 in. high. Determine the period of the sidewise oscillation from these data. Assume that the cross section of the bath tub is rectangular. Assume that the water surface stays plane, neglect the vertical component of the velocity of the water particles, and assume that the horizontal velocity is independent of the depth.

Hint: The magnitude of the velocity in a vertical cross section is determined by the condition of constant volume. Calculate the kinetic energy and the potential energy as functions of the inclination of the water surface. The kinetic energy has to be evaluated by integration. The potential energy is immediately given by the displacement of the water mass.

10. A uniform shaft free to rotate in bearings carries five equidistant disks. The moments of inertia of four disks are equal to I, while the moment of inertia of one of the end disks is equal to $2I$. Calculate the lowest frequency and the corresponding mode by using matrices.

Hint: Calling $q_1, q_2, \ldots q_5$ the angular deflections, the potential energy is $U = \frac{1}{2}c\,[(q_1 - q_2)^2 + (q_2 - q_3)^2 + (q_3 - q_4)^2 + (q_4 - q_5)^2]$ and the kinetic energy is $T = \frac{1}{2}\,I(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + \dot{q}_4^2 + 2\dot{q}_5^2)$. The zero frequency mode (free rotation) will be eliminated by the condition

$$q_1 + q_2 + q_3 + q_4 + 2q_5 = 0$$

This leaves us with four degrees of freedom.

11. In Prob. 10 calculate, by using matrices, the two highest frequencies and the corresponding modes of vibration.

12. A uniform shaft free to rotate in bearings carries four equidistant disks of equal moments of inertia I. The distance between the disks is l (Fig. P.12). At one end, at a distance $2l$ from the fourth disk, there are a flywheel and a small pinion of total moment of inertia $5I$. The pinion drives a gear connected to a rotor. The gear and rotor have a moment of inertia $100I$. The gear turns six times slower than the pinion. Determine the lowest frequency of the system.

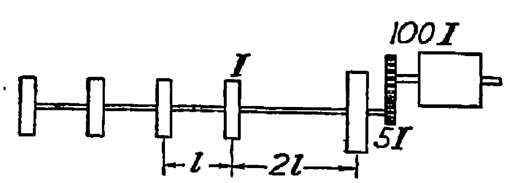

FIG. P.12.

Hint: With q_1, q_2, q_3, and q_4, as the angular deflection of the disks and q_5 as that of the flywheel, the potential energy is

$$U = \tfrac{1}{2}c\,[(q_1 - q_2)^2 + (q_2 - q_3)^2 + (q_3 - q_4)^2 + \tfrac{1}{2}(q_4 - q_5)^2]$$

The kinetic energy is

$$T = \frac{I}{2}\left[\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + \dot{q}_4^2 + \left(5 + \frac{100}{36}\right)\dot{q}_5^2\right]$$

Assume first $q_5 = 0$ for an approximate calculation. Compare the result with the result obtained by eliminating the free rotation by the condition

$$q_1 + q_2 + q_3 + q_4 + (5 + \tfrac{100}{36})q_5 = 0$$

13. The oscillations of the crankshaft of a four-cylinder engine can be calculated approximately by replacing the oscillating masses by four identical disks of the moment of inertia I. The torsional stiffness of the crankshaft is C, and the distance between the disks is equal to l. Assume that a propeller is connected with the crankshaft by an elastic coupling whose spring constant is denoted by k. The distance between the coupling and the last disk is l. Discuss the influence of k on the lowest frequency, varying k between the limits $k = 0$ (free wheeling) and $k = \infty$ (rigid coupling). Carry out the calculation under the assumption that the moment of inertia of the propeller may be taken infinite.

Hint: Calculate the value of $\omega^2 I/c$ (ω = lowest frequency) for different values of the ratio k/c.

14. A four-story building has an elastic frame structure. The mass is concentrated at each floor; one-half at the second floor; the other half equally distributed between the third and fourth floors. Assume the shearing rigidities between the floors to be the same. Determine the natural frequencies of the building and the modes of vibration.

15. What is the forced oscillation of the building (Prob. 14) if the ground vibrates horizontally in a pure harmonic motion of known amplitude and frequency?

Hint: Let the motion of the ground be $x = x_0 \sin \omega t$. The acceleration is $a = -\omega^2 x_0 \sin \omega t$. The motion of the building relative to the ground is the same as if each floor were submitted to an alternating force $-ma = m\omega^2 x_0 \sin \omega t$, where m is the mass of that floor (cf. Chapter IV, section 6). These forces may be used for $F_i \sin \omega t$ in the expressions developed in section 7 of this chapter for the forced oscillations. The coefficients $\varphi_i^{(r)}$ of the normal modes can be calculated from the results of Prob. 14.

16. Calculate the real root of the equation

$$x^3 - 6x^2 + 12x - 6 = 0$$

by Newton's method.

17. Calculate the roots of the equation

$$x^4 - 2x^3 - 13x^2 + 14x + 24 = 0$$

by the method of root squaring.

References

General:

1. Reference 2, Chapter III; references 2 to 4, Chapter IV.
2. ROUTH, E. J.: "A Treatise on the Dynamics of a System of Rigid Bodies," vol. 2, Macmillan & Company, Ltd., London, 4th ed., 1884.
3. LORD RAYLEIGH: "The Theory of Sound," Macmillan & Company, Ltd., London, 1929.
4. FÖPPL, A.: "Vorlesungen über Technische Mechanik," vol. VI, B. G. Teubner, Leipzig, 1920.

Algebraic equations and matrices:

5. Bôcher, M.: "Introduction to Higher Algebra," The Macmillan Company, New York, 1924.

6. Frazer, R. A., W. J. Duncan, and A. R. Collar: "Elementary Matrices and Some Applications to Dynamics and Differential Equations," Cambridge University Press, London, 1938.

CHAPTER VI

SMALL OSCILLATIONS OF NONCONSERVATIVE SYSTEMS

> "It has been long understood that approximate solutions of problems in the ordinary branches of Natural Philosophy may be obtained by a species of *abstractions*, or rather *limitations of the data*, such as enables us easily to solve the modified form of the question, while we are well assured that the circumstances (so modified) affect the result only in a superficial manner."
>
> —LORD KELVIN and PETER GUTHRIE TAIT, "Treatise on Natural Philosophy."

Introduction.—The mechanical systems treated in the preceding chapter are characterized completely by their kinetic and potential energies; especially the forces that represent the responses of the system to given displacements appear as derivatives of the potential energy with respect to these displacements. In this chapter we are concerned with oscillation problems that involve also forces which cannot be derived from a potential energy. We use for such forces, and also for the systems in which they are involved, the term *nonconservative*. The term *nonconservative* is strictly correct for the case of so-called *dissipative* systems, where mechanical or electrical energy is actually transformed into heat. It appears, however, that in some other cases the nonconservative character of the forces is due, to some extent, to the method of analysis. For example, we shall see that the system consisting of an airplane wing and a flap is *nonconservative* in the sense that it can absorb energy from or lose energy to the surrounding air. The total system including the air, wing, and flap—provided viscosity is neglected—is conservative. Hence, the conservative or nonconservative character depends on the way we isolate our mechanical system. Another interesting example is the vertical top. A gyroscope is a conservative system with three degrees of freedom. However, if we are interested only in the oscillations of the symmetry axis, we can ignore one of the coordinates, *viz.*, the angular displacement of the top

around this axis. In fact, this angular displacement itself does not enter into either the potential or the kinetic energy of the system. Thus, we consider the top as a system with two degrees of freedom. However, owing to the ignored rotation of the top, a deflection of the symmetry axis produces forces which depend on the velocities and of course are not derivatives of the potential energy of the system. We call these forces also nonconservative and use for them the term *gyroscopic forces*. They are in reality inertia forces. In Lagrange's equations for the complete system (*i.e.*, for the system with three degrees of freedom) the gyroscopic forces appear as derivatives of the kinetic energy with respect to the coordinates.

Whereas in the previous chapter we restricted ourselves a priori to stable systems, the conditions under which a certain system is stable are of paramount interest for the type of problems treated in this chapter. Therefore, section 6 is devoted to the stability criteria for characteristic equations of third and fourth degree, and section 7 deals with methods for the computation of the complex roots of algebraic equations.

1. Small Oscillations of Nonconservative Systems. General Remarks.—In a conservative system a set of displacements $q_1, q_2, \ldots, q_n$ produces forces $Q_1, Q_2, \ldots, Q_n$ that are derivatives of the potential energy of the system. In the vicinity of the equilibrium position $q_1 = q_2 = \cdots = q_n = 0$ they can be expressed by the linear expressions

$$Q_i = -\frac{\partial U}{\partial q_i} = -\sum_{j=1}^{n} k_{ij} q_j \tag{1.1}$$

where $i = 1, 2, \cdots, n$ and $k_{ij} = k_{ji}$.

Now there are systems for which the forces depend on the coordinates and in the neighborhood of the equilibrium position are given by linear functions of the coordinates, while the system is nevertheless not conservative. This will be the case if, in the expressions

$$Q_i = -\sum_{j=1}^{n} k_{ij} q_j \tag{1.2}$$

at least one pair of the coefficients k_{ij} and k_{ji} are unequal.

If we calculate the work done by the forces in a closed cycle, *i.e.*, in a process for which the values of the coordinates are the

same at the beginning and at the end of the process, we shall find that, if $k_{ij} \neq k_{ji}$, the work does not identically vanish; hence, the condition $k_{ij} = k_{ji}$ is *necessary* for a system to be conservative. On the other hand, if for all i's and j's, $k_{ij} = k_{ji}$, it is evident that the forces Q_i are derivatives of the function $-U = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}q_iq_j$; hence, $k_{ij} = k_{ji}$ is a *sufficient* condition.

In some cases we shall introduce forces that are *linear functions of the velocity components*. As a simple example, consider the motion of a pendulum in a very viscous fluid such as oil. The frictional drag of the pendulum can be assumed to be a linear function of the velocity $\dot{q}$. Hence, we write for the damping force

$$Q = -\beta\dot{q} \tag{1.3}$$

The work done per unit time by the damping force is equal to

$$W = -\beta\dot{q}^2 \tag{1.4}$$

The quantity $-W$ is the time rate of energy dissipation. We call

$$D = \tfrac{1}{2}\beta\dot{q}^2 \tag{1.5}$$

the *dissipation function*. Then,

$$-\frac{\partial D}{\partial \dot{q}} = Q \tag{1.6}$$

i.e., the damping force is the negative of the derivative of the dissipation function with respect to the velocity.

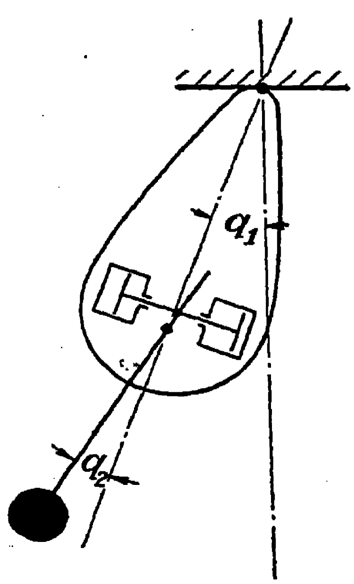

FIG. 1.1.—Double pendulum with internal damping.

Of great practical importance is the case in which the damping depends on the difference between the derivatives of two coordinates, *i.e.*, on the relative velocity between two degrees of freedom. Let us consider, for example, a double pendulum and assume that the relative displacement of the second pendulum is damped by a damping device (Fig. 1.1). If q_1 denotes the angular displacement of the first pendulum and q_2 the angular displacement of the second pendulum, the damping will be proportional to the relative angular velocity $\dot{q}_2 - \dot{q}_1$, and the dissipation function appears in the form

$$D = \frac{\beta}{2}(\dot{q}_2 - \dot{q}_1)^2$$

The generalized damping forces are

$$Q_1 = -\frac{\partial D}{\partial \dot{q}_1} = -\beta\dot{q}_1 + \beta\dot{q}_2$$
$$Q_2 = -\frac{\partial D}{\partial \dot{q}_2} = \beta\dot{q}_1 - \beta\dot{q}_2$$

If we write this in the form:

$$Q_1 = -\beta_{11}\dot{q}_1 - \beta_{12}\dot{q}_2$$
$$Q_2 = -\beta_{21}\dot{q}_1 - \beta_{22}\dot{q}_2$$

we notice that we have

$$\beta_{12} = \beta_{21} = -\beta$$

In the general case of n degrees of freedom, we assume that the forces that depend on the velocities can be written in the form:

$$Q_i = -\sum_{j=1}^{n} \beta_{ij}\dot{q}_j \tag{1.7}$$

Then the work, done per unit time by these forces is

$$W = \sum_{i=1}^{n} Q_i\dot{q}_i = -\sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij}\dot{q}_i\dot{q}_j \tag{1.8}$$

Let us now investigate two special cases:

a. Assume that $\beta_{ij} = \beta_{ji}$. Then it is seen by differentiation of (1.8) that

$$\frac{\partial W}{\partial \dot{q}_i} = -\sum_{j=1}^{n} \beta_{ij}\dot{q}_j - \sum_{j=1}^{n} \beta_{ji}\dot{q}_j = -2\sum_{j=1}^{n} \beta_{ij}\dot{q}_j$$

Therefore, in this case the function

$$D = -\tfrac{1}{2}W = \tfrac{1}{2}\sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij}\dot{q}_i\dot{q}_j \tag{1.9}$$

has the property that its derivatives with respect to the velocities taken with negative sign are equal to the dissipative forces:

$$-\frac{\partial D}{\partial \dot{q}_i} = Q_i \tag{1.10}$$

D is a positive definite quadratic function, for the work done by dissipative forces is always negative. Equation (1.9) is the general form for the *dissipation* function. The system itself is also said to be *dissipative.* The dissipative forces can be included in Lagrange's equation by adding a term $\partial D/\partial \dot{q}_i$. We then have:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q} + \frac{\partial D}{\partial \dot{q}} = 0 \tag{1.11}$$

b. Assume that $\beta_{ij} = -\beta_{ji}$, from which it follows that $\beta_{ii} = 0$. In this case $W = 0$, *i.e.*, the forces do no work. We call such forces *gyroscopic forces* because they are a generalization of certain forces produced by the rotation of rigid bodies. We shall encounter such forces in section 5 dealing with the small oscillations of the vertical top. Other examples of gyroscopic forces are the electromagnetic forces acting on a charged particle moving in a magnetic field, and the Coriolis force acting on a body moving on the surface of the earth. These forces have the characteristic property that they are perpendicular to the velocity of the point upon which they act. Let us consider, for instance, the case of a system with two degrees of freedom and interpret q_1 and q_2 as Cartesian coordinates. Denoting the velocity vector $\dot{q}_1,\dot{q}_2$ by $\bar{v}$ and the force vector Q_1,Q_2 by $\bar{Q}$, from

$$\begin{aligned} Q_1 &= \beta_{12}\dot{q}_2 \\ Q_2 &= -\beta_{12}\dot{q}_1 \end{aligned} \tag{1.12}$$

it follows that the scalar product

$$\bar{Q} \cdot \bar{v} = Q_1\dot{q}_1 + Q_2\dot{q}_2 = 0 \tag{1.13}$$

i.e., the force is perpendicular to the velocity and the work done by the forces is zero; *i.e.*, the gyroscopic forces do not cause dissipation.

2. Example of a Nonconservative System. Elementary Theory of Wing Flutter.—Our first example of a nonconservative system is a simplified model for the study of certain flutter phenomena which occur in airplane structures at high speed of flight. In certain cases and at certain speeds violent oscillations of airplane control surfaces are observed. Such oscillations begin, in general, with small amplitudes, and often increase to the extent that the structure fails. This suggests instability of the coupled mechanical system consisting of the wing and the

control surface. Let us consider a half wing and an aileron as a *mechanical system*. We consider the wing as an elastic structure, clamped to the airplane body, which is considered as a rigid base. Then the wing has two degrees of freedom; corresponding to bending and twisting. The relative deflection of the aileron, whose elastic deflection is neglected, represents a third degree of freedom. A discussion of the vibration of such a *ternary* system is rather complicated. However, in most cases the interaction between any two of the three degrees of freedom is practically

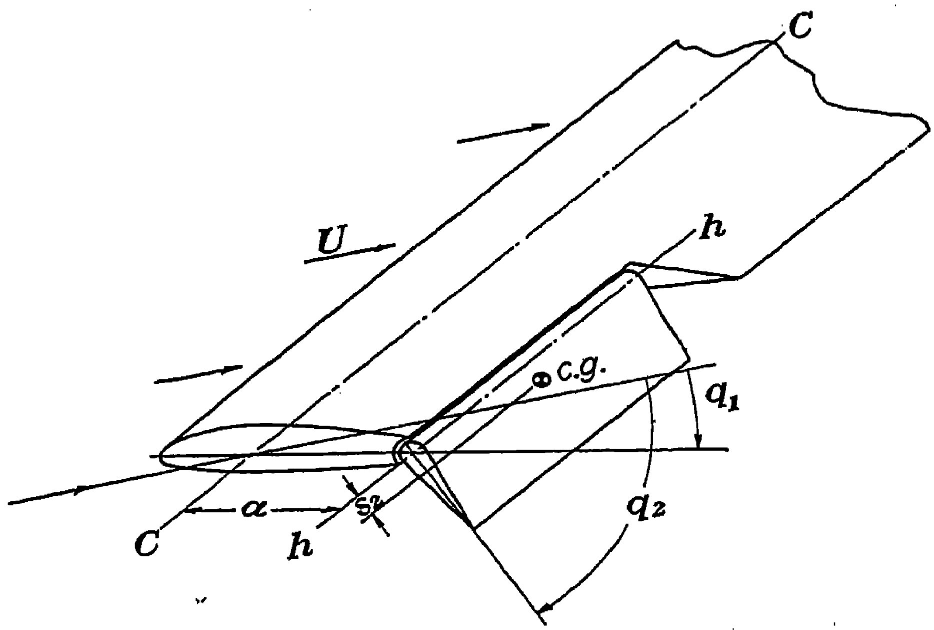

FIG. 2.1.—Diagram of wing and aileron.

independent of the third one so that the system in each case may be considered approximately as a *binary* system.

The accompanying model simulates the case of torsional vibration of a wing combined with oscillations of an aileron.

The wing (Fig. 2.1) is represented by a rigid cylindrical body with a streamline section that is elastically restrained to a fixed axis CC (corresponding to the so-called *elastic axis* of the wing structure). The aileron is hinged at the axis hh. The two axes are assumed to be parallel, and the distance between them is denoted by a. The moment of inertia of the wing about the axis is I_1, the mass of the aileron is m_2, and its moment of inertia about the hinge axis is equal to $m_2 i_2^2$ where i_2 is the radius of gyration of the aileron. Finally, the distance of the center of gravity of the aileron from the hinge axis hh is denoted by s_2, measured as positive if the center of gravity is behind the hinge axis.

We use as coordinates the angular deflection q_1 of the wing and the angular deflection q_2 of the aileron relative to the direction of flight. Then the kinetic energy is expressed in the form:

$$T = \tfrac{1}{2}(A\dot{q}_1^2 + 2B\dot{q}_1\dot{q}_2 + C\dot{q}_2^2) \tag{2.1}$$

where

$$\begin{aligned} A &= I_1 + m_2 a^2 \\ B &= m_2 a s_2 \\ C &= m_2 i_2^2 \end{aligned} \tag{2.2}$$

The expressions (2.1) and (2.2) can be obtained by applying the results of Chapter III. The wing alone is considered as a rigid body rotating around a fixed axis. Since its angular velocity is equal to $\dot{q}_1$, its kinetic energy is equal to $\frac{1}{2}I_1\dot{q}_1^2$. In order to calculate the kinetic energy of the aileron, Eq. (8.19) of Chapter III can be applied. The velocity of the center of gravity is $v_c = a\dot{q}_1 + s_2\dot{q}_2$, and the angular velocity about the center of gravity is $\omega = \dot{q}_2$. The moment of inertia of the aileron with respect to the center of gravity is given by $I = m_2(i_2^2 - s_2^2)$. Substituting these values in Eq. (8.19) of Chapter III, we obtain for the kinetic energy of the aileron

$$\tfrac{1}{2}m_2(a\dot{q}_1 + s_2\dot{q}_2)^2 + \tfrac{1}{2}m_2(i_2^2 - s_2^2)\dot{q}_2^2 = \tfrac{1}{2}m_2(a^2\dot{q}_1^2 + 2as_2\dot{q}_1\dot{q}_2 + i_2^2\dot{q}_2^2)$$

in accordance with Eqs. (2.1) and (2.2).

The generalized forces in this case are moments, and we assume that they are linear functions of the angular deflections. In addition to the moments that are proportional to the angular deflections, actually there are also moments proportional to the angular velocities and to the angular accelerations. The moments that depend on the angular velocities are governed by rather complicated physical laws, resulting from the aerodynamics of airfoils in nonstationary flight. Though it can be assumed that the moments concerned are linear functions of the angular velocities $\dot{q}_1$ and $\dot{q}_2$, the coefficients of these linear functions are complicated functions of the frequency. The forces that are proportional to the angular acceleration have their origin in the inertia of the air oscillating with the wing and are known as *apparent-mass forces*.

In the following simple analysis both the forces depending on the velocities and the apparent-mass forces are neglected. The latter could be added to the inertia forces proper, but they

do not bring any new aspect into the analysis. The inclusion of the forces proportional to the velocities would render the calculations and the discussions somewhat cumbersome. The idealized model given in the text reveals the main features of the process of flutter and does not involve complicated calculations.

Hence, we write

$$
\begin{aligned}
Q_1 &= -k_{11}q_1 - k_{12}q_2 \\
Q_2 &= -k_{21}q_1 - k_{22}q_2
\end{aligned}
\tag{2.3}
$$

The physical significance of the terms in (2.3) is the following:

$k_{11}q_1$ is the *restoring moment* about the elastic axis produced by an angular deflection q_1 of the wing; it corresponds to the torsional stiffness of the wing and to the aerodynamic moment produced by a change of angle of attack.

$k_{12}q_2$ is the moment about the elastic axis produced by the deflection of the aileron q_2; hence, $k_{12}q_2$ represents the *aileron effect.*

$k_{21}q_1 + k_{22}q_2$ is the *hinge moment,* which consists of two parts. The first part is the hinge moment due to the change of angle of attack of the wing. The second part is the hinge moment produced by a change of the angle of attack of the aileron.

It is important to consider the signs of the four coefficients and their variation with the flying speed of the airplane. As far as k_{11} is concerned, the contribution of the torsional rigidity is positive, but the aerodynamic moment is in most cases destabilizing, and, therefore, its contribution to k_{11} is negative. However, for small speeds the torsional restoring moment is certainly larger than the moment of the aerodynamic forces; hence, k_{11} is greater than zero. With increasing speed k_{11} decreases by an amount proportional to the square of the speed so that at a certain flying speed k_{11} will change in sign. At this speed the destabilizing aerodynamic moment is equal to the elastic restoring moment, and the wing becomes unstable. This speed is called the *speed of torsional divergence.*

The moment corresponding to the coefficient k_{12} is caused by aerodynamic forces only. Hence, k_{12} is proportional to the square of the flying speed. Since the aerodynamic forces for a

positive aileron setting tend to decrease the angle of attack, k_{12} is greater than zero.

We assume that the two hinge-moment coefficients k_{21} and k_{22} also result from aerodynamic forces only. This means that the aileron is considered mechanically completely free; for example, the restoring forces due to the cable connections are neglected.

In aeronautical practice the hinge moment is generally expressed as a function of the angle of attack of the wing, which is identical with our parameter q_1, and of the aileron angle α, which is the angular deflection of the aileron relative to the wing, *i.e.*, $\alpha = q_2 - q_1$. If we express the hinge moment as a linear function of α and q_1, *e.g.*, by $-c_{21}q_1 - c_{22}\alpha$, in general both c_{21} and c_{22} are positive. The coefficient c_{21} is positive, for, if we change the angle of attack of the wing without changing the aileron angle, the aerodynamic forces tend to turn the aileron into the wind direction. The coefficient c_{22} is positive with the exception of the undesirable case when the aileron is overbalanced aerodynamically.* Moreover, in practical cases $c_{22} > c_{21}$, and thus $k_{21} = c_{21} - c_{22} < 0$; but k_{22} and also $k_{22} + k_{21}$ are positive.

It is seen that the coefficients k_{12} and k_{21} have opposite signs, and, therefore, certainly $k_{12} \neq k_{21}$, which means that the forces are *nonconservative*. Let us assume that we carry out a so-called *closed cycle* during which the wing and aileron take the following successive positions:

$$
\begin{aligned}
&(1)\ q_1 = q_2 = 0\\
&(2)\ q_1 = q_1,\ q_2 = 0\\
&(3)\ q_1 = q_1, q_2 = q_2\\
&(4)\ q_1 = 0, q_2 = q_2\\
&(5)\ q_1 = 0, q_2 = 0.
\end{aligned}
$$

Then the total work done by the forces is equal to

$$
\begin{aligned}
W = -\int_0^{q_1} k_{11}q_1\,dq_1 - \int_0^{q_2}(k_{21}q_1 + k_{22}q_2)\,dq_2 - \int_{q_1}^{0}(k_{11}q_1 + k_{12}q_2)\,dq_1\\
-\int_{q_2}^{0} k_{22}q_2\,dq_2 = (k_{12} - k_{21})q_1q_2 \qquad (2.4)
\end{aligned}
$$

Since $k_{12} > 0$ and $k_{21} < 0$ the work of the forces is positive, *i.e.*, energy is

* A control surface is aerodynamically balanced if it is in neutral equilibrium in every position, *i.e.* the air forces have not the tendency either to bring it back to the zero position or to increase the angular deflection; if the aerodynamic forces tend to increase the deflection, the surface is said to be aerodynamically overbalanced.

introduced into the system during the process, whereas the initial and the end values of the coordinates are the same. It is evident that if the same process is carried out in the opposite direction, energy will be transferred from the wing-aileron unit into the surrounding air.

Substituting Eqs. (2.1) and (2.3) into Lagrange's equations, we obtain the equations of motion

$$\begin{aligned} A\ddot{q}_1 + B\ddot{q}_2 &= -k_{11}q_1 - k_{12}q_2 \\ B\ddot{q}_1 + C\ddot{q}_2 &= -k_{21}q_1 - k_{22}q_2 \end{aligned} \tag{2.5}$$

Let us first assume $s_2 = 0$; then $B = 0$, *i.e.*, there is no dynamic coupling between the coordinates q_1 and q_2. Then substituting $q_1 = c_1e^{i\omega t}$ and $q_2 = c_2e^{i\omega t}$, we obtain the frequency equation

$$\begin{vmatrix} -A\omega^2 + k_{11} & k_{12} \\ k_{21} & -C\omega^2 + k_{22} \end{vmatrix} = 0 \tag{2.6}$$

or

$$AC\omega^4 - \omega^2(Ak_{22} + Ck_{11}) + k_{11}k_{22} - k_{12}k_{21} = 0$$

Dividing the left side by AC and writing $\omega_{11}^2 = k_{11}/A$, $\omega_{22}^2 = k_{22}/C$,

$$\omega^4 - (\omega_{11}^2 + \omega_{22}^2)\omega^2 + \omega_{11}^2\omega_{22}^2 - \frac{k_{12}k_{21}}{AC} = 0 \tag{2.7}$$

Solving (2.7) for ω^2, we obtain

$$\omega^2 = \frac{\omega_{11}^2 + \omega_{22}^2}{2} \pm \sqrt{\left(\frac{\omega_{11}^2 + \omega_{22}^2}{2}\right)^2 - \omega_{11}^2\omega_{22}^2 + \frac{k_{12}k_{21}}{AC}} \tag{2.8}$$

The stability of the system depends on the exponent in the formula $q = \text{const. } e^{i\omega t}$. There are two cases possible: (*a*) if ω is real, then we obtain undamped simple harmonic oscillations and the system is stable; (*b*) if ω is complex, say $\omega = \alpha + i\beta$, then $-\omega = -\alpha - i\beta$ is also a root of Eq. (2.7), and either $i\omega$ or $-i\omega$ has a positive real part, *i.e.*, we certainly obtain at least one mode of oscillation with increasing amplitude, and the system is unstable. Hence, the system is stable only if ω is real, *i.e.*, ω^2 is real and positive.

According to Eq. (2.8), ω^2 is real if the radical is real, and ω^2 is positive if the first term is larger than the radical. Hence, the first condition of stability is that the radical in Eq. (2.8) is real. This will be the case when

$$(\omega_{11}^2 - \omega_{22}^2)^2 + 4\,\frac{k_{12}k_{21}}{AC} > 0 \tag{2.9}$$

It is seen that if $k_{12}k_{21} > 0$, condition (2.9) is always satisfied. Thus, it is identically satisfied if the system is conservative, as for such a system $k_{12} = k_{21}$. In our problem $k_{12} > 0$, and $k_{21} < 0$. Hence, the system is unstable in the neighborhood of $\omega_{11} - \omega_{22} = 0$, *i.e.*, if the two frequencies ω_{11} and ω_{22} are equal or nearly equal. The frequencies ω_{11} and ω_{22} are defined by $\omega_{11}^2 = k_{11}/A$ and $\omega_{22}^2 = k_{22}/C$. Their physical significance is the following: The frequency ω_{11} is the torsional frequency

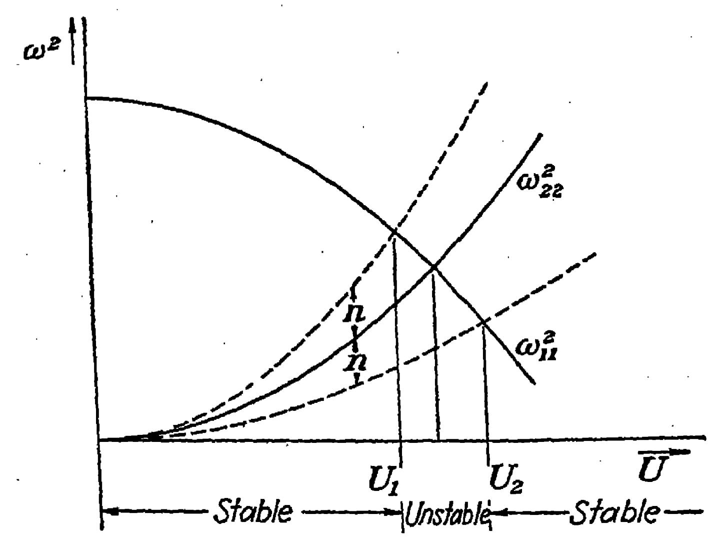

FIG. 2.2.—Diagram of frequencies vs. flight speed, showing the range in which flutter occurs.

of the system, provided that the mass of the aileron is concentrated on the hinge axis. As k_{11} decreases with the flying speed, ω_{11}^2 also decreases with the same speed. The frequency ω_{22} is equal to the frequency of the aileron oscillating around the hinge as a fixed axis. As k_{22} is proportional to the square of the speed, ω_{22}^2 increases with the speed. At a certain speed the two frequencies ω_{11} and ω_{22} coincide. Figure 2.2 shows ω_{11}^2 and ω_{22}^2 as functions of the flying speed U. Let us denote the quantity $-4\frac{k_{12}k_{21}}{AC}$ by n^2 (we remember that $k_{12}k_{21} < 0$). Then the stability condition (2.9) is given by

$$(\omega_{11}^2 - \omega_{22}^2)^2 > n^2$$

This is satisfied if either $\omega_{11}^2 - \omega_{22}^2 > |n|$ or $\omega_{22}^2 - \omega_{11}^2 > |n|$. Now, both k_{12} and k_{21} are proportional to the square of the speed U; hence, $|n|$ is proportional to U^2. In Fig. 2.2 $\omega_{22}^2 + |n|$ and $\omega_{22}^2 - |n|$ are represented by the two dotted parabolic curves. The limits of the range of instability U_1 and U_2 are determined

by the intersection of these parabolas with the curve representing ω_{11}^2. The lower limit of the range of instability is called the *critical speed.*

The second condition of stability requires that

$$\frac{\omega_{11}^2 + \omega_{22}^2}{2} > \sqrt{\left(\frac{\omega_{11}^2 + \omega_{22}^2}{2}\right)^2 - \omega_{11}^2\omega_{22}^2 + \frac{k_{12}k_{21}}{AC}} \tag{2.10}$$

It is seen that since $k_{12}k_{21} < 0$, this condition is satisfied if ω_{11}^2 and ω_{22}^2 are positive, *i.e.*, k_{11} and k_{22} are positive. This holds in all practical cases. To be sure, k_{11} becomes negative if the flying speed exceeds the speed of torsional divergence. However, the torsional divergence occurs, in general, beyond the maximum velocity that the airplane can actually reach.

Thus far we have considered the case $s_2 = 0$, *i.e.*, we have assumed that the center of gravity lies on the hinge axis, so that the aileron is *statically balanced.* We find that if the oscillation frequency of the aileron coincides with the torsional frequency of the wing, flutter will occur in spite of the static balance of the aileron. We shall now vary the location of the center of gravity relative to the hinge axis and investigate its influence on stability.

If $s_2 \neq 0$, the coordinates are not only statically, but also dynamically coupled, because the term $2m_2as_2\dot{q}_1\dot{q}_2$ appears in the expression (2.1) for the kinetic energy. Therefore, we introduce new coordinates ξ_1 and ξ_2 such that the kinetic energy becomes again a sum of squares.

Let us use as new coordinates the linear combinations:

$$\xi_1 = q_1 \qquad \text{and} \qquad \xi_2 = q_2 - \beta q_1 \tag{2.11}$$

and determine the coefficient β so that the kinetic energy becomes

$$T = \tfrac{1}{2}(M\xi_1^2 + N\xi_2^2) \tag{2.12}$$

Substituting $q_1 = \xi_1$, $q_2 = \xi_2 + \beta\xi_1$ into the expression (2.1) for T, we see that the coefficient of the product $\xi_1\xi_2$ becomes $m_2(as_2 + \beta i_2^2)$. Hence, if we put $\beta = -(as_2/i_2^2)$, we obtain T in the form (2.12).

We now transform the generalized forces. The forces Ξ_1 and Ξ_2 corresponding to the coordinates ξ_1 and ξ_2 are defined by the relation (Chapter III, section 11)

$$\Xi_1\, d\xi_1 + \Xi_2\, d\xi_2 = Q_1\, dq_1 + Q_2\, dq_2 \tag{2.13}$$

Hence, using (2.11),

$$\Xi_1 = Q_1 + \beta Q_2, \qquad \Xi_2 = Q_2$$

Substituting Q_1 and Q_2 from Eq. (2.3), we have

$$\begin{aligned} -\Xi_1 &= (k_{11} + \beta k_{21})q_1 + (k_{12} + \beta k_{22})q_2 \\ -\Xi_2 &= k_{21}q_1 + k_{22}q_2 \end{aligned}$$

and finally

$$\begin{aligned} -\Xi_1 &= [k_{11} + \beta(k_{21} + k_{12}) + \beta^2 k_{22}]\,\xi_1 + (k_{12} + \beta k_{22})\xi_2 \\ -\Xi_2 &= (k_{21} + \beta k_{22})\xi_1 + k_{22}\xi_2 \end{aligned} \tag{2.14}$$

The stability condition $k_{12}k_{21} > 0$ [cf. discussion of Eq. (2.9)] is replaced now by

$$(k_{12} + \beta k_{22})(k_{21} + \beta k_{22}) > 0 \tag{2.15}$$

Since k_{12} and k_{22} are positive and $k_{21} < 0$, the condition (2.15) is satisfied if $\beta > 0$ and $k_{21} + \beta k_{22} > 0$ or $\beta > -(k_{21}/k_{22})$. If $\beta > 0$, $s_2 < 0$, *i.e.*, the center of gravity of the aileron lies ahead of the hinge axis. The limiting value of s_2 is given by

$$-s_2 = \beta\frac{i_2^2}{a} = \frac{i_2^2}{a}\left(\frac{-k_{21}}{k_{22}}\right) \tag{2.16}$$

We remember that $k_{22} > 0$ and $k_{21} + k_{22} > 0$, so that

$$-k_{21} < k_{22}.$$

Therefore, we are on the safe side if we put $\beta = 1$, or

$$-s_2 = \frac{i_2^2}{a} \tag{2.17}$$

The coordinates of the system for $\beta = 1$ are $\xi_1 = q_1$ and $\xi_2 = q_1 - q_2$. The geometrical meaning of $q_2 - q_1$ is the angular displacement α of the aileron relative to the wing (the *aileron angle*). If there is no dynamic coupling between the coordinates q_1 and α, we say that the aileron is *dynamically balanced*, since in this case a rotation of the aileron around the hinge axis does not contribute to the moment of momentum with respect to the axis of suspension of the wing. The condition of dynamic balance is according to Eq. (2.17) $i_2^2 + as_2 = 0$. If s_2 satisfies this condition, the instability is avoided.

3. Dissipative Systems. The Analogy between Mechanical and Electrical Oscillations.—The general form for the equation of motion of a mechanical system with one degree of freedom

subjected to elastic restraint, damping, and an external periodic force is the following [cf. Eq. (7.1), Chapter IV]:

$$m\frac{d^2q}{dt^2} + \beta\frac{dq}{dt} + kq = F = F_0 \sin \omega t \tag{3.1}$$

Of the parameters of this equation, m depends on the *inertia* of the system, β determines the *damping* and k the *elastic restraint*, and F_0 is the amplitude of the periodic external force. The coordinate q denotes a suitably chosen displacement. We will show that the oscillation produced by a periodic electromotive force in an electric circuit consisting of a *coil*, of a *resistor*, and of a *capacitor* is governed by an equation of similar form.

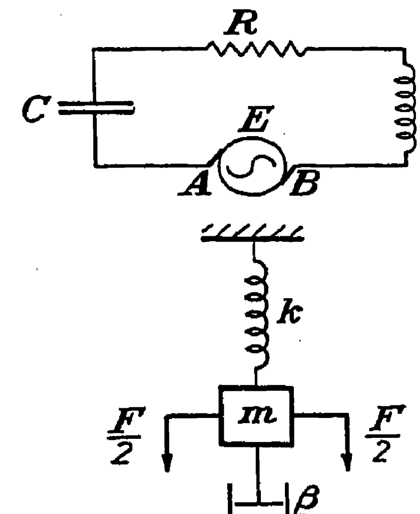

FIG. 3.1.—Schematic diagram illustrating the analogy between a mechanical system and an electric circuit.

In fact, let us consider the circuit shown in Fig. 3.1 and denote the electric charge of the capacitor (of capacity C) by Q, the current through the capacitor by I, the ohmic resistance by R, and the inductance of the coil by L. We assume that a periodic electromotive force of the magnitude $E = E_0 \sin \omega t$ is acting between the poles A and B of a generating device. Then the voltage drop between A and B must be equal to the voltage produced by the generator. The voltage drop consists of the drop due to the inductance, the ohmic resistance, and the capacity. Let us calculate these three contributions.

a. The voltage drop due to the inductance is equal to the product of the inductance L of the coil and the rate of change of the current, *i.e.*, equal to $L\dfrac{dI}{dt}$.

b. The voltage drop due to the ohmic resistance is equal to the product RI.

c. The voltage drop through the capacitor is equal to Q/C. Hence, the differential equation of the oscillation becomes

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E_0 \sin \omega t \tag{3.2}$$

Now the two quantities Q and I are connected by a simple relation since the rate of change of the charge of the capacitor

is equal to the current. Hence, $I = dQ/dt$ and $dI/dt = d^2Q/dt^2$. Substituting these relations in (3.2), we obtain

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E_0 \sin \omega t \tag{3.3}$$

Comparing Eqs. (3.1) and (3.3), it is seen that the electrical and mechanical oscillations are analogous if we substitute inductance for inertia, resistance for damping, the reciprocal value of the capacity for elastic restraint, and the impressed electromotive force for the mechanical force impressed on the body.

The same analogy can be applied to systems with an arbitrary number of degrees of freedom. The number of the degrees of freedom is equal to the minimum number of closed circuits which include all elements, *i.e.*, all resistors, capacitors, inductances, and generating devices of the system. The elements which belong to more than one closed circuit may be considered as the coupling elements.

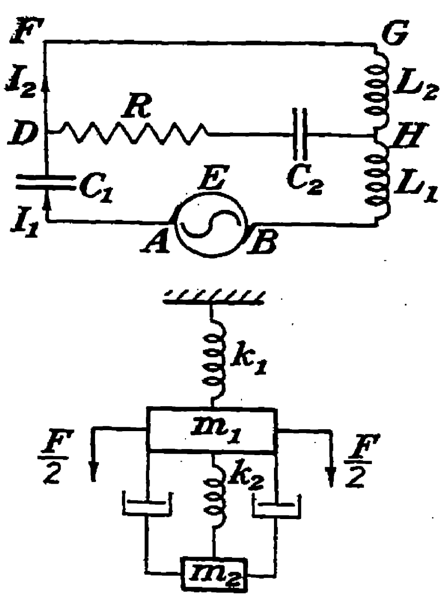

FIG. 3.2.—Schematic diagram of a mechanical vibration damper and its electrical equivalent.

As an example, analyze the coupled circuit system in Fig. 3.2. The first circuit contains the electromotive force E, the inductance L_1, the two capacities C_1 and C_2, and the resistance R. The mechanical equivalent of this circuit is a mass m_1 subjected to the action of an external force F, two elastic springs k_1 and k_2, and a dashpot β. The second circuit contains the inductance L_2, the capacity C_2, and the resistance R; the mechanical equivalent consists of the mass m_2 attached to the spring k_2 and the dashpot β.

In the simple example of one closed electric circuit we chose as generalized coordinate the charge of the capacitor Q. Owing to the relation $dQ/dt = I$, Q can be replaced by the integral $\int_0^t I\,dt$. For coupled circuits it is more convenient to replace the charge Q by this integral. Hence, in the example corresponding to Fig. 3.2, we denote by I_1 the current flowing through L_1, and by I_2, the current flowing through L_2, positive in

the direction indicated by the arrows. Then the current passing through R is equal to $I_1 - I_2$, according to Kirchhoff's law. Writing $Q_1 = \int_0^t I_1\,dt$ and $Q_2 = \int_0^t I_2\,dt$, the charge of the capacitor C_2 will be equal to $\int_0^t (I_1 - I_2)\,dt = Q_1 - Q_2$.

Hence, the drop of voltage E between A and B can be calculated along $ADFGHB$ thus:

$$E = \frac{Q_1}{C_1} + L_1 \frac{d^2Q_1}{dt^2} + L_2 \frac{d^2Q_2}{dt^2} \tag{3.4}$$

For the closed circuit $DFGH$ the drop of voltage is equal to zero; hence,

$$L_2 \frac{d^2Q_2}{dt^2} - \frac{1}{C_2}(Q_1 - Q_2) - R\frac{d(Q_1 - Q_2)}{dt} = 0 \tag{3.5}$$

The statement corresponding to (3.4) for the mechanical system expresses the equilibrium among the external force F, the elastic force $-k_1q_1$, and the mass forces $-m_1 \frac{d^2q_1}{dt^2}$ and $-m_2 \frac{d^2q_2}{dt^2}$ in the form:

$$F = k_1q_1 + m_1 \frac{d^2q_1}{dt^2} + m_2 \frac{d^2q_2}{dt^2} \tag{3.6}$$

This relation is perfectly analogous to (3.4) and is independent of the internal forces produced by the spring and dashpot inserted between the two masses. The latter enter into the second relation which determines the dynamic equilibrium for the mass m_2:

$$m_2 \frac{d^2q_2}{dt^2} - k_2(q_1 - q_2) - \beta \frac{d}{dt}(q_1 - q_2) = 0 \tag{3.7}$$

Let us now set up the balance between the forces acting on the mass m_1. We have

$$m_1 \frac{d^2q_1}{dt^2} + k_1q_1 + k_2(q_1 - q_2) + \beta \frac{d}{dt}(q_1 - q_2) = F \tag{3.8}$$

The corresponding equation

$$L_1 \frac{d^2Q_1}{dt^2} + \frac{1}{C_1}Q_1 + \frac{1}{C_2}(Q_1 - Q_2) + R\frac{d}{dt}(Q_1 - Q_2) = E \tag{3.9}$$

determines the drop of voltage along the circuit $ADHB$.

We recognize in the mechanical system the model of a *vibration damper*, *i.e.*, of a device used for the absorption of energy in case of resonance. Assume that resonance is due to the coincidence between the frequency of the force F_1 imposed on the mass m_1 and the natural frequency of the mass m_1. To absorb the energy produced by the resonance, we attach to m_1 another mass m_2 and insert a spring and dashpot between the two masses. The corresponding electrical device is called a *wave trap;* it is able to damp the resonance induced in a circuit by an impressed voltage whose frequency coincides with the natural frequency of the circuit.

In the next section the theory of the vibration damper is carried out for the mechanical system.

The equations for the small oscillations of a dissipative mechanical system with n degrees of freedom can be written in the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial U}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = 0 \tag{3.10}$$

where D is the dissipation function defined in section 1 and $i = 1, 2, \cdots, n$.

The equations for the small oscillations for an electric network consisting of n circuits can be set up in an analogous way, if we introduce the following definitions:

a. The magnetic energy of the system is

$$T = \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} L_{ij}\dot{Q}_i\dot{Q}_j \tag{3.11}$$

where $\dot{Q}_i = I_i$ is the current in the ith circuit, and the L_{ij}'s are inductance coefficients where $L_{ij} = L_{ji}$.

b. The electrostatic energy of the system is

$$U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{C_{ij}}Q_iQ_j \tag{3.12}$$

where $C_{ij} = C_{ji}$; the C_{ij}'s are coefficients of capacitance.

c. The dissipation function is

$$D = \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} R_{ij}\dot{Q}_i\dot{Q}_j \tag{3.13}$$

where D is equal to half the energy transformed into heat per unit time.

With these notations the equations for the n circuits [*e.g.*, Eqs. (3.5) and (3.9)] can be obtained according to the following scheme:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}_i}\right) + \frac{\partial U}{\partial Q_i} + \frac{\partial D}{\partial \dot{Q}_i} = 0 \tag{3.14}$$

which is formally identical with (3.10).

4. The Theory of the Vibration Damper.—We start from the two equations of motion (3.6) and (3.7) given in the previous section. We assume that the frequency of the impressed force is equal to ω and write $F = F_0 e^{i\omega t}$. Then substituting into (3.6) and (3.7), $q_1 = c_1 e^{i\omega t}$ and $q_2 = c_2 e^{i\omega t}$, we obtain

$$\begin{aligned} F_0 &= (k_1 - m_1\omega^2)c_1 - m_2\omega^2 c_2 \\ 0 &= (-k_2 - \beta i\omega)c_1 + (k_2 + \beta i\omega - m_2\omega^2)c_2 \end{aligned} \tag{4.1}$$

The main interest is directed toward finding the magnitude of c_1, *i.e.*, the amplitude of the vibration of the mass m_1. If $\beta \neq 0$, in general, c_1 and c_2 will be complex.

Let us first assume that $\beta = 0$, *i.e.*, consider the vibration damper without dissipation. Then,

$$c_1 = F_0 \frac{k_2 - m_2\omega^2}{(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) - m_2 k_2 \omega^2} \tag{4.2}$$

It is seen that the amplitude c_1 is no longer infinite if the frequency of the force and the natural frequency of the mass coincide, although $c_1 = \infty$ for two other values of ω_1, *viz.*, when

$$(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) = m_2 k_2 \omega^2 \tag{4.3}$$

We write (4.3) in the form:

$$\omega^4 - \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)\omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0$$

or with

$$\omega_{11}^2 = \frac{k_1}{m_1}, \qquad \omega_{22}^2 = \frac{k_2}{m_2}$$

$$\omega^4 - \left[\omega_{11}^2 + \left(1 + \frac{m_2}{m_1}\right)\omega_{22}^2\right]\omega^2 + \omega_{11}^2\omega_{22}^2 = 0 \tag{4.4}$$

Let us assume $\omega_{11} = \omega_{22}$. Then the product of the two frequencies for which $c_1 = \infty$ is equal to ω_{11}^2. Therefore, one of these frequencies is larger, the other is smaller than ω_{11}. In

any case, by attaching to the vibrating mass m_1 another mass m_2 by a suitable spring, the resonance is *split* into two frequencies but is not removed.

Let us now introduce damping. From (4.1) we obtain

$$c_1 = \frac{k_2 - m_2\omega^2 + \beta i\omega}{(k_1 - m_1\omega^2)(k_2 + \beta i\omega - m_2\omega^2) - m_2\omega^2(k_2 + \beta i\omega)} F_0 \quad (4.5)$$

or

$$\frac{c_1}{F_0} = \frac{k_2 - m_2\omega^2 + \beta i\omega}{(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2 m_2\omega^2 + \beta i\omega(k_1 - m_1\omega^2 - m_2\omega^2)} \quad (4.6)$$

We shall now show that there are two frequencies ω for which the absolute magnitude of the amplitude c_1 is independent of the magnitude of the damping. The absolute value of c_1/F_0 is, in fact, independent of β, if

$$\left[\frac{k_2 - m_2\omega^2}{(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2 m_2\omega^2}\right]^2 = \frac{1}{[k_1 - (m_1 + m_2)\omega^2]^2}$$

or

$$(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2 m_2\omega^2 = \pm[(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) - m_2\omega^2(k_2 - m_2\omega^2)] \quad (4.7)$$

It is seen that if we use the positive sign, relation (4.7) can be satisfied only by $\omega = 0$. This case is without practical interest. The negative sign leads to the relation

$$2(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) = m_2\omega^2(2k_2 - m_2\omega^2)$$

or with $\omega_{11}^2 = k_1/m_1$, $\omega_{22}^2 = k_2/m_2$, $m_2/m_1 = \mu$,

$$(\omega_{11}^2 - \omega^2)(\omega_{22}^2 - \omega^2) = \mu\omega^2\left(\omega_{22}^2 - \frac{\omega^2}{2}\right) \quad (4.8)$$

Let us denote the two roots of this equation by Ω_1^2 and Ω_2^2. Then the corresponding values of $|c_1/F_0|$ can be expressed easily by (4.6). Since $|c_1/F_0|$ is independent of the magnitude of the damping coefficient β for $\omega^2 = \Omega_1^2$ and $\omega^2 = \Omega_2^2$, we can put $\beta = \infty$ and obtain from Eq. (4.6)

$$\left|\frac{c_1}{F_0}\right| = \frac{1}{|k_1 - (m_1 + m_2)\Omega^2|}$$

or with $k_1 = m_1\omega_{11}^2$,

$$\left|\frac{c_1}{F_0}\right| = \frac{1}{k_1}\frac{1}{\left|1 - (1+\mu)\dfrac{\Omega^2}{\omega_{11}^2}\right|} \tag{4.9}$$

where Ω^2 is equal to either Ω_1^2 or Ω_2^2. For $\omega \to 0$, $c_1 = F_0/k_1$ [cf. Eq. (4.6)], hence, F_0/k_1 is the static deflection of the mass m_1 corresponding to a force F_0. Let us denote it by c_0 and call the ratio c_1/c_0 the amplitude ratio η. From (4.9) we have

$$\eta_1 = \frac{1}{|1 - (1+\mu)(\Omega_1^2/\omega_{11}^2)|}, \qquad \eta_2 = \frac{1}{|1 - (1+\mu)(\Omega_2^2/\omega_{11}^2)|} \tag{4.10}$$

whereas for all other values of ω, η is k_1 times the absolute value of the expression on the right side of (4.6). It is a function of ω with ω_{11}, ω_{22}, μ, and β as parameters.

We now propose to find the best performance of the damper for a given ω_{11} and for a given mass ratio $\mu = m_2/m_1$. In other words, we shall find the value of ω_{22}, which makes the maximum value of η in the range $0 < \omega < \infty$ as small as possible. Since the values η_1 and η_2 are independent of β, the maximum amplitude ratio cannot be smaller than the larger of the two values obtained from Eq. (4.10). By appropriate choice of the damping factor β we can keep the amplitude ratio for the whole frequency range near this limit.

Let us vary ω_{22}^2 and investigate first its influence on Ω_1^2 and Ω_2^2. These quantities are roots of the Eq. (4.8). We write Eq. (4.8) in the following form:

$$\left(1 + \frac{\mu}{2}\right)\Omega^4 - [\omega_{11}^2 + (1+\mu)\omega_{22}^2]\,\Omega^2 + \omega_{11}^2\omega_{22}^2 = 0 \tag{4.11}$$

It is easily seen that Eq. (4.11) has no double roots, *i.e.*, $\Omega_1 \neq \Omega_2$. We shall, therefore, assume that $\Omega_1^2 > \Omega_2^2$. Then for $\omega_{22} = 0$ we have $\Omega_1^2 = \dfrac{\omega_{11}^2}{1 + \frac{\mu}{2}}$ and $\Omega_2^2 = 0$; for $\omega_{22} \to \infty$, $\Omega_1^2 \to \infty$ and $\Omega_2^2 \to \dfrac{\omega_{11}^2}{1+\mu}$. The two roots Ω_1^2 and Ω_2^2 behave in the following way (Fig. 4.1): Ω_1^2 decreases with increasing ω_{22}^2 till it reaches a minimum. The value of this minimum can be found by differentiating the left side of (4.11) with respect to ω_{22}^2, considering Ω^2

as a function of ω_{22}^2 and putting $d(\Omega^2)/d(\omega_{22}^2) = 0$. We have, therefore,

$$-(1 + \mu)\Omega_1^2 + \omega_{11}^2 = 0$$

which means that the minimum of Ω_1^2 is equal to $\dfrac{\omega_{11}^2}{1 + \mu}$. For $\omega_{22}^2 \to \infty$, Ω_1^2 increases to infinity. The value of Ω_2^2 increases from 0 to $\dfrac{\omega_{11}^2}{1 + \mu}$ if ω_{22}^2 varies from 0 to ∞.

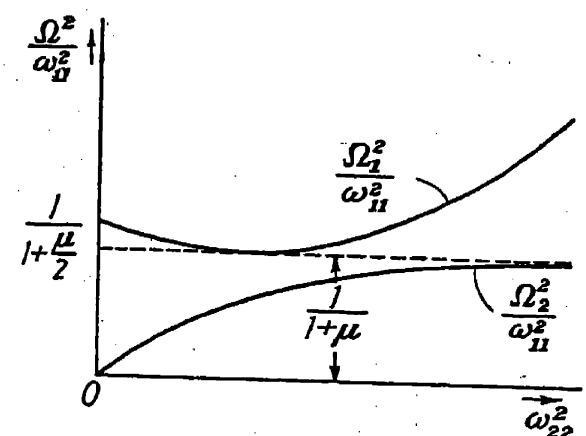

FIG. 4.1.—Theory of the vibration damper: Diagram showing the frequencies Ω_1 and Ω_2 for which the amplitude ratios are independent of the damping factor.

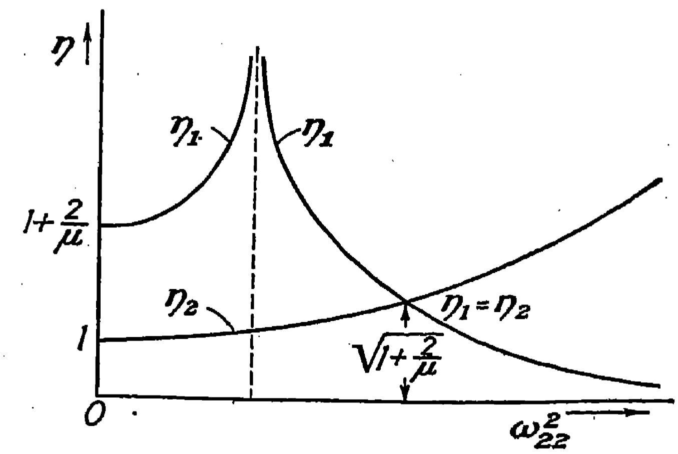

FIG. 4.2.—Theory of the vibration damper: Diagram showing the amplitude ratios η_1 and η_2 corresponding to the frequencies Ω_1 and Ω_2 plotted in Fig. 4.1. The larger of η_1 and η_2 determines the best performance of the damper.

If we now plot (Fig. 4.2) η_1 and η_2—corresponding to $\Omega^2 = \Omega_1^2$ and $\Omega^2 = \Omega_2^2$, respectively—we see that for $\omega_{22}^2 = 0$, $\eta_1 = 1 + \dfrac{2}{\mu}$, and when Ω_1^2 reaches its minimum, η_1 increases to infinity; then it decreases from ∞ to 0. The second value, *viz.*, η_2, varies from 1 to ∞ if ω_{22}^2 varies from 0 to ∞. It is seen that the optimum value for η is reached when $\eta_1 = \eta_2$. According to Eq. (4.10) $\eta_1 = \eta_2$ if either

$$1 - (1 + \mu)\frac{\Omega_1^2}{\omega_{11}^2} = 1 - (1 + \mu)\frac{\Omega_2^2}{\omega_{11}^2}$$

or

$$1 - (1 + \mu)\frac{\Omega_1^2}{\omega_{11}^2} = -\left[1 - (1 + \mu)\frac{\Omega_2^2}{\omega_{11}^2}\right]$$

The first case is excluded since $\Omega_1 \neq \Omega_2$. Therefore, we obtain

$$\Omega_1^2 + \Omega_2^2 = \frac{2}{1 + \mu}\omega_{11}^2 \tag{4.12}$$

However, from (4.11) it follows that

$$\Omega_1^2 + \Omega_2^2 = \frac{1}{1 + (\mu/2)} [\omega_{11}^2 + (1 + \mu)\omega_{22}^2] \tag{4.13}$$

and, therefore, by comparison of (4.12) and (4.13),

$$\omega_{22} = \frac{\omega_{11}}{1 + \mu} \tag{4.14}$$

Since in practical cases $\mu << 1$, the best tuning is reached at a

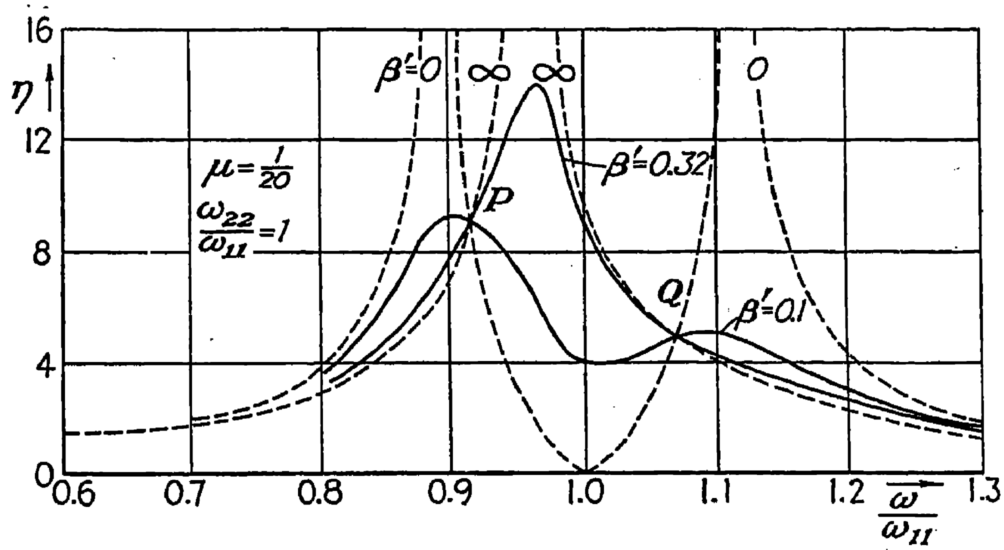

FIG. 4.3.—The amplitude ratio η for the forced oscillations of a mass equipped with a vibration damper, plotted as function of the frequency of the external force for different values of the damping factor. Note that all curves pass through the points P and Q, which correspond to the frequencies Ω_1 and Ω_2. (The diagram is reproduced from J. P. Den Hartog, "Mechanical Vibrations," Fig. 107.)

frequency ω_{22}, which is slightly lower than ω_{11}. Substituting the value of ω_{22} from (4.14) in Eq. (4.11), we obtain

$$\Omega^2 = \frac{\omega_{11}^2}{1 + \mu} \left[1 + \sqrt{\frac{\mu}{2 + \mu}} \right]$$

and, therefore,

$$\eta_1 = \eta_2 = \sqrt{\frac{2 + \mu}{\mu}} = \sqrt{1 + \frac{2}{\mu}} \tag{4.15}$$

In Fig. 4.3 the amplitude ratio $\eta = |c_1k/F_0|$ is plotted as function of ω/ω_{11} for $\omega_{22}/\omega_{11} = 1$, $\mu = \frac{1}{20}$, and for various values of the dimensionless parameter $\beta' = \beta/2m_1\omega_{11}$. It is seen that we have infinite amplitude not only for $\beta' = 0$ but also for $\beta' = \infty$, since if the damping is excessive the two masses are almost rigidly connected and no energy is dissipated. The points P and Q correspond to $\omega = \Omega_1$ and $\omega = \Omega_2$.

5. The Stability of Uniform Rotation. The Vertical Top.—In the examples of the previous sections the small oscillations of a system in the neighborhood of its equilibrium position were investigated. In this section we investigate the small oscillations of a rotating top in the neighborhood of a so-called *stationary* state of motion. Let us consider a symmetrical top rotating uniformly around its axis of symmetry. If the axis of symmetry is vertical and we neglect friction, the uniform rotation will persist indefinitely, since gravity, which is the only external force acting on the system, exerts no moment with respect to the fixed point. Let us now investigate the motion in the neighborhood of the vertical position. We determine the position of the top by two angles (Fig. 5.1). The angle θ is the inclination between the vertical z-axis and the axis of symmetry ζ. We pass a plane through the z- and the ζ-axes and call the intersection of this plane with the xy plane the x'-axis. The angle between the x- and the x'-axes is denoted by φ. The axis normal to x' and z is called the y'-axis. The angular velocity around the ζ-axis is denoted—as in Chapter III, section 7—by Ω, and the angular velocity around the z-axis, *i.e.*, $\dot{\varphi} = d\varphi/dt$, by μ. We remember that the angular velocity μ is called the *velocity of precession* (cf. Chapter III). We must consider a third component of the angular velocity, *viz.*, $\dot{\theta} = d\theta/dt$; this angular velocity is called the *velocity of nutation.* The angular velocity $\dot{\theta}$ is represented by a vector perpendicular to the $z\zeta$ plane; hence, it lies on the y'-axis. We consider θ and $\dot{\theta}$ small in comparison with unity; however, φ and $\dot{\varphi}$ are not necessarily small.

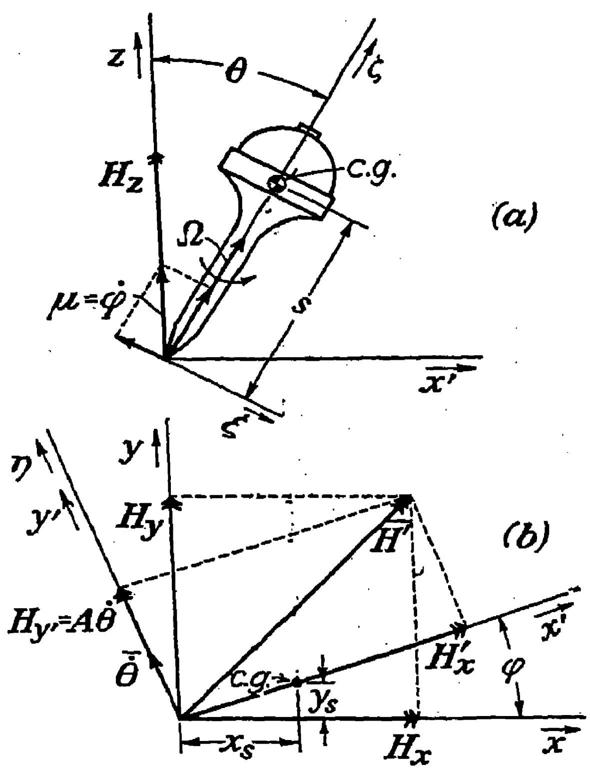

FIG. 5.1.—Diagram of a vertical top. (a) Side view perpendicular to the $z\zeta$ plane.

In order to obtain the equations of motion for the small oscillations of the top in the neighborhood of the vertical position, we evaluate the components of the moment of momentum. We first calculate the components with respect to the $x'y'z$

coordinate system. The contributions of Ω and μ were already calculated in Chapter III, section 7. Let us use, as in that chapter, the notation C for the moment of inertia with respect to the axis of symmetry ζ and A for the moment of inertia with respect to an arbitrary axis normal to ζ. Then the moment of momentum consists of a component $C(\Omega + \mu \cos \theta)$ in the ζ-direction and a component $A\mu \sin \theta$ directed normal to ζ in the $z\zeta$ plane. Therefore, the component of the moment of momentum in the z-direction is

$$H_z = C(\Omega + \mu \cos \theta) \cos \theta + A\mu \sin^2 \theta$$

and in the x'-direction,

$$H_{x'} = C(\Omega + \mu \cos \theta) \sin \theta - A\mu \sin \theta \cos \theta$$

The moment of momentum corresponding to the angular velocity $\dot{\theta}$ is equal to $A\dot{\theta}$ and is directed parallel to the y'-axis.

Dealing with small oscillations, we neglect the higher order terms of θ and $\dot{\theta}$. Thus, we obtain for the components of the moment of momentum the following expressions:

$$H_{x'} = C(\Omega + \mu)\theta - A\mu\theta, \qquad H_{y'} = A\dot{\theta}, \qquad H_z = C(\Omega + \mu) \tag{5.1}$$

In order to obtain the equations of motion, we resolve the $\bar{H}$ vector into components parallel to the fixed axes H_x, H_y, H_z. The equations of transformation are (Fig. 5.1*b*)

$$\begin{aligned} H_x &= H_{x'} \cos \varphi - H_{y'} \sin \varphi \\ H_y &= H_{x'} \sin \varphi + H_{y'} \cos \varphi \end{aligned} \tag{5.2}$$

or substituting the expressions (5.1), we obtain

$$\begin{aligned} H_x &= C(\Omega + \mu)\theta \cos \varphi - A(\mu\theta \cos \varphi + \dot{\theta} \sin \varphi) \\ H_y &= C(\Omega + \mu)\theta \sin \varphi - A(\mu\theta \sin \varphi - \dot{\theta} \cos \varphi) \\ H_z &= C(\Omega + \mu) \end{aligned} \tag{5.3}$$

Taking into account that $\mu = \dot{\varphi}$, the expressions (5.3) can be written in the form:

$$\begin{aligned} H_x &= C(\Omega + \mu)\theta \cos \varphi - A \frac{d}{dt}(\theta \sin \varphi) \\ H_y &= C(\Omega + \mu)\theta \sin \varphi + A \frac{d}{dt}(\theta \cos \varphi) \\ H_z &= C(\Omega + \mu) \end{aligned} \tag{5.4}$$

Denoting the components of the moment of gravity M_x, M_y, M_z, the equations of motion are

$$\frac{dH_x}{dt} = M_x, \qquad \frac{dH_y}{dt} = M_y, \qquad \text{and} \qquad \frac{dH_z}{dt} = M_z \quad (\text{5}$$

If the center of gravity is located on the ζ-axis at a dista s from the fixed point and the mass of the top is equal to m, components of the moment of gravity are:

$$M_x = -mgs\theta \sin \varphi, \qquad M_y = mgs\theta \cos \varphi, \qquad \text{and} \qquad M_z =$$

Hence, from the third equation of (5.5) it follows t $H_z = C(\Omega + \mu) \doteq$ const., and substituting M_x, M_y, H_x, H_y in (5.5), we obtain

$$\begin{aligned} H_z \frac{d}{dt}(\theta \cos \varphi) - A \frac{d^2}{dt^2}(\theta \sin \varphi) &= -mgs\theta \sin \varphi \\ H_z \frac{d}{dt}(\theta \sin \varphi) + A \frac{d^2}{dt^2}(\theta \cos \varphi) &= mgs\theta \cos \varphi \end{aligned} \quad (5$$

Let us now introduce the x and y coordinates of the cen of gravity: $x_s = s\theta \cos \varphi$ and $y_s = s\theta \sin \varphi$. Then Eqs. (5 can be written in the form:

$$\begin{aligned} \ddot{x}_s &= \frac{mgs}{A} x_s - \frac{H_z}{A} \dot{y}_s \\ \ddot{y}_s &= \frac{mgs}{A} y_s + \frac{H_z}{A} \dot{x}_s \end{aligned} \quad (5$$

Equations (5.7) are identical with the equations of motion fo particle of unit mass moving in the xy plane subjected tc repulsive radial force of the magnitude $\frac{mg}{A} r$, where r is the distai of the point from the origin, and to a force of the magnitu $\frac{H_z}{A}\sqrt{\dot{x}^2 + \dot{y}^2}$ which is perpendicular to the instantanec velocity $v = \sqrt{\dot{x}^2 + \dot{y}^2}$.

For completeness of the analysis we show how Eq. (5.7) can be obtair from Lagrange's equation of motion. The kinetic energy of the top equal to half the sum of the products of the components of the angu velocity and the components of the moment of momentum. We shall refer t components to axes fixed to the top and choose as such axes the ζ-, ξ-, η-a: where ξ lies in the $z\zeta$ plane and is normal to ζ, η is normal to ξ and ζ ai therefore, identical with y'. The components of the angular velocit

with respect to these axes are: $\Omega + \mu \cos \theta$, $-\mu \sin \theta$, and $\dot{\theta}$; therefore, the components of the moment of momentum are $C(\Omega + \mu \cos \theta)$, $-A\mu \sin \theta$, and $A\dot{\theta}$. Thus, the kinetic energy is given by

$$T = \tfrac{1}{2}[C(\Omega + \mu \cos \theta)^2 + A\mu^2 \sin^2 \theta + A\dot{\theta}^2] \tag{5.8}$$

We introduce the coordinate β, which we ignored up to this point, *viz.*, the angular displacement of the top around the ζ-axis. We have $\Omega = \dot{\beta}$. Substituting $\mu = \dot{\varphi}$ and $\Omega = \dot{\beta}$, we have

$$T = \tfrac{1}{2}[C(\dot{\beta} + \dot{\varphi} \cos \theta)^2 + A\dot{\varphi}^2 \sin^2 \theta + A\dot{\theta}^2] \tag{5.9}$$

and the potential energy is given by

$$U = mgs \cos \theta \tag{5.10}$$

Hence, Lagrange's equations of motion for the generalized coordinates β, φ, and θ are

$$\frac{d}{dt}[C(\dot{\beta} + \dot{\varphi} \cos \theta)] = 0 \tag{5.11}$$

$$\frac{d}{dt}[C(\dot{\beta} + \dot{\varphi} \cos \theta) \cos \theta + A\dot{\varphi} \sin^2 \theta] = 0 \tag{5.12}$$

$$\frac{d}{dt}(A\dot{\theta}) + C(\dot{\beta} + \dot{\varphi} \cos \theta)\dot{\varphi} \sin \theta - A\dot{\varphi}^2 \sin \theta \cos \theta = mgs \sin \theta \tag{5.13}$$

From Eq. (5.11) it follows that $C(\dot{\beta} + \dot{\varphi} \cos \theta) = \text{const.} = H_z$. Then we have

$$-H_z \sin \theta \cdot \dot{\theta} + A\ddot{\varphi} \sin^2 \theta + 2A\dot{\varphi}\dot{\theta} \sin \theta \cos \theta = 0 \tag{5.14}$$
$$A\ddot{\theta} + H_z\dot{\varphi} \sin \theta - A\dot{\varphi}^2 \sin \theta \cos \theta = mgs \sin \theta \tag{5.15}$$

We put $s\theta = r$ and take into account that θ and r are small. Then, multiplying (5.14) by s, we obtain

$$-H_z\dot{r} + A\ddot{\varphi}r + 2A\dot{\varphi}\dot{r} = 0 \tag{5.16}$$

and from (5.15)

$$A\ddot{r} + H_z r\dot{\varphi} - Ar\dot{\varphi}^2 = mgsr \tag{5.17}$$

or

$$\begin{aligned} \frac{d}{dt}(r^2\dot{\varphi}) &= \frac{H_z}{A} r\dot{r} \\ \ddot{r} - r^2\dot{\varphi}^2 &= \frac{mgs}{A} r - \frac{H_z}{A} r\dot{\varphi} \end{aligned} \tag{5.18}$$

It is seen that Eqs. (5.18) represent the equations of motion of a particle of unit mass whose polar coordinates in a plane are r and φ under the action of a repulsive radial force $\frac{mgs}{A}r$ and a force $\frac{H_z}{A}\sqrt{\dot{r}^2 + r^2\dot{\varphi}^2}$ whose direction is normal to the velocity of the mass point. Hence, Eqs. (5.18) are equivalent to (5.7).

We shall investigate the stability of the motion defined by Eqs. (5.7). Substituting $x_s = Xe^{\lambda t}$ and $y_s = Ye^{\lambda t}$, we obtain

$$X\left(\lambda^2 - \frac{mgs}{A}\right) = -Y\frac{H_z}{A}\lambda \qquad Y\left(\lambda^2 - \frac{mgs}{A}\right) = X\frac{H_z}{A}\lambda \tag{5.19}$$

or by multiplication,

$$\left(\lambda^2 - \frac{mgs}{A}\right)^2 = -\left(\frac{H_z}{A}\right)^2\lambda^2 \tag{5.20}$$

The roots of Eq. (5.20) are

$$\lambda = \pm\frac{H_z}{2A}i \pm \sqrt{\frac{mgs}{A} - \left(\frac{H_z}{2A}\right)^2} \tag{5.21}$$

It is seen that if $(H_z/2A)^2 > mgs/A$, all four roots of Eq. (5.20) are pure imaginary quantities, and we obtain oscillations with constant amplitude. If $(H_z/2A)^2 < mgs/A$, two roots have positive real parts and two have negative. In this case the motion is unstable. The condition for stability is, therefore,

$$H_z^2 > 4Amgs \tag{5.22}$$

Hence, the rotation of a symmetrical top around a vertical axis is stable if its moment of momentum is sufficiently large. The minimum amount required is given by Eq. (5.22).

If s is less than zero, the center of gravity is below the fixed point of the top. Then Eqs. (5.7) give the small oscillations of a so-called *gyroscopic pendulum*. The reader will be able to discuss this case, which is also quite interesting, without difficulty.

6. Stability Conditions for Oscillating Systems.—A motion of the type $q =$ const. $e^{\lambda t}$ is stable if the exponent λ has no real positive part. In the two examples (sections 2 and 5) in which we discussed the stability of systems, the characteristic equations were *biquadratic*. In such cases the character of the roots can easily be decided. In this section we discuss the case in which the equation for λ is a cubic or an arbitrary quartic. A few remarks will be devoted to higher degree equations, which we encounter when dealing with three or more degrees of freedom.

Stability Conditions for the Cubic Equation.—We assume that the equation

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \tag{6.1}$$

is the characteristic equation of a system; we shall find the condition for the stability of the system. For stability it is necessary that none of the roots has a positive real part. Now, if λ_1, λ_2, λ_3 are the roots, the left side of Eq. (6.1) is equal to

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

Then it is seen that $-a$ is equal to the sum of the real parts of the roots. Hence, $-a$ must be negative, *i.e.*, $a > 0$. The same holds for the coefficient c, because $-c$ is equal to the product of the three roots. If the three roots are all real and negative, their product is certainly negative, and $c > 0$. If two roots are complex conjugates, *e.g.*, $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, $-c$ is the product of the real root λ_3 and of the product $\lambda_1\lambda_2 = \alpha^2 + \beta^2$. Hence, if λ_3 is negative, again $c > 0$. Finally, consider the coefficient $b = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$. If all roots are real and negative, $b > 0$; if, for instance, λ_1 and λ_2 are complex conjugates, we write $b = \lambda_1\lambda_2 + \lambda_3(\lambda_1 + \lambda_2)$. Now $\lambda_1\lambda_2$ is always positive; if the real parts of λ_1 and λ_3 are negative, $\lambda_1 + \lambda_2$ is negative and multiplied by the negative λ_3 again makes b positive. It follows that for stability all coefficients of the equation must be positive. However, this condition is not sufficient.

If $a > 0$, $b > 0$, $c > 0$, one of the roots, say λ_1, is certainly real and negative; however, the two other roots, say λ_2 and λ_3, could be conjugate complex with a positive real part. For example, if c is very large compared to a and b, λ will have a positive real part, since, approximately, $\lambda^3 + c = 0$, or $\lambda = c^{1/3}\sqrt[3]{-1}$, *i.e.*, $\lambda_1 \cong -c^{1/3}$, $\lambda_2 \cong \left(\frac{1}{2} + \frac{i}{2}\sqrt{3}\right)c^{1/3}$, $\lambda_3 \cong \left(\frac{1}{2} - \frac{i}{2}\sqrt{3}\right)c^{1/3}$. If we now vary the coefficient c; we pass through a value of c for which the real parts of λ_2 and λ_3 become negative. For this limiting value of c, the roots λ_2 and λ_3 are pure imaginary, for example, $\lambda_2 = \beta i$, $\lambda_3 = -\beta i$. Physically speaking the system is at the stability limit, carrying out pure harmonic vibration of the frequency β. Substituting $\lambda = \pm i\beta$ in Eq. (6.1), we have

$$\begin{aligned} -\beta^3 + b\beta &= 0 \\ -a\beta^2 + c &= 0 \end{aligned} \tag{6.2}$$

or eliminating β,

$$\frac{c}{a} = b \tag{6.3}$$

If we consider a and b as fixed values and c variable, the roots are continuous functions of c. Then it follows that the complex roots will have a positive real part for

$$c > ab \tag{6.4}$$

and a negative real part if

$$c < ab \tag{6.5}$$

The condition (6.5), together with $a > 0$, $b > 0$, $c > 0$, is necessary and sufficient for stability.

Stability Conditions for the Quartic Equation.—Let us write the characteristic quartic equation in the form:

$$f(\lambda) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0 \tag{6.6}$$

Then a reasoning analogous to that used in the case of the cubic shows that in case of stability $a > 0$ and $d > 0$ because the sum of the roots must be negative and their product must be positive. The coefficient $b = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4$ can be written in the form:

$$b = (\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4) + \lambda_1\lambda_2 + \lambda_3\lambda_4$$

Now, if all roots are real and negative, $b > 0$. If λ_1 and λ_2 are a pair of conjugate complex roots, $\lambda_1 + \lambda_2$ is real and negative, and $\lambda_1\lambda_2$ is real and positive; hence, again $b > 0$. If both λ_1, λ_2 and λ_3, λ_4 are pairs of conjugate complex roots, both $\lambda_1 + \lambda_2$ and $\lambda_3 + \lambda_4$ are real and negative, and $\lambda_1\lambda_2$ and $\lambda_3\lambda_4$ are real and positive; consequently, again $b > 0$ holds. It is easy to show in a similar way that under the assumption of negative real parts $c > 0$. If the real parts of all roots are equal to zero, $a = c = 0$. This is the case of the biquadratic equation discussed in the previous section.

Hence, for stability it is necessary that none of the coefficients of (6.6) should be negative. However, this condition is not sufficient. There is an additional condition which can be obtained by the method of the so-called *test function.**

Let us consider the equation

$$g(\lambda) = \lambda^4 + a(\alpha)\lambda^3 + b(\alpha)\lambda^2 + c(\alpha)\lambda + d(\alpha) = 0 \tag{6.7}$$

* The reasoning applied above to the cubic equation can be applied also to the quartic and yields immediately the condition (6.10). However, we prefer to introduce the test function in view of later applications.

whose coefficients are functions of a parameter α and satisfy the following conditions:

a. The values $a(0)$, $b(0)$, $c(0)$, and $d(0)$ for $\alpha = 0$ are equal to the values of the coefficients in Eq. (6.6).

b. The roots of (6.7) are equal to $\lambda_1 + \alpha$, $\lambda_2 + \alpha$, $\lambda_3 + \alpha$, $\lambda_4 + \alpha$, where λ_1, λ_2, λ_3, and λ_4 are the roots of Eq. (6.6). [How $a(\alpha)$, $b(\alpha)$, . . . can be calculated is shown later in Eq. (7.2)].

It is seen that if, for example, $\lambda_1 = -\alpha_1 + i\beta_1$, Eq. (6.7) has for $\alpha = \alpha_1$ two pure imaginary roots, *viz.*, $\pm\beta_1 i$. For such a value of α, the functions $a(\alpha)$, $b(\alpha)$, $c(\alpha)$, and $d(\alpha)$ must satisfy simple conditions. Specifically, if (6.7) has a pure imaginary root βi, we have

$$\beta^4 - b\beta^2 + d = 0 \tag{6.8}$$

and

$$a\beta^2 - c = 0$$

since the real and imaginary part of (6.6) must vanish separately.

Eliminating β^2, we have

$$\frac{c^2}{a^2} - b\frac{c}{a} + d = 0$$

or

$$c^2 - abc + a^2d = 0 \tag{6.9}$$

We call the function

$$F(\alpha) = c^2 - abc + a^2d$$

where a, b, c, and d are functions of α, as defined above, the *test function* of the quartic (6.6).

It is evident that as often as $-\alpha$ is equal to the real part of a root, of $f(\lambda) = 0$, we have $F(\alpha) = 0$, *i.e.*, this equation has a real root. If the real parts of all the roots of $f(\lambda)$ are negative, all real roots of $F(\alpha)$ must be positive, *i.e.*, must lie between $\alpha = 0$ and $\alpha = +\infty$, and no root of $F(\alpha)$ can lie between $\alpha = -\infty$ and $\alpha = 0$. Now if α is a very large negative number, say $-N$, the four roots of the polynomial $g(\lambda)$ will be approximately equal to $-N$, and $g(\lambda)$ will have the approximate form $g(\lambda) \cong (\lambda + N)^4$. Expanding this expression, we have $a(\alpha) \cong 4N$, $b(\alpha) \cong 6N^2$, $c(\alpha) \cong 4N^3$, and $d(\alpha) = N^4$. Hence, $F(-N) = c^2 - abc + a^2d \cong -64N^6$, and therefore $F(-\infty)$ is certainly negative. Consequently, if $F(\alpha)$ has no negative real roots, $F(0)$ must be negative too.

We obtain the result that if the roots of $f(\lambda) = 0$ have no positive real part,

$$F(0) = c^2 - abc + a^2d < 0^* \tag{6.10}$$

where a, b, c, and d, are the values of $a(\alpha)$, $b(\alpha)$, $c(\alpha)$, and $d(\alpha)$ for $\alpha = 0$, *i.e.*, the coefficients of $f(\lambda)$.

Condition (6.10) is necessary for stability and, together with the rules which refer to the signs of the coefficients, is sufficient.

That the conditions $a > 0$, $b > 0$, $c > 0$, $d > 0$, and $F(0) < \alpha$ are sufficient for stability of the system whose characteristic equation is $f(\lambda) = 0$, can be shown in the following way: If all coefficients of $f(\lambda)$ are positive, $f(\lambda) = 0$ can have no real positive roots; however, it would be possible to have complex roots with positive real parts. Now, since $F(-\infty)$ and $F(0)$ have identical signs, as was shown above, if $F(\alpha) = 0$ has real roots between $\alpha = -\infty$ and $\alpha = 0$, it must have an even number of them. Hence, there must be two pairs of conjugate complex roots with positive real parts, if any, *e.g.*, $\lambda_{1,2} = \alpha_1 \pm i\beta_1$ and $\lambda_{3,4} = \alpha_2 \pm i\beta_2$, where α_1 and α_2 are positive. But in this case $a = -(2\alpha_1 + 2\alpha_2)$ would be negative which contradicts the condition $a > 0$. Hence, if $a > 0$, $b > 0$, $c > 0$, $d > 0$, and $F(0) < \alpha$, none of the roots can have positive real parts.

The test-function method can also be applied to equations of higher degree; for instance, in the case of a sextic, the expression $F(0)$ is the so-called *resolvent* of two algebraic equations resulting from the substitution of $\lambda = i\beta$ and the separation of the real and imaginary terms.

7. Calculation of Complex Roots of Algebraic Equations.—There are exact formulas for the calculation of the roots of *cubic equations*. However, in general, it will be simpler to calculate real roots by one of the methods given in Chapter V, or if there is a pair of conjugate complex roots to calculate first the real root, λ_1, divide the equation by $\lambda - \lambda_1$, and solve the remaining equation of second degree.

In the case of *quartics* the following methods are recommended:

a. The Test-function Method.—The test function introduced in the last section can also be used for numerical computation of the complex roots by plotting $F(\alpha)$ as function of α and determining the real roots of $F(\alpha) = 0$. The roots of $F(\alpha) = 0$ determine

* Equation (6.10) for $d = 0$ becomes identical with Eq. (6.5).

the real parts of the roots of $f(\lambda) = 0$. Then the complex parts are determined by one of the equations (6.8), *e.g.*, by the relation $\beta^2 = c/a$.

In order to plot $F(\alpha)$, the coefficients $a(\alpha)$, $b(\alpha)$, $c(\alpha)$, and $d(\alpha)$ must be computed as functions of α. Let us denote the roots of $f(\lambda) = 0$, by λ_1, λ_2, λ_3, and λ_4; then the polynomial whose roots are equal to $\lambda_1 + \alpha$, $\lambda_2 + \alpha$, $\lambda_3 + \alpha$, and $\lambda_4 + \alpha$ is given by

$$g(\lambda) = f(\lambda - \alpha) = f(-\alpha) + f'(-\alpha)\lambda + \frac{f''(-\alpha)}{1 \cdot 2}\lambda^2 + \frac{f'''(-\alpha)}{1 \cdot 2 \cdot 3}\lambda^3 + \frac{f^{\mathrm{IV}}(-\alpha)}{1 \cdot 2 \cdot 3 \cdot 4}\lambda^4 \quad (7.1)$$

For example, $g(\lambda_1 + \alpha) = f(\lambda_1 + \alpha - \alpha) = f(\lambda_1) = 0$. We obtain

$$g(\lambda) = \lambda^4 + (a - 4\alpha)\lambda^3 + (b - 3a\alpha + 6\alpha^2)\lambda^2 + (c - 2b\alpha + 3\alpha^2 - 4\alpha^3)\lambda + (d - c\alpha + b\alpha^2 - a\alpha^3 + \alpha^4) \quad (7.2)$$

The coefficients of λ^3, λ^2, λ, 1 are the functions $a(\alpha)$, $b(\alpha)$, $c(\alpha)$, $d(\alpha)$ and, substituting them in $F(\alpha) = c^2 - a(bc - ad)$, we obtain the test function sought for.

b. Approximate Factorization.—Let us assume that the quartic equation

$$f(\lambda) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0 \quad (7.3)$$

has two pairs of conjugate complex roots λ_1, λ_2 and λ_3, λ_4 and in addition, that $|\lambda_1| = |\lambda_2| >> |\lambda_3| = |\lambda_4|$. Then it is easily shown by a reasoning quite similar to that applied in Chapter V to equations with real roots that λ_1 and λ_2 are well approximated by the roots of the equation $\lambda^2 + a\lambda + b = 0$ and the small roots λ_3 and λ_4 by the roots of the equation $b\lambda^2 + c\lambda + d = 0$. Hence, the expression

$$f(\lambda) \cong (\lambda^2 + a\lambda + b)\left(\lambda^2 + \frac{c}{b}\lambda + \frac{d}{b}\right) = 0 \quad (7.4)$$

represents an *approximate factorization* of Eq. (7.3).

If the ratio $|\lambda_1|/|\lambda_3|$ is not sufficiently large, the method of squaring the roots can be applied in a manner similar to that used in the case of real roots. The method is not very efficient in the case of complex roots because, if we apply the squaring n times, we obtain an equation for λ^{2n}; and if λ is complex, it is rather hard to determine which of the $2n$ roots of λ^{2n} is the correct

root of the original equation (7.3). In the case of real roots, only the sign of λ is dubious; in the case of complex roots we obtain $2n$ different complex numbers as possible values for λ.

The following general theorem, which is quoted here without proof, is useful for equations of higher degree and for equations with some real and some complex roots:

If the absolute values of r roots (real or complex) of the equation

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0 \tag{7.5}$$

are very large in comparison with the rest of the roots, the r *large* roots are approximately given by the roots of the equation:

$$\lambda^r + a_1\lambda^{r-1} + \cdots + a_r = 0$$

and the rest of the roots are approximated by the roots of the equation

$$a_r\lambda^{n-r} + a_{r+1}\lambda^{n-r-1} + \cdots + a_n = 0$$

c. Extension of Newton's Method to Complex Roots.—Newton's original method can be applied to the calculation of complex roots if we start from an approximate complex value. Starting from a real value, the method can never lead to complex roots as is seen by Eq. (8.1) in Chapter V. However, the method can be modified by approximating the function $f(\lambda)$ by a quadratic function instead of a linear one. Using Taylor's expansion, we write

$$f(\lambda) = f(\lambda_1) + f'(\lambda_1)(\lambda - \lambda_1) + \tfrac{1}{2}f''(\lambda_1)(\lambda - \lambda_1)^2 \tag{7.6}$$

Choosing now an arbitrary value of λ_1 as first approximation, we obtain, solving Eq. (7.6) for λ,

$$\lambda = \lambda_1 - \frac{f'(\lambda_1)}{f''(\lambda_1)} \pm \frac{\sqrt{f'(\lambda_1)^2 - 2f(\lambda_1)f''(\lambda_1)}}{f''(\lambda_1)} \tag{7.7}$$

If the radical in the last term is negative we obtain a complex value for the next approximation of the root. By the process of iteration we are able to obtain successive approximations for a pair of conjugate complex roots.

This method has the disadvantage that very complicated numerical work has to be done with complex quantities. It is an advantage to have schemes for the numerical calculation which

involve only real quantities. If $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ are one pair of complex roots of the equation $f(\lambda) = 0$, the polynomial $f(\lambda)$ is divisible by the quadratic factor

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2$$

Hence, if we assume approximate values for the coefficients 2α and $\alpha^2 + \beta^2$ and improve these coefficients by the process of iteration, we also improve the approximations for the corresponding roots. Methods for successive correction of the coefficients of a quadratic factor have been worked out by L. Bairstow, J. I. Craig, and others.

8. Longitudinal Stability of an Airplane.—Assume that every point of an airplane moves in a vertical plane parallel to its plane of symmetry; we call such a motion the *longitudinal motion* of the airplane. The longitudinal motion of the airplane is described by the translation of its center of gravity and the rotation of the airplane body around an axis normal to the plane of motion. The equations of motion were given in Chapter IV [Eqs. (9.1) to (9.3)]

$$\begin{aligned} m\frac{dv}{dt} &= -D + mg\sin\theta \\ mv\frac{d\theta}{dt} &= -L + mg\cos\theta \\ I\frac{d^2\varphi}{dt^2} &= -M \end{aligned} \tag{8.1}$$

In these equations the following notations are used:

v the magnitude of the velocity of the center of gravity.
θ the inclination of the flight path, positive downward.
φ the angle between the longitudinal axis of the airplane and the horizontal, positive upward.
m the mass of the airplane.
I the moment of inertia of the airplane around a transversal axis through the center of gravity.
D the drag, *i.e.*, the aerodynamic force opposite to the flight direction.
L the lift, *i.e.*, the aerodynamic force normal to the flight direction.
M the moment of the aerodynamic forces, diving moment being measured positive.

To investigate the small oscillations of the airplane in the neighborhood of uniform level flight with the velocity v_0 we assume that the velocity $v = v_0 + u$, where u is small compared to v_0 and θ, φ are small angles. Furthermore, we assume that the drag D can be neglected or that it is balanced at every instant by a propeller thrust of equal magnitude. Then Eqs. (8.1) upon neglecting terms of higher order give us the following relations:

$$\begin{aligned} \frac{1}{g}\dot{u} &= \theta \\ \frac{v_0}{g}\dot{\theta} &= -\frac{L}{mg} + 1 \\ i^2\ddot{\varphi} &= -\frac{M}{m} \end{aligned} \tag{8.2}$$

where $i^2 = I/m$, *i.e.*, i is equal to the radius of gyration of the airplane around its transversal axis.

In level flight $L = mg$ and $M = 0$. Hence, if ΔL and ΔM are the increments of the lift and the moment, we have

$$L = mg + \Delta L, \qquad M = \Delta M$$

Let us calculate the increments ΔL and ΔM as functions of u, θ, and φ. It is assumed in general that the lift is proportional to the square of the velocity of flight and is a linear function of the angle of attack α. We measure the angle of attack from the value corresponding to zero lift and assume that $\alpha = \alpha_0$ and $\varphi = 0$ correspond to level flight. Then we write

$$L = mg\left(\frac{v}{v_0}\right)^2\frac{\alpha}{\alpha_0} \tag{8.3}$$

or substituting $v = v_0 + u$ and $\alpha = \alpha_0 + \Delta\alpha$ and neglecting terms of higher order

$$L = mg\left(1 + 2\frac{u}{v_0} + \frac{\Delta\alpha}{\alpha_0}\right) \tag{8.4}$$

Taking into account that $\Delta\alpha = \theta + \varphi$,

$$L = mg\left(1 + 2\frac{u}{v_0} + \frac{\theta + \varphi}{\alpha_0}\right) \tag{8.5}$$

The aerodynamic moment M consists of the wing moment and the moment of the aerodynamic force acting on the tail surface. The tail force consists of two parts: one due to the change of the angle of attack, the other due to the rotation of the airplane. We combine the first part of the tail force and the wing moment into one term, in that we write for the total stabilizing moment produced by a change $\Delta\alpha$ of the angle of attack

$$\Delta M_1 = mgck_m\Delta\alpha \tag{8.6}$$

where c is the mean chord length of the wing and k_m is a numerical factor. We say that k_m determines the *static stability* of the airplane, *i.e.*, the magnitude of the restoring or stabilizing moment.

The contribution of the rotation to the tail force corresponds to an apparent increase of the angle of attack. If the airplane rotates with the angular velocity $d\varphi/dt$ and the distance between the center of pressure of the tail and the center of gravity of the airplane is l, the vertical velocity of the tail is equal to $l\dfrac{d\varphi}{dt}$. This is equivalent to a change in angle of attack of the amount $\dfrac{1}{v_0}l\dfrac{d\varphi}{dt}$. Thus the rotation produces a force on the tail which is proportional to $\dfrac{1}{v_0}l\dfrac{d\varphi}{dt}$, and we can write the corresponding moment about the center of gravity of the airplane in the form:

$$\Delta M_2 = k_t mgl^2\dot{\varphi}\frac{1}{v_0} \tag{8.7}$$

where k_t is a numerical constant that depends especially on the ratio between tail and wing surface.

Substituting (8.5), (8.6), and (8.7) into Eq. (8.2), we obtain

$$\begin{aligned}
\frac{1}{g}\dot{u} &= \theta \\
\frac{v_0}{g}\dot{\theta} &= -2\frac{u}{v_0} - \frac{1}{\alpha_0}(\theta + \varphi) \\
\frac{1}{g}\ddot{\varphi} &= -\frac{k_m c}{i^2}(\varphi + \theta) - k_t\left(\frac{l}{i}\right)^2\frac{1}{v_0}\dot{\varphi}
\end{aligned} \tag{8.8}$$

We eliminate u and obtain the following two equations for θ and φ:

$$\begin{aligned}\frac{v_0^2}{g^2}\ddot{\theta} &= -2\theta - \frac{v_0}{g\alpha_0}(\dot{\theta} + \dot{\varphi}) \\ \frac{v_0^2}{g^2}\ddot{\varphi} &= -k_m\frac{cv_0^2}{gi^2}(\varphi + \theta) - k_t\frac{l^2}{i^2}\frac{v_0}{g}\dot{\varphi}\end{aligned} \tag{8.9}$$

In order to obtain a dimensionless form of these equations we introduce the dimensionless time parameter $\tau = tg/v_0$. This amounts to using v_0/g as an appropriate time unit. Then we have

$$\begin{aligned}\frac{d^2\theta}{d\tau^2} &= -2\theta - \frac{1}{\alpha_0}\frac{d\theta}{d\tau} - \frac{1}{\alpha_0}\frac{d\varphi}{d\tau} \\ \frac{d^2\varphi}{d\tau^2} &= -\sigma(\varphi + \theta) - \delta\frac{d\varphi}{d\tau}\end{aligned} \tag{8.10}$$

where σ and δ are used as abbreviations for the two dimensionless quantities occurring in the second of Eqs. (8.9). It is seen that the solution of the stability problem depends on the following three dimensionless parameters:

α_0 is a characteristic aerodynamic parameter of the airplane; it depends on the lift coefficient employed in level flight.

$\sigma = k_m\dfrac{cv_0^2}{gi^2}$ is a parameter which is characteristic for the static stability of the airplane. In fact, $\sigma > 0$ means that a rotation of the airplane by an angle φ produces a restoring moment. If $\sigma < 0$, the moment tends further to increase the angle φ. Hence, we say that if $\sigma > 0$, the airplane is *statically stable*, if $\sigma < 0$, it is *statically unstable;* if $\sigma = 0$, the airplane has *neutral static stability.*

$\delta = k_t\dfrac{l^2}{i^2}$ is a parameter which determines the damping effect of the tail.

Substituting $\theta = Ae^{\lambda\tau}$, $\varphi = Be^{\lambda\tau}$, we obtain the following frequency equation:

$$\begin{vmatrix}\lambda^2 + \dfrac{\lambda}{\alpha_0} + 2 & \dfrac{1}{\alpha_0}\lambda \\ \sigma & \lambda^2 + \delta\lambda + \sigma\end{vmatrix} = 0 \tag{8.11}$$

or, expanding the determinant,

$$\lambda^4 + \left(\frac{1}{\alpha_0} + \delta\right)\lambda^3 + \left(2 + \sigma + \frac{\delta}{\alpha_0}\right)\lambda^2 + 2\delta\lambda + 2\sigma = 0 \tag{8.12}$$

Before we investigate the stability criteria, let us assume that σ is very large, so that all terms which do not contain σ are small compared with the terms containing σ. Then, neglecting all the rest of the terms, we obtain

$$\lambda^2 + 2 = 0 \tag{8.13}$$

or $\lambda = \pm\sqrt{-2}$. Taking into account that $\tau = tg/v_0$, this value of λ corresponds to harmonic oscillations with the period

$$T = 2\pi\frac{v_0}{g\sqrt{2}}$$

In addition to the roots $\lambda = \pm\sqrt{-2}$ Eq. (8.12) must have two other roots. It is seen that these roots must be of the order of magnitude $\sqrt{\sigma}$. In that case λ^4 and $\sigma\lambda^2$ are of the same order and are large compared with the rest of the terms. Thus, we have approximately

$$\lambda^2 + \sigma = 0 \tag{8.14}$$

or $\lambda = \pm\sqrt{-\sigma}$. The corresponding period is equal to

$$T = 2\pi\frac{v_0}{g\sqrt{\sigma}} = 2\pi\sqrt{\frac{i^2}{gck_m}}$$

Let us investigate the corresponding modes of oscillation: If σ is very large compared with δ, the two equations (8.10) are reduced to

$$\begin{aligned} \frac{d^2\theta}{d\tau^2} + 2\theta &= -\frac{1}{\alpha_0}\frac{d}{d\tau}(\theta + \varphi) \\ \frac{d^2\varphi}{d\tau^2} + \sigma\varphi &= -\sigma\theta \end{aligned} \tag{8.15}$$

If $\lambda = \pm\sqrt{-2}$, the second equation is satisfied with sufficient approximation by $(\varphi + \theta) = 0$, and the first equation gives (with $\tau = gt/v_0$)

$$\theta = C\sin\frac{\sqrt{2}gt}{v_0} + D\cos\frac{\sqrt{2}gt}{v_0} \tag{8.16}$$

This result corresponds exactly to the result obtained in Chapter IV for the so-called *phugoid motion*. In fact, there the assumption was made that the airplane is so stable that every deviation

in the angle of attack is instantaneously corrected. This means that σ is very large and $\Delta\alpha = \varphi + \theta = 0$.

For $\lambda = \pm\sqrt{-\sigma}$, the first equation is satisfied with sufficient approximation by $\theta = 0$, and we are left with the equation

$$\frac{d^2\varphi}{d\tau^2} + \sigma\varphi = 0 \tag{8.17}$$

which means that the airplane oscillates about its center of gravity as a pendulum oscillates about its axis of suspension.

Hence, in the case of excessive static stability the motion of the airplane consists of a phugoid-like motion of the center of gravity and of an oscillation of the airplane body around the center of gravity.* The wave length of the phugoid motion is equal to $2\pi\frac{v_0^2}{g\sqrt{2}}$, where v_0 is the speed of level flight, and the period of the rotational oscillation is equal to $2\pi\sqrt{i^2/gck_m}$.

Let us now drop the assumption of large σ and investigate the conditions for the stability of the oscillations defined by the frequency equation (8.12). We found in section 6 that in the case of stability, the coefficients

$$a = \frac{1}{\alpha_0} + \delta, \qquad b = 2 + \sigma + \frac{\delta}{\alpha_0}, \qquad c = 2\delta, \qquad d = 2\sigma$$

of Eq. (8.12) must be positive. In addition, the expression

$$F = c^2 - a(bc - ad) \tag{8.18}$$

has to be negative.

It is seen that the coefficients a, b, c, and d are positive if $\sigma > 0$, since δ and α_0 are positive by their definition. Hence, an airplane can be stable only if $\sigma > 0$, *i.e.*, if it has static stability.

Substituting the values of a, b, c, and d into (8.18), we obtain as an additional stability criterion

$$4\delta^2 - \left(\frac{1}{\alpha_0} + \delta\right)\left(4\delta + \frac{2\delta^2}{\alpha_0} - \frac{2\sigma}{\alpha_0}\right) < 0$$

or

$$\sigma < \delta^2 + \frac{2\delta\alpha_0}{1 + \delta\alpha_0} \tag{8.19}$$

* As a matter of fact, both modes of oscillation are slightly damped if the stability criterion (8.19) is satisfied; the amplitude of the first mode increases slowly if the stability criterion is violated.

The line $\sigma = \delta^2 + \frac{2\delta\alpha_0}{1 + \delta\alpha_0}$ is plotted in Fig. 8.1 corresponding to $\alpha_0 = \frac{1}{12}$. The abscissa in this diagram is δ (damping parameter), the ordinate is σ (parameter of static stability). It is seen that the airplane is stable only if it has a certain damping. The magnitude of δ which is required for stability increases with increasing static stability. If the drag D is taken into account, the limiting curve which separates the stable and unstable regions in the $\delta\sigma$ plane intersects the positive σ-axis. Hence, in reality, if the parameter of the static stability σ is beyond a certain limit, the airplane becomes dynamically stable also for $\delta = 0$.

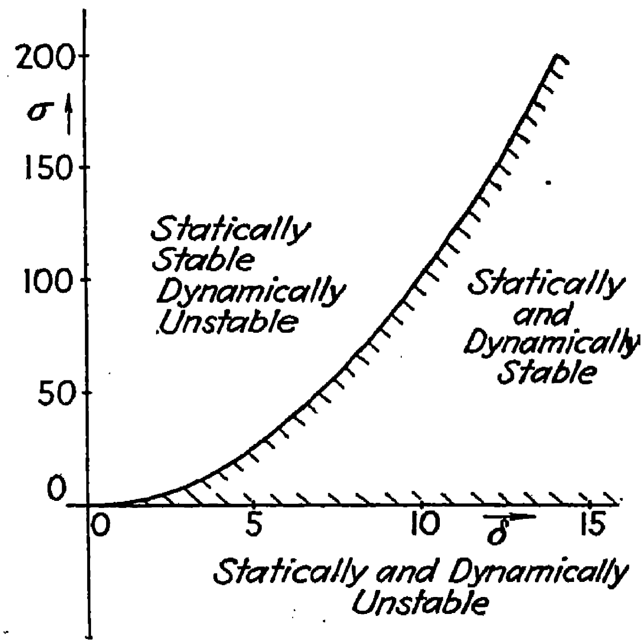

FIG. 8.1.—Diagram for the longitudinal stability of an airplane.

Problems

1. A shaft rotates in an abundantly lubricated sleeve bearing. Assume that a displacement ρ of the center of the shaft from the center of the bearing produces a reaction of the magnitude R directed at an angle of $180° - \theta$ with the direction of the displacement (Fig. P.1). Also assume that the shaft is unloaded so that in the equilibrium position the center of the shaft coincides with the center of the bearing. Show that if damping is neglected, this equilibrium position is unstable unless $\theta = 0$.

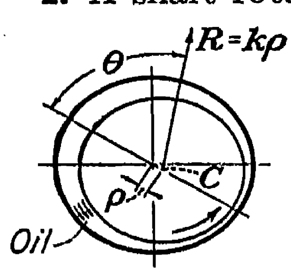

FIG. P.1.

2. A U-tube pressure gauge consists of two vertical tubes of diameter d_1 and a horizontal capillary tube of diameter $d_2 = d_1/10$ and length l. The height of the liquid in the columns is equal to h_0 when the same pressure acts on both columns. The specific weight of the fluid is γ, the coefficient of viscosity μ. Find the amplitude of the oscillations of the liquid columns if a variable pressure acts on one of the columns, when the pressure difference is given by $p\left(1 + \epsilon \sin \frac{2\pi t}{T}\right)$ and $\epsilon \ll 1$. The fluid resistance in the vertical tubes can be neglected.

Hint: The pressure drop in a capillary tube of diameter d and length l is, according to Poiseuille's law, $\Delta p = \frac{128}{\pi} \frac{\mu l}{d^4} Q$ where Q is the quantity of fluid flowing through per unit time.

3. *Condenser Microphone.*—The electromechanical setup known as the *condenser microphone* is shown schematically in Fig. P.3. The circuit consists of a battery of constant voltage E, a coil of inductance L, a resistance R, and a condenser of variable capacity C. The condenser has one fixed plate to which a movable plate of mass m is attached elastically. This movable electrode is the receiving membrane of the microphone. It vibrates under the variable pressure F of the acoustic waves. The system has two degrees of freedom. The charge Q of the condenser and the displacement q of the membrane are the two generalized coordinates. Set up the differential equations for these quantities using Lagrange's equations.

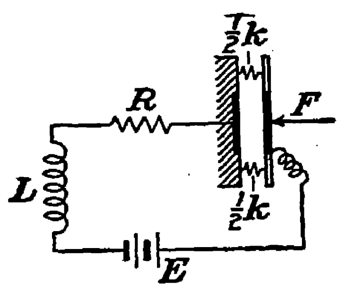

FIG. P.3.

Solution: This is a mixed problem in which one of the variables is a mechanical and the other an electrical quantity. However, we know that no distinction needs to be made between them in setting up Lagrange's equations. When the system is at rest, the condenser is charged by an amount Q_0, therefore the electrodes are attracted and compress the springs by an amount q_0. When the system oscillates, the charge and the displacement are $Q_0 + Q$ and $q_0 + q$. The capacity of the condenser is inversely proportional to the distance between the electrodes; we put $C = \dfrac{A}{a - q}$ where A and a are constants. The total potential energy (electrostatic and elastic) is

$$U = \frac{1}{2}\left(\frac{a - q}{A}\right)(Q_0 + Q)^2 + \frac{1}{2}k(q_0 + q)^2 - EQ$$

Taking into account the equilibrium conditions $\partial U/\partial Q = \partial U/\partial q = 0$, for $Q = q = 0$, and dropping terms higher than quadratic in Q and q, we write $U = \dfrac{1}{2}\left(\dfrac{a}{A}Q^2 - 2\dfrac{Q_0}{A}Qq + kq^2\right) + \text{const.}$ The term $\dfrac{2Q_0}{A}Qq$ represents the electromechanical coupling. The reader will notice that the coupling coefficient $2Q_0/A$ is proportional to E. The sum of the magnetic and kinetic energies is $T = \frac{1}{2}L\dot{Q}^2 + \frac{1}{2}m\dot{q}^2$; the dissipation function is $D = \frac{1}{2}R\dot{Q}^2$. Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = -\frac{\partial U}{\partial q} + F; \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}}\right) = -\frac{\partial D}{\partial \dot{Q}} - \frac{\partial U}{\partial Q}$$

4. A short electric train is made up of three units: a locomotive and two passenger cars. The weight of each unit is 40,000 lb., the spring constants of the springs connecting them are equal to 15,000 lb./in. Determine the smallest value of the damping factor for two identical shock absorbers placed between the three units and acting by viscous friction such that the relative motion of the cars is not oscillatory.

Solution: The characteristic equation of the system is

$$(m\lambda^2 + \beta\lambda + k)(m\lambda^2 + 3\beta\lambda + 3k) = 0$$

where m is the mass of one car, k is the spring factor, and β the damping factor. The general solution is the superposition of two damped oscillations. The critical damping for the oscillation corresponding to the first factor of the characteristic equation is equal to

$$\beta = 2\sqrt{km} = 2\sqrt{\frac{40{,}000 \times 15{,}000}{386}} \text{ lb.-sec./in.}$$

The condition for the nonoscillatory character of the motion corresponding to the second factor of the characteristic equation is $3\beta > 2\sqrt{3km}$. This condition is satisfied by the above value of β.

5. The vertical axis of symmetry of the gyroscope shown in Fig. P.5 is free to rotate about the point A and is restrained by two springs at the point B. The axes of the springs are perpendicular to each other; their spring constants are k_1 and k_2. Calculate the magnitude of the moment of momentum of the gyroscope required for stable rotation.

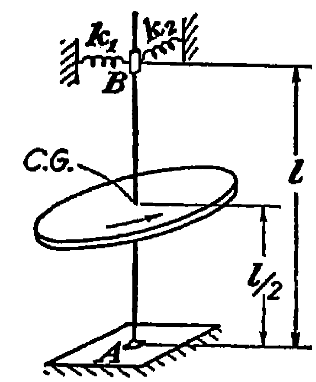

FIG. P. 5.

6. An airplane engine is suspended in an elastic mounting as shown in Chapter V, Prob. 8, in such a manner that the three translatory and rotational degrees of freedom are not coupled. Show that the gyroscopic moment of the propeller introduces a coupling between the rotations, and calculate the precession of the axis of the engine for the case of a four-bladed propeller.

7. The stability of a monorail car is obtained by a vertical gyroscope. The axis of the gyroscope is hinged to the car at point P in such a way that it can move only in the plane of symmetry of the car. We have the following data:

W_c total weight of the car and the gyroscope.
W_g weight of the gyroscope.
I moment of inertia of the car with its mounted gyroscope about the rail.
C moment of inertia of the gyroscope about its spin axis.
A moment of inertia of the gyroscope about an axis through P and perpendicular to the plane of symmetry of the car.
h height of the center of gravity of the gyroscope above P.
s height of the center of gravity of the car and its gyroscope above the rail.

Set up the equations of motion for small oscillations of the gyroscope around the vertical and determine the spin velocity Ω of the gyroscope required for stability.

Hint: Calculate the rate of change of the moment of momentum of the gyroscope about a horizontal axis passing through P. Because the oscillations are small, this axis may be considered as fixed. We have

$$A\ddot{\theta} = C\Omega\dot{\varphi} + W_g h\theta \ .$$

where θ is the inclination of the gyropscope in the plane of symmetry of the car and φ the inclination of the car. Similarly, taking moments about the rail, $I\ddot{\varphi} = -C\Omega\dot{\theta} + W_c s\varphi$.

8. The equation of motion for the *sleeve* of a centrifugal steam-engine *governor* is given by

$$m\ddot{x} + \beta\dot{x} + kx = C_1(\omega - \omega_0)$$

where m is an inertia factor which accounts for the mass of the flyballs, the link arms, and the sleeve, β is a damping factor, k the constant of the governor spring, ω the instantaneous velocity, ω_0 the mean angular velocity of the engine, and C_1 a characteristic constant of the governor, which determines the force acting on the sleeve if the angular velocity is different from its mean value. The equation of the engine is of the form:

$$I\frac{d\omega}{dt} = -C_2x$$

where I is the equivalent moment of inertia of the moving parts of the engine and C_2 is a characteristic constant of the engine, which determines the torque as function of the displacement x of the sleeve of the governor. Find the condition of stability of the coupled system consisting of the engine and the governor.

9. Solve the algebraic equation

$$x^4 - 9x^3 + 30x^2 - 51x + 26 = 0$$

10. Solve the algebraic equation

$$x^5 - 7x^4 + 22x^3 - 38x^2 + 37x - 16 = 0$$

References

General:

1. Reference 2, Chapter III; references 2 to 4, Chapter IV; and references 2 and 4, Chapter V.

Special problems:

2. References 5 and 6, Chapter III (for section 5).
3. JONES, B. M.: "Dynamics of the Airplane." W. F. Durand, "Aerodynamic Theory," Vol. V, Verlag Julius Springer, Berlin, 1935.

CHAPTER VII

THE DIFFERENTIAL EQUATIONS OF THE THEORY OF STRUCTURES

> *Salviati.*—In this discussion I shall take for granted the well-known mechanical principle, which has been shown to govern the behavior of a bar, which we call a lever, namely, that the force bears to the resistance the inverse ratio of the distances which separate the fulcrum from the force and resistance, respectively.
>
> *Simplicio.*—This was demonstrated first of all by Aristotle in his "Mechanics."
>
> *Salviati.*—Yes, I am willing to concede his priority in point of time; but as regards rigor of demonstration the first place must be given to Archimedes.
>
> —GALILEO GALILEI, "Dialogues concerning Two New Sciences —The Second Day" (1638).

Introduction.—This chapter is concerned with the equilibrium and with the harmonic oscillations of cords and beams. Practically all problems of this chapter are boundary problems, *i.e.*, they ask for particular solutions which satisfy certain conditions at the boundaries. We treat in detail differential equations which have constant coefficients or can be reduced to the standard form of Bessel's differential equation. However, methods are shown for solving problems involving arbitrary differential equations also.

In the differential equations treated in Chapters IV to VI, the *time* appeared as an independent variable. In the differential equations encountered in this chapter the independent variable is a *space coordinate*. The unknown function is in most cases the deflection of a one-dimensional structure, such as a string or a beam. The deflection under given loads is governed by *nonhomogeneous* differential equations; the particular physical setup, *e.g.*, the type of support or the type of connection with other parts of the structure, furnishes the *boundary conditions*.

The next group of problems deals with the determination of the natural frequencies of one-dimensional structures. In the oscillation problems of Chapter V we were concerned with the

oscillations of a finite number of masses, mass points, or rigid bodies, whereas we shall now consider the masses continuously distributed over the whole elastic structure. To be sure, we restrict ourselves to harmonic oscillations of structures, *i.e.*, we do not treat *transient* vibrations which are governed by partial differential equations where the time and at least one space coordinate enter as independent variables.

Returning to equilibrium problems, we treat the classical problem of elastic instability or buckling of columns—a problem first treated by Euler. There are two methods of approach to this problem. In one method the existence of equilibrium positions in the vicinity of the undeflected shape of the column is investigated, *i.e.*, the existence of solutions of the differential equation in addition to the *trivial solution* which represents the undeflected shape. The second method consists of the investigation of the stability of the undeflected shape by comparison of the total potential energy of the system in undeflected and various deflected positions. This method leads to the *calculus of variations*. This chapter contains a short discussion of the stability problem from this point of view, and we give in Chapter VIII a practical, so-called *direct method* for solving simple variation problems.

The free oscillation and buckling problems lead to *homogeneous* differential equations containing unknown parameters and require the determination of certain *characteristic values* of these parameters for which the given boundary conditions can be satisfied. These characteristic values represent the *natural frequencies*, the *critical speeds*, or the *critical loads* of the structure.

Several examples of forced oscillations and combined axial and lateral loadings of structures are included. These problems lead to *nonhomogeneous* equations, however, with a parameter in each equation representing either the frequency of the impressed force or the axial load. The character of the solution depends greatly on the value of this parameter. If certain characteristic values are approached, the structure develops resonance or may collapse owing to excessive deflections.

1. Deflection of a String under Vertical Load.—We consider a perfectly flexible string submitted to vertical loads (Fig. 1.1a). Since such a string has no resistance to bending, the only internal force is the tension F acting in the direction of the tangent of the

deflection curve. Let us denote the ordinates of the deflection curve by $w(x)$, the angle between the horizontal x-direction and the tangent to the deflection curve by θ, the load per unit length of the horizontal projection of the deflection curve by $p(x)$.

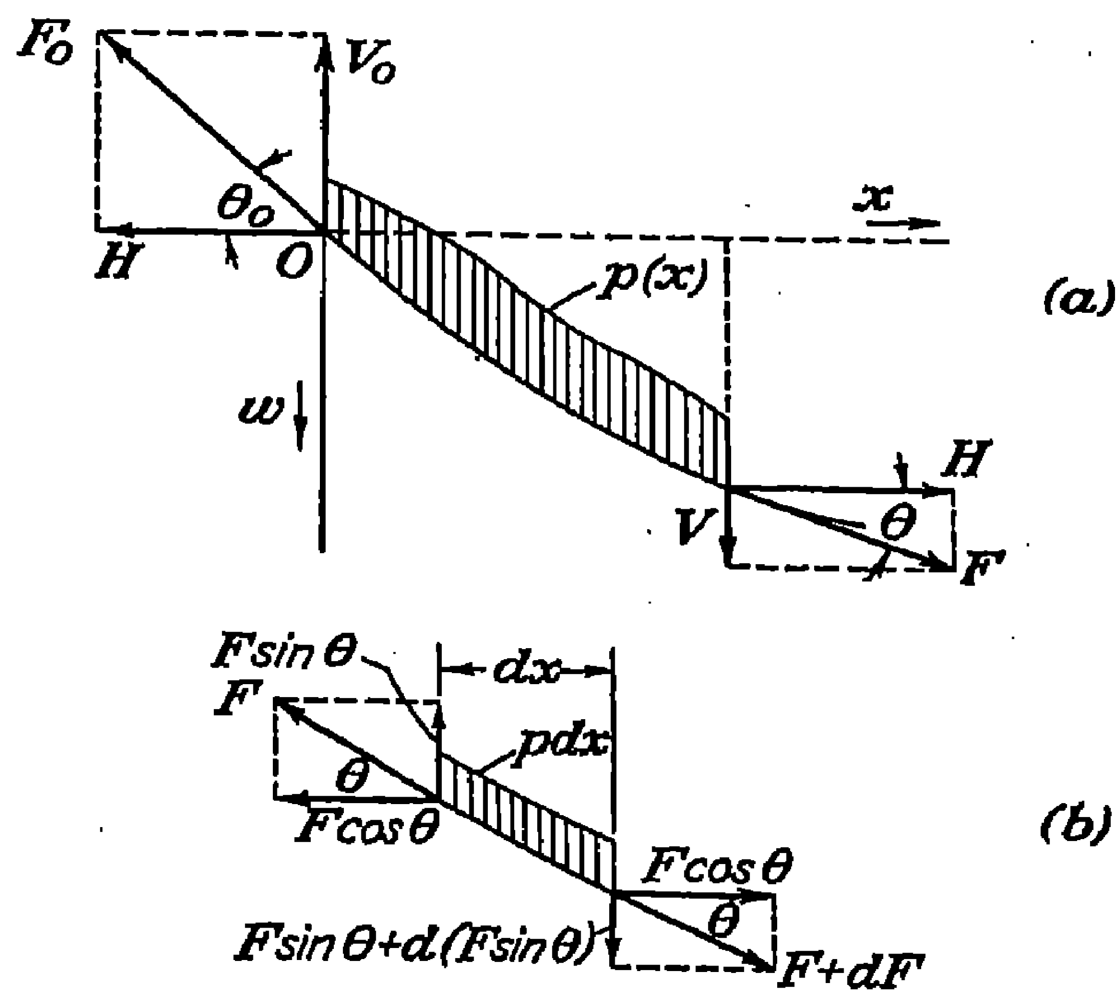

FIG. 1.1.—Equilibrium of a string under a vertical distributed load.

Then the equilibrium equations for an element between x and $x + dx$ are (Fig. 1.1b)

$$\begin{aligned} \frac{d}{dx}(F\cos\theta) &= 0 \\ \frac{d}{dx}(F\sin\theta)\,dx + p\,dx &= 0 \end{aligned} \tag{1.1}$$

According to the first equation, the horizontal component of the tension, $H = F\cos\theta$ is constant, and substituting H in the second equation, we have

$$H\frac{d}{dx}(\tan\theta) = -p(x) \tag{1.2}$$

or with $\tan\theta = dw/dx$

$$H\frac{d^2w}{dx^2} = -p(x) \tag{1.3}$$

Let us assume that $w = 0$ and $dw/dx = \tan\theta_0$ for $x = 0$. Then by repeated integration

$$w = -\frac{1}{H}\int_0^x d\xi \int_0^\xi p(\xi)\, d\xi + x \tan \theta_0 \tag{1.4}$$

Integrating by parts, we have

$$\int_0^x d\xi \int_0^\xi p(\xi)\, d\xi = \int_0^x xp(\xi)\, d\xi - \int_0^x \xi p(\xi)\, d\xi$$

and substituting this expression in Eq. (1.4),

$$w = -\frac{1}{H}\int_0^x (x - \xi)p(\xi)\, d\xi + x \tan \theta_0 \tag{1.5}$$

If we write Eq. (1.5) in the form:

$$-Hw + Hx \tan \theta_0 + \int_0^x (\xi - x)p(\xi)\, d\xi = 0 \tag{1.6}$$

it is easily seen that Eq. (1.6) expresses the condition that the resulting moment of the forces acting on the string to the left of an arbitrary point whose coordinates are x,w (cf. Fig. 1.1), is equal to zero. In fact, $-Hw$ represents the moment of the horizontal component H of the tension; the term $Hx \tan \theta_0$ represents the moment of the vertical reaction V, which is equal to $H \tan \theta_0$; and finally, the integral $\int_0^x (\xi - x)p(\xi)\, d\xi$ represents the moment of the load p.

Let us assume that a concentrated force P_i acts at the point $x = \xi_i$. This means that the distributed load $p(x)$ per unit length has very large values in a small interval between $x = \xi_i - \epsilon$ and $x = \xi_i + \epsilon$. Let us write $\xi = \xi_i + \xi'$; then

$$\int_{\xi_i-\epsilon}^{\xi_i+\epsilon} (\xi - x)p(\xi)\, d\xi = \int_{\xi_i-\epsilon}^{\xi_i+\epsilon} (\xi_i - x)p(\xi)\, d\xi + \int_{-\epsilon}^{\epsilon} \xi' p(\xi')\, d\xi' \tag{1.7}$$

If we proceed to the limit $\epsilon \to 0$, the first integral on the right side of Eq. (1.7) becomes $(\xi_i - x)\int_{\xi_i-\epsilon}^{\xi_i+\epsilon} p(\xi)\, d\xi = (x - \xi_i)P_i$, and the second term vanishes, as ξ' is of the order of ϵ. The expression $(\xi_i - x)P_i$ is equal to the moment of P_i with respect to the point x. Hence, if we include concentrated forces, Eq. (1.6) becomes

$$-Hw + Hx \tan \theta_0 + \sum_i (\xi_i - x)P_i + \int_0^x (\xi - x)p(\xi)\, d\xi = 0 \tag{1.8}$$

It follows from Eq. (1.8) that the deflection w of a string loaded by vertical forces—distributed or concentrated—multiplied by the horizontal tension H gives the value of the moment of the vertical forces whose lines of action are to the left of x, including the vertical reaction V, the concentrated forces P_i, and the distributed load $p(\xi)$.

The *string polygon method* commonly used for the construction of bending-moment diagrams of beams is based on Eq. (1.8). The string polygon is the actual equilibrium configuration of a string under the action of the loads acting on the beam and of

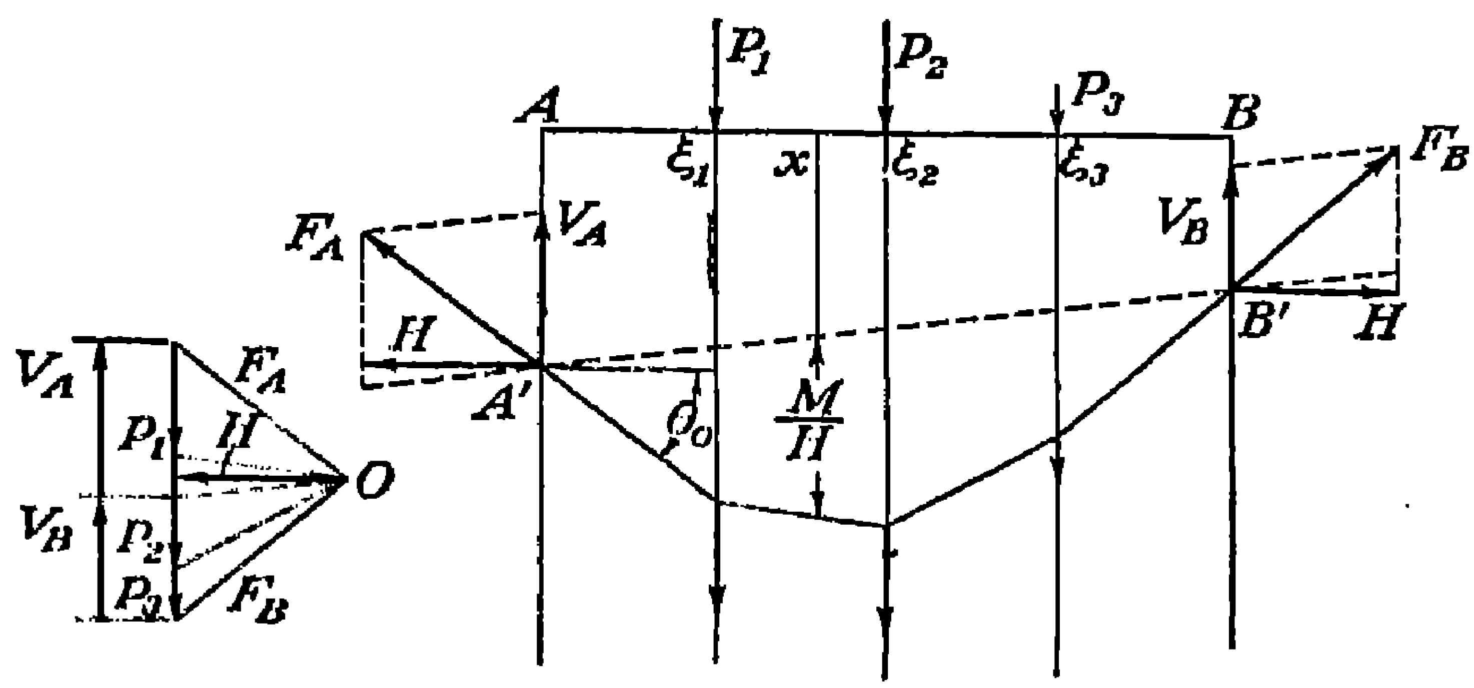

FIG. 1.2.—Force diagram and string polygon.

two end tensions F_A and F_B, whose horizontal component H is arbitrary. The integral in Eq. (1.8) can be evaluated graphically by replacing the distributed loads by concentrated forces whose magnitudes P_i are given by the areas of the corresponding sections of the load diagram $p(x)$ and whose points of application are the centroids of those sections.

Figure 1.2 shows the construction of the string polygon for a beam simply supported at the two ends and acted upon by the vertical loads P_1, P_2, and P_3. The ordinates w of the string polygon correspond to the equation (cf. Fig. 1.1.)

$$w = x \tan \theta_0 + \frac{1}{H} \sum_i (\xi_i - x) P_i \tag{1.9}$$

The summation includes the forces acting to left of $\xi = x$. To construct the polygon we first draw a *force diagram* with an arbitrarily chosen value of H. The sides of the string polygon

are parallel to the lines connecting the *pole* O with the end points of the forces. Since the vertical reaction at $x = 0$ is unknown, we start with an arbitrary value of θ_0. Then, of course, $w(l)$ is not necessarily zero, and to satisfy this condition we connect the end points A' and B' of the string polygon by a straight line. Then the ordinates between the sides of the polygon and the line $A'B'$ are equal to the moment divided by H. The vertical reactions at the supports are obtained from the force diagram by drawing a straight line from the pole parallel to $A'B'$.

Small Deflections of a String Held under Tension.—Let us now assume that the deflection w of a horizontal string is very small. In this case $\cos\theta \cong 1$, and in Eq. (1.3) H can be replaced by the resultant tension F. Hence Eq. (1.3) becomes

$$F\frac{d^2w}{dx^2} = -p(x) \tag{1.10}$$

and the solution with $w = 0$ for $x = 0$ is

$$w = -\frac{1}{F}\int_0^x (x - \xi)p(\xi)\,d\xi + Cx \tag{1.11}$$

where C is a constant of integration, which is determined by the second boundary condition. In the case of a constant load p_0,

$$w = -\frac{p_0x^2}{2F} + Cx$$

and if $w = 0$ for $x = l$, $C = p_0l/2F$ and

$$w = \frac{p_0x}{2F}(l - x)$$

The maximum deflection occurs at $x = l/2$ and is equal to $w_{max} = p_0l^2/8F$.

In the case of an arbitrary load $p(x)$, putting $w = 0$ for $x = l$ in Eq. (1.11), we have

$$-\frac{1}{F}\int_0^l (l - \xi)p(\xi)\,d\xi + Cl = 0$$

and substituting the value of C in (1.11),

$$w = \frac{1}{F}\int_0^x \xi\left(1 - \frac{x}{l}\right)p(\xi)\,d\xi + \frac{1}{F}\int_x^l x\left(1 - \frac{\xi}{l}\right)p(\xi)\,d\xi \tag{1.12}$$

This equation can be interpreted in the following way: Let us assume as the only load a concentrated load P acting at the point ξ (Fig. 1.3). Then $d^2w/dx^2 = 0$, except for $x = \xi$, and the solu-

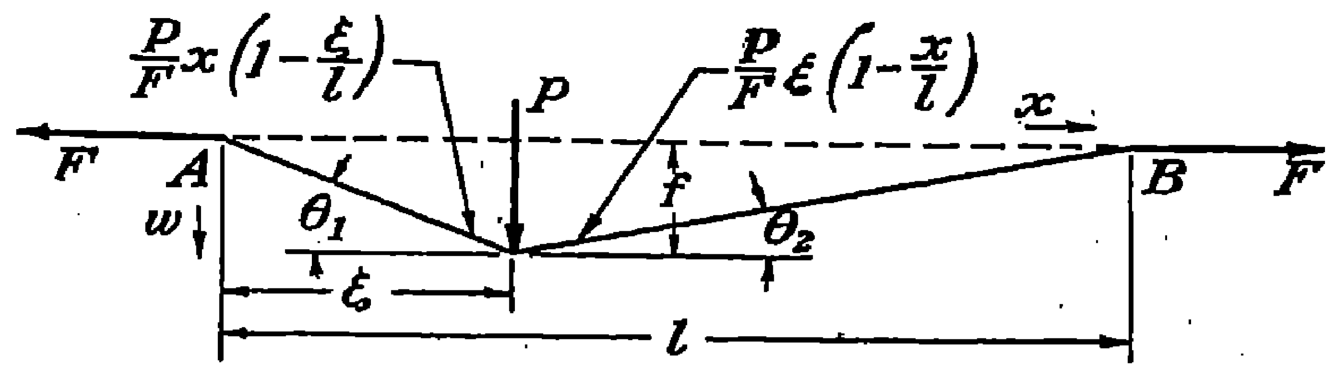

FIG. 1.3.—Small deflection of a string under action of a horizontal tension and a vertical concentrated load.

tion $w(x)$ consists of two linear functions. The deflection f at the point $x = \xi$ is given by the condition that

$$F(\tan \theta_2 + \tan \theta_1) = P$$

or

$$F\left(\frac{f}{\xi} + \frac{f}{l - \xi}\right) = P \tag{1.13}$$

$$f = \frac{P}{lF}\,\xi(l - \xi) \tag{1.14}$$

Hence, the solution for $x < \xi$ is given by

$$w = f\frac{x}{\xi} = \frac{P}{F}x\left(1 - \frac{\xi}{l}\right)$$

and for $x > \xi$ (1.15)

$$w = f\frac{l - x}{l - \xi} = \frac{P}{F}\xi\left(1 - \frac{x}{l}\right)$$

We call the function obtained by substituting $P = 1$ in (1.15) the *influence function* $g(x,\xi)$ (Fig. 1.4*b*). It gives the deflection at x for a unit load applied to the point ξ. It is a function of two variables x and ξ, and its derivative with respect to x is discontinuous for $x = \xi$.

Let us now subdivide the load $p(\xi)$ into sections $p(\xi)\,\Delta\xi$ and replace $p(\xi)\,\Delta\xi$ by concentrated forces (Fig. 1.4*a*). Then the contribution of the loads between $\xi = 0$ and $\xi = x$ to the deflection amounts to $\frac{1}{F}\sum_{\xi=0}^{\xi=x} p(\xi)\,\Delta\xi\,\xi\left(1 - \frac{x}{l}\right)$ and that of the loads

between $\xi = x$ and $\xi = l$ to $\frac{1}{F}\sum_{\xi=x}^{\xi=l} p(\xi)\,\Delta\xi\, x\left(1 - \frac{\xi}{l}\right)$. Proceeding to the limit $\Delta\xi \to 0$, we obtain the expression (1.12), which we write in the form (Fig. 1.4c):

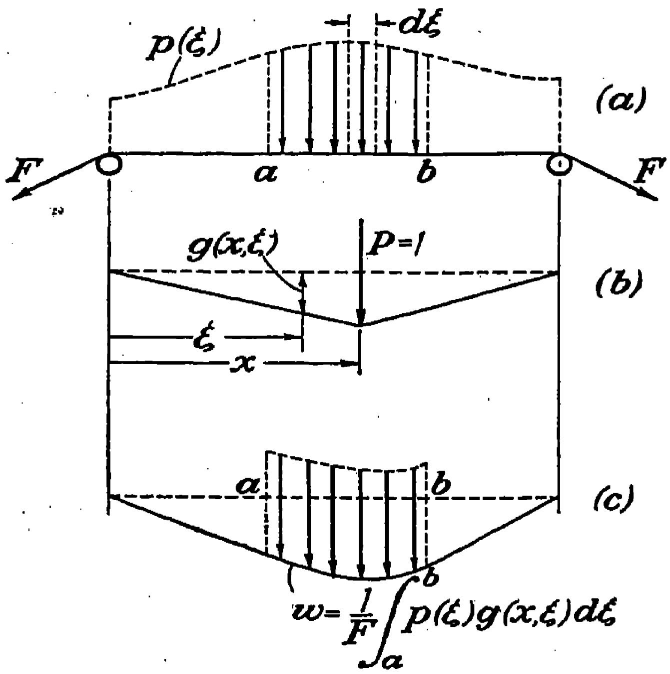

FIG. 1.4.—Deflection of a loaded string determined by means of the influence function.

$$w = \frac{1}{F}\int_0^l p(\xi)g(x,\xi)\,d\xi \qquad (1.16)$$

In Fig. 1.4c the application of Eq. (1.16) is shown for the case in which the load is applied between two points, $\xi = a$ and $\xi = b$. This method of solution is called the *method of the influence function.*

2. String with Elastic Support.—We assume that the vertical deflection of the string is restrained by a large number of springs (Fig. 2.1) such that their action can be replaced by a distributed restoring force per unit length equal to kw, where k is a proportionality factor and w is the deflection. In this case we must add the amount $-kw$ to the vertical load and obtain the differential equation

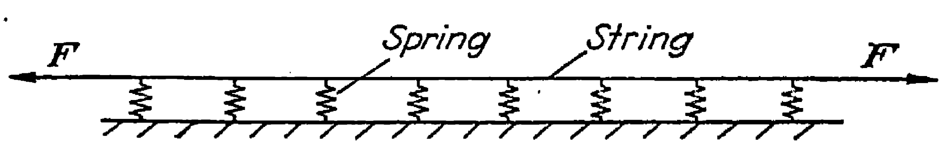

FIG. 2.1.—String held under horizontal tension and restrained elastically in the vertical direction.

$$F\frac{d^2w}{dx^2} = -p(x) + kw$$

or

$$F\frac{d^2w}{dx^2} - kw = -p(x) \qquad (2.1)$$

The solution of (2.1) consists of the general solution of the homogeneous equation, w_1, and an arbitrary particular solution w_2 of the nonhomogeneous equation. The general solution of the homogeneous equation is

$$w_1 = Ae^{\sqrt{\frac{k}{F}}x} + Be^{-\sqrt{\frac{k}{F}}x} \tag{2.2}$$

If $p(x)$ has a simple form, the particular solution can often be easily guessed or calculated. For example, if $p(x)$ is equal to a constant p_0, then $w_2 = p_0/k$ is a particular solution.

As an example, we give the solution corresponding to the deflection of an infinite string under the action of a concentrated

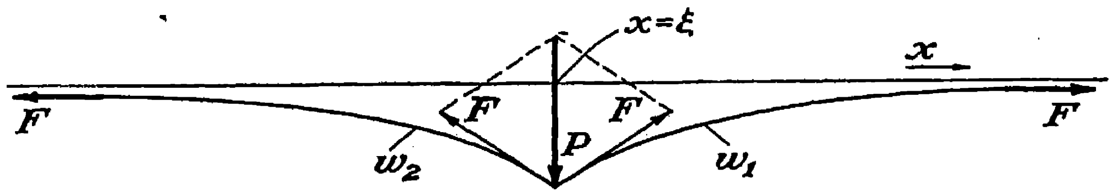

FIG. 2.2.—Deflection of an elastically restrained infinite string under a load P.

force P at the point ξ (Fig. 2.2). In this case, for all points $x \neq \xi$, Eq. (2.1) becomes $F\dfrac{d^2w}{dx^2} - kw = 0$. If we assume that $w = 0$ for $x = \pm\infty$, we have $w = C_1e^{-\sqrt{\frac{k}{F}}x}$ for $x > \xi$ and $w = C_2e^{\sqrt{\frac{k}{F}}x}$ for $x < \xi$. Denoting the deflection for $x = \xi$ by f, we have

$$C_1e^{-\sqrt{\frac{k}{F}}\xi} = C_2e^{\sqrt{\frac{k}{F}}\xi} = f \tag{2.3}$$

or $w = fe^{-\sqrt{\frac{k}{F}}(x-\xi)}$ for $x > \xi$ and $w = fe^{\sqrt{\frac{k}{F}}(x-\xi)}$ for $x < \xi$. The deflection f is determined by the condition that

$$\lim_{\epsilon\to 0} F\left[\left(\frac{dw}{dx}\right)_{\xi+\epsilon} - \left(\frac{dw}{dx}\right)_{\xi-\epsilon}\right] = -P \tag{2.4}$$

This condition yields the relation $2f\sqrt{kF} = P$. Hence, the deflection of the string is given by the following expressions:

$$\begin{aligned} w_1 &= \frac{P}{2\sqrt{kF}}e^{-\sqrt{\frac{k}{F}}(x-\xi)} \qquad &\text{for} \qquad x > \xi \\ w_2 &= \frac{P}{2\sqrt{kF}}e^{\sqrt{\frac{k}{F}}(x-\xi)} \qquad &\text{for} \qquad x < \xi \end{aligned} \tag{2.5}$$

3. Bending of Beams. General Theory.—A *beam* is a prismatical or approximately prismatical body with a resistance to bending and twisting. A beam is called a *straight beam* when

the centers of gravity of all cross sections lie on a straight line which is called the *axis* of the beam.

We give in this section a short review of the results of the elastic theory of straight beams. We assume that the x-axis of a rectangular coordinate system coincides with the beam axis and the cross sections are parallel to the yz plane. We consider an arbitrary cross section and denote the components of the resultant and of the resultant moment of the external forces acting on the portion of the beam to the left of the cross section considered by X, Y, Z, M_x, M_y, M_z. The center of gravity of the cross section is chosen as base point for the moments. We call X the *axial thrust;* Y and Z, the *shear forces;* M_x, the *twisting moment;* M_y and M_z, the *bending moments* acting around the y- and z-axes, respectively.

Let us first assume that the cross section is symmetrical with respect to the xz plane and the lines of action of all forces are parallel to the z-axis. Hence, $X = Y = 0$ and $M_x = M_z = 0$. In this case we shall call $M_y = M$ simply the *bending moment;* the resultant force $Z = S$, the *shear force.* We measure the normal deflection w positive downward, M positive clockwise, and S positive upward. We denote the load per unit length of the beam by $p(x)$, the concentrated loads by $P_1, P_2, \ldots, P_n$; p and the P's are taken positive downward.

The equilibrium condition for the vertical forces acting on an element dx of the beam requires that

$$\frac{dS}{dx}\,dx + p\,dx = 0 \tag{3.1}$$

The equilibrium condition for the moments acting on the same element is given by

$$\frac{dM}{dx}\,dx - S\,dx = 0 \tag{3.2}$$

Hence, $S = \dfrac{dM}{dx}$, $p = -\dfrac{dS}{dx}$, and, therefore,

$$p = -\frac{d^2M}{dx^2} \tag{3.3}$$

The analysis of the deformation produced by the bending leads to the result that the *curvature* of the beam axis is equal to

$$\frac{1}{R} = \frac{M}{EI}$$

where R is the radius of curvature (positive if the deflected axis is convex seen from below), I is the *moment of inertia* of the cross section with respect to the y-axis, and E denotes *Young's modulus*.

The curvature of the line $w = w(x)$ is given by

$$\frac{1}{R} = -\frac{d^2w}{dx^2}\left[1 + \left(\frac{dw}{dx}\right)^2\right]^{-3/2}$$

or for small deflections approximately by $\frac{1}{R} = -\frac{d^2w}{dx^2}$. Hence,

$$\frac{d^2w}{dx^2} = -\frac{M}{EI} \tag{3.4}$$

Substituting M from (3.4) into Eqs. (3.2) and (3.3), we obtain

$$-\frac{d}{dx}\left(EI\,\frac{d^2w}{dx^2}\right) = S \tag{3.5}$$

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2w}{dx^2}\right) = p(x) \tag{3.6}$$

Equation (3.6) is called the differential equation for the deflection of the beam. The quantity EI is called the *flexural rigidity*.

We consider the following boundary conditions:

a. Hinged Support.—If the hinged support is at one end of the beam, at that point $w = 0$, and the moment is equal to zero, and, therefore, $d^2w/dx^2 = 0$. If the beam extends beyond a hinged support, at the support $w = 0$, and the moment is continuous.

b. Clamped Support.—$w = 0$ and $dw/dx = 0$.

c. Free End.—$M = S = 0$ and, therefore, $EI\,\frac{d^2w}{dx^2} = 0$, and $\frac{d}{dx}\left(EI\,\frac{d^2w}{dx^2}\right) = 0$.

A hinged support involves a reaction; a clamped support, a reaction and a reaction moment. The beam is statically determinate if the total number of reactions and reaction moments is equal to the number of equilibrium equations. In this case, the moment M can be determined from the equilibrium conditions, and instead of Eq. (3.6) we can use Eq. (3.4) for the computation of the deflection w.

The elastic energy per unit length of the beam is equal to

$$\frac{1}{2}M\frac{1}{R} = \frac{1}{2}EI\left(\frac{1}{R}\right)^2 = \frac{1}{2}EI\left(\frac{d^2w}{dx^2}\right)^2 \tag{3.7}$$

In general, the resistance of a section of the beam to bending is determined by its two *moments of inertia* $I_y = \int_A z^2\,dA$ and $I_z = \int_A y^2\,dA$, and by its *product of inertia* $I_{yz} = \int_A yz\,dA$, where dA is an element of the cross-sectional area. If M_y and M_z are the bending moments with respect to the y- and z-axes, respectively, if v and w are the deflections in the negative y- and z-directions, and if $\frac{1}{R_z} = -\frac{d^2v}{dx^2}$ and $\frac{1}{R_y} = -\frac{d^2w}{dx^2}$ are the curvatures of the deflection curve in the xy and xz planes, we have

$$\begin{aligned} M_y &= -EI_y\frac{d^2w}{dx^2} - EI_{yz}\frac{d^2v}{dx^2} \\ M_z &= -EI_z\frac{d^2v}{dx^2} + EI_{yz}\frac{d^2w}{dx^2} \end{aligned} \tag{3.8}$$

If $I_{yz} = 0$, the y- and z-axes are called the *principal inertia axes* of the cross section. In this case

$$M_y = -EI_y\frac{d^2w}{dx^2} \qquad M_z = -EI_z\frac{d^2v}{dx^2} \tag{3.9}$$

and the quantities EI_y and EI_z are called the *principal flexural rigidities* of the beam. It is seen that the planes of the resultant moment and the resultant deflection coincide only if

(a) $EI_y = EI_z$,

or

(b) either M_y or M_z is equal to zero.

The influence of an axial thrust X on the bending of beams will be taken into account in the sections dealing with the theory of the suspension bridge, the theory of buckling, and the theory of a combined axial and lateral load.

The twisting moment is supposed to be proportional to the *angle of twist* per unit length of the beam. If the angle of rotation of an arbitrary cross section is $\theta(x)$,

$$M_x = -C\frac{d\theta}{dx} \tag{3.10}$$

where C is called the *torsional rigidity of the beam.* It is the

product of the *shear modulus* G and a quantity which has the dimension of a moment of inertia of the cross section.

4. Deflection of Beams. Beams on Elastic Foundation.—We assume that the y- and z-axes are principal axes of the beam and the load is acting in the xz plane. Then, according to Eq. (3.6), the deflection is given by the equation

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2w}{dx^2}\right) = p(x) \tag{4.1}$$

If $p(x)$ is a given function of x, w can be calculated by repeated quadratures. Several graphical and numerical methods have

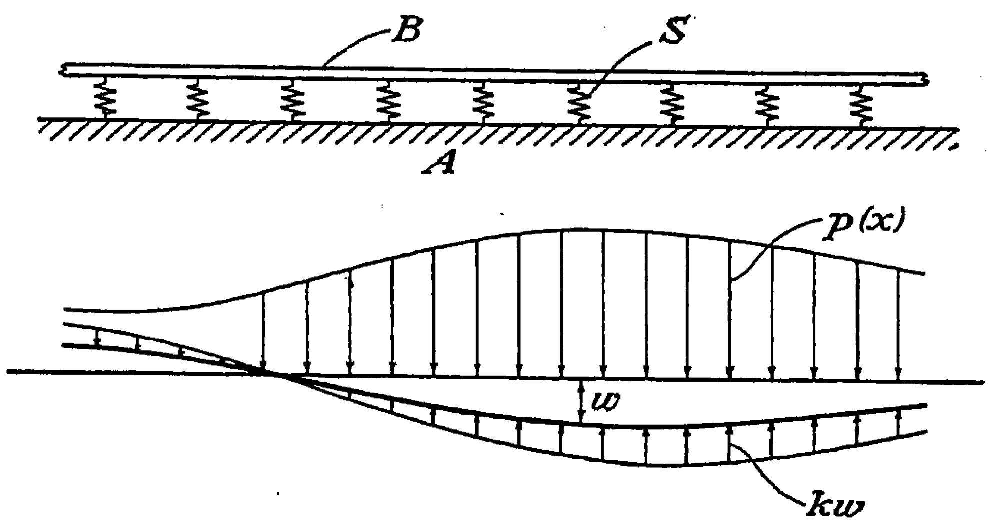

FIG. 4.1.—Beam on elastic foundation. Diagram showing the deflection w, the load $p(x)$, and the elastic restoring force kw.

been developed for solving Eq. (4.1). We remember that $\dfrac{d^2w}{dx^2} = -\dfrac{M}{EI}$ and $d^2M/dx^2 = -p$. If we consider the function M/EI as a load applied to the beam, we may consider the deflection w as the bending moment corresponding to this load. Hence, the construction of the string polygon explained in section 1 can be applied also for the determination of the deflection.

Let us assume now that a uniform beam B is attached to a fixed base A by means of a large number of springs S. We shall take into account their action—as was done in section 2—by a restoring force $-kw$ per unit length acting in a direction opposite to w (Fig. 4.1). The factor k is called the *modulus of the foundation.* If a distributed external load $p(x)$ acts on the beam, the total load will be $p(x) - kw$, and we obtain from Eq. (4.1)

$$EI\,\frac{d^4w}{dx^4} = p(x) - kw \tag{4.2}$$

We first calculate the general solution of the homogeneous equation

$$\frac{d^4w}{dx^4} + \beta^4 w = 0 \tag{4.3}$$

where $\beta^4 = k/EI$. The characteristic equation of (4.3) is $\lambda^4 + \beta^4 = 0$ and we have

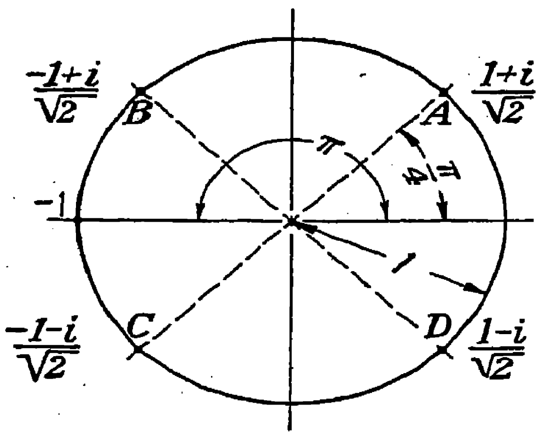

FIG. 4.2.—The four values of $\sqrt[4]{-1}$.

$$\lambda = \beta\sqrt[4]{-1}.$$

Let us write $-1 = e^{(2k+1)\pi i}$ where k is an integer. Then it is seen that the four values of $\sqrt[4]{-1}$ are $e^{\frac{i\pi}{4}}$, $e^{\frac{3\pi i}{4}}$, $e^{\frac{5\pi i}{4}}$, and $e^{\frac{7\pi i}{4}}$, *i.e.*, the four values of the fourth root of -1 are represented in the complex plane by points located at A, B, C, and D on the unit circle (Fig. 4.2). The general solution of Eq. (4.3) is, therefore,

$$w = C_1 e^{\frac{\beta}{\sqrt{2}}(1+i)x} + C_2 e^{\frac{\beta}{\sqrt{2}}(-1+i)x} + C_3 e^{\frac{\beta}{\sqrt{2}}(-1-i)x} + C_4 e^{\frac{\beta}{\sqrt{2}}(1-i)x} \tag{4.4}$$

or in real form

$$w = e^{\frac{\beta}{\sqrt{2}}x}\left(A \cos \frac{\beta}{\sqrt{2}}x + B \sin \frac{\beta}{\sqrt{2}}x\right) + e^{-\frac{\beta}{\sqrt{2}}x}\left(A' \cos \frac{\beta}{\sqrt{2}}x + B' \sin \frac{\beta}{\sqrt{2}}x\right) \tag{4.5}$$

To find w for a given $p(x)$, a particular solution of Eq. (4.2) must be added to Eq. (4.5). The four arbitrary constants are determined by four boundary conditions.

Let us in particular investigate the case of a concentrated load P applied at a point ξ, *i.e.*, $p(x) = 0$ for $x \neq \xi$. We assume the length of the beam unlimited in both directions. Then, owing to symmetry, $dw/dx = 0$ at $x = \xi$; also w must be finite (or zero) at $x = \pm\infty$. The solution satisfying these conditions is given by

$$w = Ce^{\pm\frac{\beta}{\sqrt{2}}(x-\xi)}\left[\cos \frac{\beta}{\sqrt{2}}(x - \xi) \mp \sin \frac{\beta}{\sqrt{2}}(x - \xi)\right] \tag{4.6}$$

where the upper sign holds for $x < \xi$ and the lower sign for $x > \xi$. The constant C is determined by the condition that the difference between the shearing force S at $x = \xi - \epsilon$ and $x = \xi + \epsilon$ is equal to the load if $\epsilon \to 0$. The shearing force is given by $S = -EI \dfrac{d^3w}{dx^3}$. We obtain by differentiation

$$\left(\frac{d^3w}{dx^3}\right)_{x=\xi-\epsilon} = -\sqrt{2}\beta^3C, \qquad \left(\frac{d^3w}{dx^3}\right)_{x=\xi+\epsilon} = +\sqrt{2}\beta^3C$$

Hence, $P = 2\sqrt{2}\beta^3CEI$, from which

$$C = \frac{P}{\sqrt{8}\beta^3EI} \tag{4.7}$$

It is seen that the deflection curve has the shape of *damped waves* with a discontinuous third derivative at the point $x = \xi$

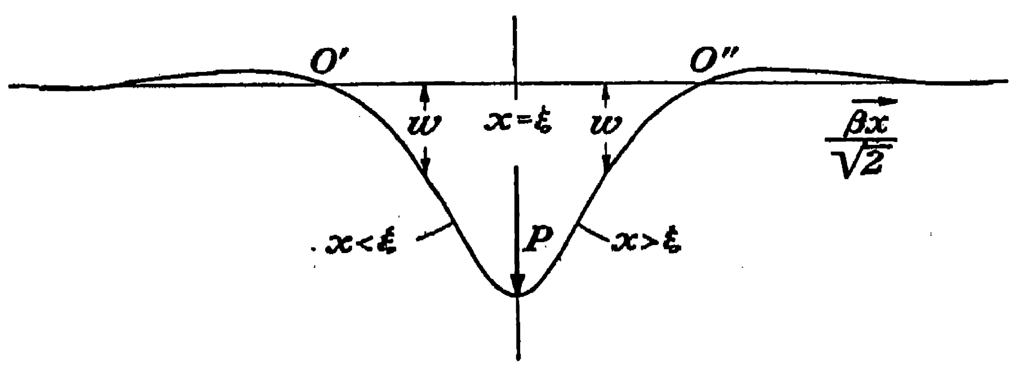

FIG. 4.3.—Deflection of an elastically supported beam under a concentrated load.

(Fig. 4.3). The distance of the first zero points O' and O'' of the deflection from the point of action of the concentrated load P is equal to $(3\pi/2\beta)\sqrt{2}$. The values of the function

$$\phi(z) = e^{-z}(\cos z + \sin z) \tag{4.8}$$

and its second derivative, which is needed for computation of the bending moment, are tabulated in Timoshenko's "Strength of Materials," vol. II, pages 405–406.

There are many classes of engineering problems leading to the equation treated in this section; among the important problems that can be reduced to that of the bending of a beam on an elastic foundation is the axially symmetrical deformation of a circular pipe.

Circular Pipe with Reinforcing Ring.—As an example of this second application, we consider a circular pipe of thickness t and mean radius r; the latter is defined as the mean value of the internal and the external radii. The pressures acting on the pipe and, therefore, the deformations are assumed symmetrical about the x-axis (Fig. 4.4). We consider a strip of the pipe between two planes passing through the x-axis, with the small angle $\Delta\alpha$ between them. Such a strip will behave like a beam of

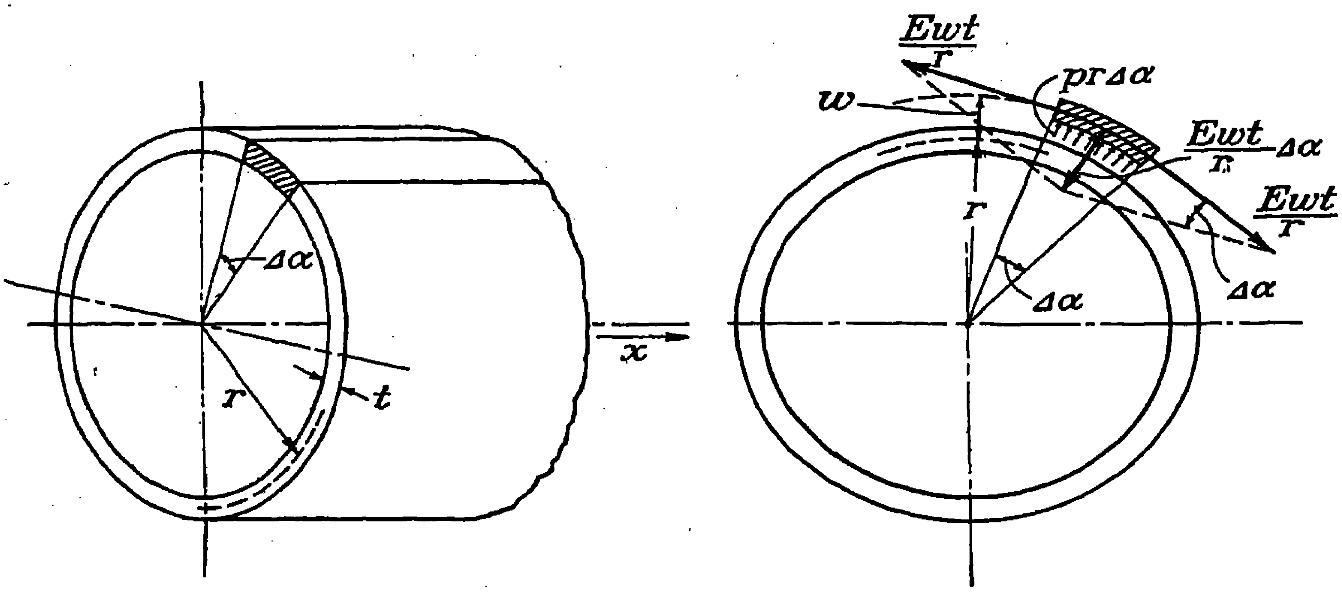

FIG. 4.4.—Axially symmetrical deformation of a circular pipe under internal pressure.

width $r\ \Delta\alpha$ and height t, provided we consider the strip subjected not only to the pressure acting on the pipe, but also to the forces acting on the two sides of the strip, owing to the elastic shrinkage or expansion of the annular sections of the pipe. The resultant of these forces is directed radially and represents an elastic restoring force for the beam considered. Let us measure the radial deflection w of the strip positive in the outward direction. The expansion of an annular section from the radius r to $r + w$ produces a tensile stress in the annulus equal to $E\frac{w}{r}$, where E is a modulus of elasticity of the material.* Then the forces acting on the side faces per unit length of the strip are equal to Ewt/r, and the radial resultant acting toward the axis is equal to $E\frac{w}{r}t\ \Delta\alpha$ per unit length of the strip. It is equivalent to a restoring

* Actually E is equal to Young's modulus divided by $1 - \mu^2$, where μ is Poisson's ratio.

force whose spring constant is equal to

$$k = \frac{Et}{r}\,\Delta\alpha$$

The external load per unit length is equal to $pr\,\Delta\alpha$ where p is the internal pressure per unit area of the cylindrical surface. (The exact value of the load would be $\left(r - \frac{t}{2}\right)p\,\Delta\alpha$; however, in the case of thin-walled pipes $t/2$ can be neglected in comparison with r.) Finally, the moment of inertia of the cross section of the strip is equal to $r\,\Delta\alpha\frac{t^3}{12}$.

With these values Eq. (4.2) becomes

$$\frac{d^4w}{dx^4} + \frac{12}{r^2t^2}w = \frac{12}{Et^3}p \tag{4.9}$$

We introduce $a = \sqrt{rt}$ as a characteristic length parameter of the problem. Its physical meaning will be illustrated by the application given below.

With this notation Eq. (4.9) becomes

$$\frac{d^4w}{dx^4} + \frac{12w}{a^4} = \frac{12}{Et^3}p \tag{4.10}$$

Let us apply Eq. (4.10) to a pipe of infinite length subjected to a uniform internal pressure p and reinforced at the section $x = 0$ by a stiffener ring whose stiffness is so excessive that the change of the diameter of the reinforced section can be neglected (Fig. 4.5). Then we have the boundary conditions $x = 0$ and $dw/dx = 0$ at $x = 0$; at infinity w is finite and $dw/dx = 0$.

The general solution of (4.10) contains a particular solution of the nonhomogeneous equation. For this particular solution we use

$$w = \frac{a^4p}{Et^3} = \frac{pr^2}{Et} \tag{4.11}$$

The solution of the homogeneous equation which together with (4.11) satisfies the boundary conditions at infinity and the condition $dw/dx = 0$ at $x = 0$ can be obtained from Eq. (4.6)

by substituting $\beta = \sqrt[4]{12}/a$ and $\xi = 0$. Hence, we have

$$w = \frac{pr^2}{Et} + Ce^{\pm\frac{x}{a}\sqrt[4]{3}}\left[\cos\left(\frac{x}{a}\sqrt[4]{3}\right) \mp \sin\left(\frac{x}{a}\sqrt[4]{3}\right)\right] \quad (4.12)$$

where the upper signs hold for $x < 0$ and the lower signs for $x > 0$. The remaining constant C is determined by the condition that $w = 0$ at $x = 0$. We finally obtain

$$w = \frac{pr^2}{Et}\left\{1 - e^{\pm\frac{x}{a}\sqrt[4]{3}}\left[\cos\left(\frac{x}{a}\sqrt[4]{3}\right) \mp \sin\left(\frac{x}{a}\sqrt[4]{3}\right)\right]\right\} \quad (4.13)$$

The deflection w is plotted as function of x in Fig. 4.5. The radial deflection at infinity is equal to $w_\infty = pr^2/Et$, *i.e.*, to the

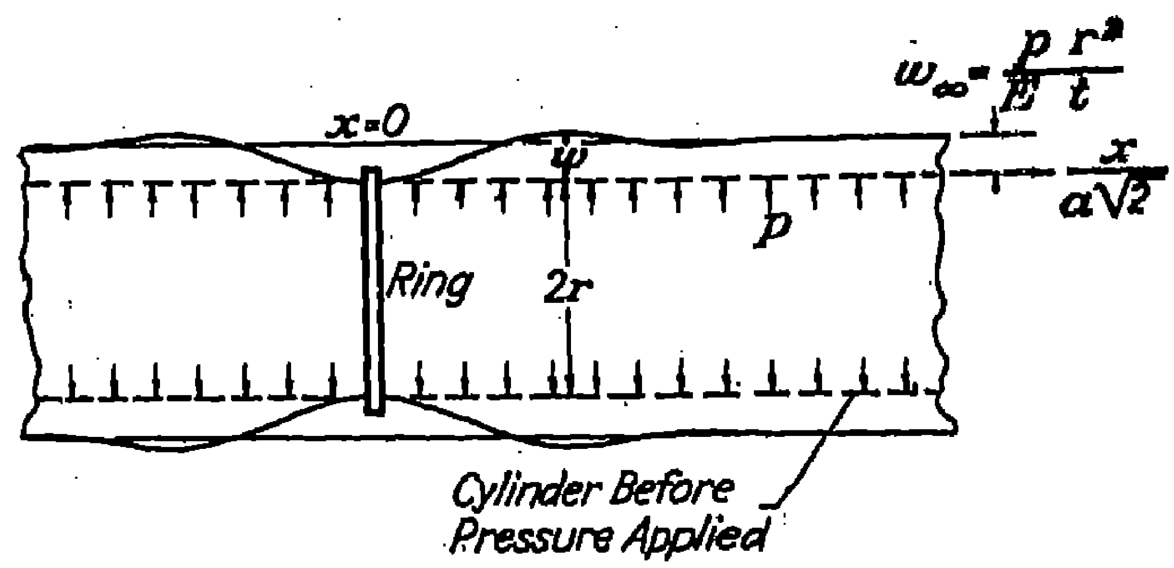

FIG. 4.5.—Deflection of a reinforced circular pipe due to internal pressure.

deflection that would occur at any section of the pipe without the stiffener ring.

If we use the stiffener ring as a stress-relieving device it is important to know the length of the pipe that is materially affected by the ring. To estimate this length we calculate the load carried by the ring. This can be done by computing the shearing forces for $x = \pm\epsilon$, as we have done on page 273 or by calculating for each section the partial pressure p_1 which is in equilibrium with the annular stresses in the pipe. If p is the total pressure, the rest of the load, *i.e.*,

$$P = \int_{-\infty}^{\infty}(p - p_1)\,dx = 2\int_0^{\infty}(p - p_1)\,dx \quad (4.14)$$

must be carried by a unit length of the ring. The annular stress in the pipe is equal to $E\frac{w}{r}$; hence, $p_1 = E\frac{wt}{r^2}$. Substituting this value into Eq. (4.14) and taking w from Eq. (4.13), we obtain

$$P = 2p \int_0^\infty e^{-\frac{x\sqrt[4]{3}}{a}} \left(\cos \frac{x\sqrt[4]{3}}{a} + \sin \frac{x\sqrt[4]{3}}{a} \right) dx \tag{4.15}$$

Carrying out the integration, we have

$$P = \frac{2pa}{\sqrt[4]{3}} = 2p \frac{\sqrt{rt}}{\sqrt[4]{3}} \tag{4.16}$$

Hence, the pressure acting over a length $2l = 2\sqrt{rt}/\sqrt[4]{3}$ is carried by the stiffener. Therefore, to reduce the stress in a long pipe materially by a series of stiffener rings, the spacing must be of the order of magnitude $2l$. It is seen that this length is proportional to the square root of the radius of the pipe and to the square root of the wall thickness.

5. The Theory of the Suspension Bridge.—The so-called *deflection theory of the suspension bridge* considers the bridge

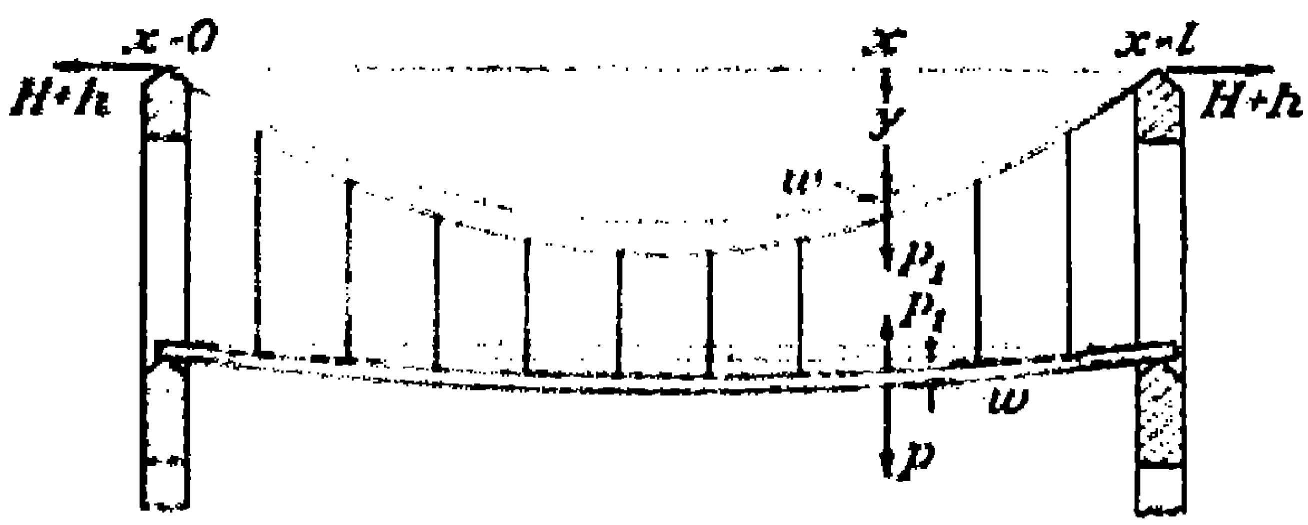

FIG. 5.1. Deflection of the truss and cable of a suspension bridge.

structure as a combination of a string (the suspension cable) and a beam (the bridge truss) (Fig. 5.1). The peculiar feature of this combination is that, whereas the deflection of the beam may be considered small, the deflection of the string, *i.e.*, the deviation of its shape from a straight line, has to be considered as of finite magnitude. However, if we assume—as is usually done in the analysis of suspension bridges—that the cable is adjusted in such a way that it carries its own weight, the weight of the hangers, and the dead weight of the truss without producing a bending moment in the beam, then all additional deformations of the cable and the truss due to the live load are of small magnitude and can be calculated by means of linear equations.

We shall use the following notations:

a. The dead load of the system per unit length is equal to q; the live load applied to the truss is equal to p.

b. The horizontal tension in the cable, when loaded by the dead weight only, is denoted by H; the additional tension produced by application of the live load by h. The shape of the cable corresponding to the initial loading condition is given by $y = y(x)$, y being considered positive downward from the horizontal line connecting the end points of the cable. The additional deflection of the cable is $w(x)$. It will be assumed that the elastic deformation of the hangers can be neglected, and the vertical deflection of the truss at the point x is equal to the vertical deflection of the cable at the same point x. It is understood that this implies, in addition to the assumption of rigid connection between cable and truss, disregard of the horizontal deflections of the cable.

c. The moment of inertia of the cross section of the truss is assumed to be constant and equal to I; Young's modulus is denoted by E.

We consider a cable extended between $x = 0$ and $x = l$ and a beam hinged at $x = 0$ and $x = l$.

The differential equation for the shape of the cable in the initial loading condition is, according to Eq. (1.3),

$$H \frac{d^2y}{dx^2} = -q \tag{5.1}$$

If a live load p is added, a certain portion p_1 of the live load is carried by the cable, while $p - p_1$ is carried by the bending stiffness of the truss. The horizontal cable tension is increased to $H + h$, and the deflection w is added to the ordinate y. Hence, the equation for this condition is

$$(H + h) \frac{d^2}{dx^2}(y + w) = -q - p_1 \tag{5.2}$$

On the other hand, the differential equation of the beam is, according to (3.6),

$$EI \frac{d^4w}{dx^4} = p - p_1 \tag{5.3}$$

Three unknown quantities occur in Eqs. (5.2) and (5.3), *viz.*, the two functions $w(x)$ and $p_1(x)$ and the parameter h. Hence, an additional relation is necessary to determine the parameter h. This is given by the following condition: obviously, if $w(x)$ is known, the change in the length is determined by the additional

tension h and the elasticity of the cable. Calculating the change in length in both ways, and comparing the two expressions, we obtain the relation which determines h.

Let us assume first that h is known. Then by substituting p_1 from Eq. (5.2) into Eq. (5.3), we obtain

$$EI\frac{d^4w}{dx^4} - (H + h)\left(\frac{d^2y}{dx^2} + \frac{d^2w}{dx^2}\right) = p + q \tag{5.4}$$

Taking into account Eq. (5.1), this reduces to

$$EI\frac{d^4w}{dx^4} - (H + h)\frac{d^2w}{dx^2} = p + h\frac{d^2y}{dx^2}$$

or substituting again $\frac{q}{H}$ for $-\frac{d^2y}{dx^2}$, we have

$$EI\frac{d^4w}{dx^4} - (H + h)\frac{d^2w}{dx^2} = p - q\frac{h}{H} \tag{5.5}$$

This is the fundamental equation of the theory of the suspension bridge. It is seen by comparison with Eq. (5.3) that the portion of the live load that is transmitted to the cable and not carried directly by the truss is equal to

$$p_1 = q\frac{h}{H} - (H + h)\frac{d^2w}{dx^2} \tag{5.6}$$

The meaning of the first term on the right side of Eq. (5.6) is easily expressed: The truss is relieved by a certain portion of the dead load q; the reduction in per cent involved is equal to the increase in per cent of the cable tension.

The second term also allows a physical interpretation. Let us assume that an axial force $X = H + h$ acts along the axis of the truss beam. If we assume that the beam is deflected into the shape $w = w(x)$ and its radius of curvature is equal to $1/R \cong -d^2w/dx^2$, the action of an axial tension X is equivalent to that of a normal load of the amount X/R, and it represents a restoring force if X is positive. Hence, the effect of the hangers is the same as if an axial tension of the magnitude of the cable tension were applied along the axis of the beam.

The general solution of Eq. (5.5) is, with $\mu = \sqrt{\frac{H + h}{EI}}$,

$$w = A + Bx + Ce^{\mu x} + De^{-\mu x} + w_p(x) \tag{5.7}$$

where $w_p(x)$ is a particular solution of Eq. (5.5).

Let us assume $p - q\frac{h}{H}$ constant. Then it is more convenient to choose the center of the beam as the origin of the coordinate system and assume $w = 0$ and $\frac{d^2w}{dx^2} = 0$ for $x = \pm\frac{l}{2}$. If the symmetry of the problem is taken into account, only an even function of w can occur in the solution. Hence, we write

$$w = C_1 + C_2 \cosh \mu x - \frac{1}{H+h}\left(p - q\frac{h}{H}\right)\frac{x^2}{2} \tag{5.8}$$

It is readily shown by differentiation that the last term in Eq. (5.8) is a particular solution of Eq. (5.5). The boundary conditions call for

$$C_1 + C_2 \cosh \mu l/2 - \frac{p - q\frac{h}{H}}{H+h}\frac{l^2}{8} = 0$$

$$-\mu^2 C_2 \cosh \mu l/2 - \frac{p - q\frac{h}{H}}{H+h} = 0$$

Hence,

$$\begin{aligned} \mu^2 C_1 &= \frac{p - q(h/H)}{H+h}\left(\frac{\mu^2 l^2}{8} + 1\right) \\ -\mu^2 C_2 &= \frac{p - q(h/H)}{H+h}\,\frac{1}{\cosh \mu l/2} \end{aligned} \tag{5.9}$$

Let us now calculate the bending moment at the center. We find from Eq. (5.8)

$$M_{\max} = -EI\frac{d^2w}{dx^2} = EI\left(\mu^2 C_2 + \frac{p - q(h/H)}{H+h}\right)$$

or if C_2 is substituted from Eq. (5.9),

$$M_{\max} = EI\frac{p - q(h/H)}{H+h}\left(1 - \frac{1}{\cosh \mu l/2}\right) \tag{5.10}$$

Substituting $\frac{H+h}{EI} = \mu^2$, we have

$$M_{\max} = pl^2\left(1 - \frac{q}{p}\frac{h}{H}\right)\frac{1}{(\mu l)^2}\left(1 - \frac{1}{\cosh \mu l/2}\right) \tag{5.11}$$

It will be shown later that h/H is a function of the dimensionless quantity Hl^2/EI and the ratio p/q. Hence, the moment at the center is expressed in the form:

$$M_{\max} = pl^2 f\left(\frac{p}{q}, \frac{Hl^2}{EI}\right) \tag{5.12}$$

The ratio of the live load to the dead load and the dimensionless quantity Hl^2/EI are the two governing parameters of the problem.

The particular solution of Eq. (5.5) for arbitrary loads p can be found either by superposition of concentrated loads or by expanding p in Fourier series. The latter method will be discussed in detail in Chapter VIII. The method of superposition is explained at the end of this chapter in connection with Prob. 6.

The Calculation of the Additional Cable Tension.—We consider two cases. In the first, we assume that the cable is inextensible, and, in the second, we assume a certain given modulus of elasticity for the extension of the cable. The initial total length of the cable is equal to $L = \int_0^l \sqrt{1 + (dy/dx)^2}\, dx$ where $y(x)$ is the deflection in the initial loading condition. (Note that the origin $x = 0$ is again moved to the left tower.) If we replace $y(x)$ by $y(x) + w(x)$, expand the radical in a Taylor series, and neglect higher terms in w, the variation of L will be*

$$\Delta L = \int_0^l \frac{(dy/dx)\,(dw/dx)}{\sqrt{1 + (dy/dx)^2}}\, dx \tag{5.13}$$

We integrate Eq. (5.13) by parts and write $y' = dy/dx$,

$$y'' = \frac{d^2y}{dx^2}$$

Then we have

$$\Delta L = \left[w \frac{y'}{\sqrt{1 + y'^2}}\right]_0^l - \int_0^l \frac{wy''}{(1 + y'^2)^{3/2}}\, dx \tag{5.14}$$

The first term on the right side of Eq. (5.14) vanishes, because w vanishes at the end points. If we now neglect y'^2 in comparison

* We remember that $\sqrt{1 + (x + \epsilon)^2} - \sqrt{1 + x^2}$ can be written $\dfrac{1 + (x + \epsilon)^2 - (1 + x^2)}{\sqrt{1 + (x + \epsilon)^2} + \sqrt{1 + x^2}}$ or approximately $\dfrac{x\epsilon}{\sqrt{1 + x^2}}$ for small ϵ.

with unity and substitute $y'' = -\frac{q}{H}$ from Eq. (5.1), we obtain

$$\Delta L = \int_0^l \frac{wq}{H}\,dx \tag{5.15}$$

or

$$H\Delta L = \int_0^l wq\,dx \tag{5.16}$$

Equation (5.16) can be interpreted as the expression of the theorem of the conservation of energy, applied to the cable. In the initial loading condition the normal load on the cable is equal to q, the horizontal, or approximately the total, tension is equal to H. If now the deflection is increased by the small quantity w, the expression on the right side gives the work done by the load q; the expression on the left side is the work done by the tension H if the cable is stretched by the amount ΔL. The two amounts of work must be equal, contributions of the deflection involving higher than the first power being neglected.

If we assume the cable inextensible, we obtain the following relation:

$$\int_0^l wq\,dx = 0 \tag{5.17}$$

This means that we have to substitute in Eq. (5.17) for w the solution of Eq. (5.5), which itself contains the parameter h, and in this way, we obtain an equation for the determination of h. This equation is a *transcendental* equation, because h is involved also in μ, and, therefore, in cosh $\mu l/2$. In order to obtain a first approximation, we put $\mu = \sqrt{H/EI}$, neglecting h in comparison with H. Then, we calculate h from (5.17), correct μ, and repeat the calculation. It is seen that, as was said above, h/H will be determined in this way as function of Hl^2/EI and p/q.

If the cable is extensible, we assume that $\Delta L = hL/EA$, where A is the effective cross-sectional area of the cable and E the modulus of elasticity of the material. Then, substituting ΔL in Eq. (5.16), we obtain

$$\frac{HhL}{EA} = \int_0^l wq\,dx \tag{5.18}$$

This relation gives again an equation for the calculation of h. We remember that h occurs also in this case in the expression for

w. In practical applications it is necessary to take into account also the extension of the cable due to the change of temperature.

6. Problems of Harmonic Vibrations Reduced to Statical Problems.—Sections 7 to 11 will deal with harmonic vibrations of one-dimensional continuous systems. As a matter of fact, the equation of motion of such a system is a partial differential equation with the time t and the space coordinate x as independent variables. Consider, for example, the small vibrations of a string. We denote the deflection by ζ and the mass of the string per unit length by ρ. According to d'Alembert's principle (Chapter III, section 10), any dynamical problem may be treated as a statical one by adding the appropriate inertia forces to the given external forces. The inertia force to be added per unit length of the string is equal to $-\rho \dfrac{\partial^2\zeta}{\partial t^2}$, and we obtain the equation of motion for the free vibrations by substituting this quantity for the load p in Eq. (1.10):

$$F \frac{\partial^2\zeta}{\partial x^2} = \rho \frac{\partial^2\zeta}{\partial t^2} \qquad (6.1)$$

This is a partial differential equation for the deflection $\zeta(x,t)$. However, if we restrict ourselves to harmonic motion by putting

$$\zeta(x,t) = w(x) \sin \omega t$$

the partial differential equation (6.1) is reduced to the ordinary differential equation

$$F \frac{d^2w}{dx^2} + \rho\omega^2 w = 0 \qquad (6.2)$$

The new variable $w(x)$ can be interpreted as the maximum deflection or the *amplitude* of the harmonic motion and $\rho\omega^2 w$ as the inertia force per unit length at the instant of maximum deflection.

To find the equation for the free vibration of a beam, we consider again the inertia force $-\rho\dfrac{\partial^2\zeta}{\partial t^2}$ as the load, and using Eq. (3.6), we find

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2\zeta}{\partial x^2}\right) + \rho \frac{\partial^2\zeta}{\partial t^2} = 0 \qquad (6.3)$$

This partial differential equation is reduced to an ordinary differ-

ential equation if we limit ourselves to harmonic motion by putting $\zeta(x,t) = w(x) \sin \omega t$. We find

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2w}{dx^2}\right) - \rho\omega^2 w = 0 \tag{6.4}$$

In the following sections we use the Eqs. (6.2) and (6.4) to determine the frequencies and the modes of vibration of strings and beams under given boundary conditions.

7. Harmonic Vibration of a String under Tension.—Taking up first the problem of the vibrating string, we find the general solution of Eq. (6.2), assuming that ρ is constant,

$$w = C_1 \cos \sqrt{\frac{\rho}{F}}\omega x + C_2 \sin \sqrt{\frac{\rho}{F}}\omega x \tag{7.1}$$

If the string extends from $x = 0$ to $x = l$ and is fixed at the two ends, we have the boundary conditions $w = 0$ for $x = 0$ and $x = l$. The condition $w = 0$ for $x = 0$ requires that $C_1 = 0$. We are then left with the condition

$$w = C_2 \sin \sqrt{\frac{\rho}{F}}\omega l = 0 \tag{7.2}$$

This can be satisfied only if either $C_2 = 0$ or $\sqrt{\frac{\rho}{F}}\omega l = k\pi$ where k is an integer. Hence the problem, in addition to the trivial solution $w = 0$ (position of rest), has an infinite number of solutions, each corresponding to a definite value of the parameter ω. The values of ω which satisfy the condition (7.2) are

$$\omega_k = \frac{k\pi}{l}\sqrt{\frac{F}{\rho}} \tag{7.3}$$

The corresponding solutions of Eq. (6.2) are

$$w_k = C_k \sin k\pi\frac{x}{l}, \qquad k = 1, 2, 3, \ldots, \infty \tag{7.4}$$

where the amplitude C_k remains undetermined. The string can vibrate in an infinite number of sinusoidal shapes called *modes of vibration* (Fig. 7.1*a*); each mode corresponds to a certain frequency.

The lowest frequency ($n = 1$) is called the *fundamental tone* of the string, the corresponding mode of vibration is called the fundamental mode of vibration. The modes corresponding to higher frequencies are known as the *harmonics*, or *overtones*. Figure 7.1*b* shows the *spectrum* of the string, *i.e.*, the distribution

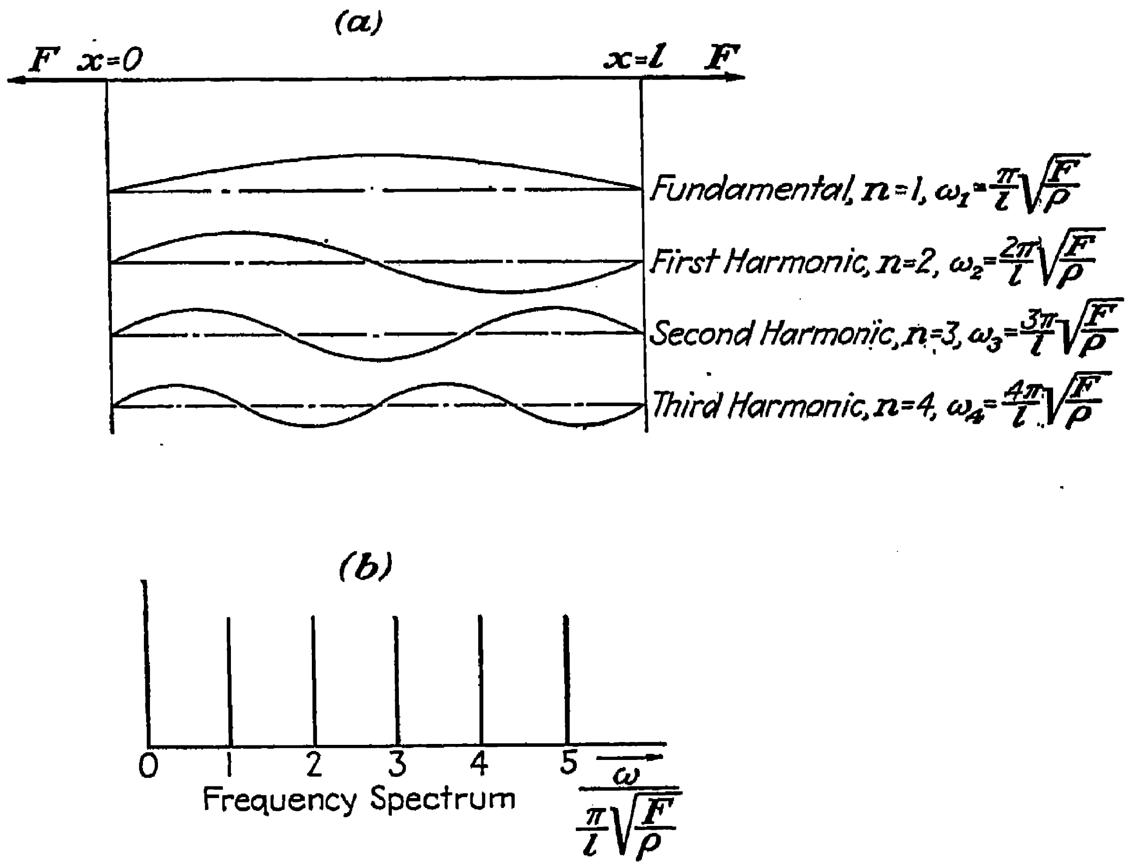

FIG. 7.1.—The modes of oscillation and the frequency spectrum for a string under tension.

of the natural frequencies of the string over the frequency scale.

Let us go back to Eq. (2.1) for the string with elastic support and put $p = 0$:

$$F \frac{d^2w}{dx^2} - kw = 0 \tag{7.5}$$

Comparing this equation with Eq. (6.2), it is seen that the sign of the coefficient of w has a decisive influence on the character of the respective solutions. In the case of a string with fixed ends and elastic support, $w = 0$ is the only possible shape of the string when no load is applied, whereas the equation for a vibrating string has additional solutions for certain definite values of the parameter ω. The mathematical reason for the different behavior of the two equations is that the general solution of the Eq.

(6.2) is of the oscillatory type and goes through zero an infinite number of times between $-\infty$ and $+\infty$, while the general solution of Eq. (7.5) cannot have more than one zero point between the same limits; hence, if the solution of Eq. (7.5) has to satisfy the boundary condition $w = 0$ at more than one point, w must be zero everywhere.

The modes of vibration of the string are analogous to the modes of principal oscillations of a system which has a finite number of degrees of freedom (Chapter V). Every mode of vibration is a pure harmonic oscillation. If we replace the string by n masses, the amplitudes of these masses would be determined by the coefficients of the normal modes. It is seen that the function $w(x)$ which determines the shape of a mode of vibration replaces a set of such coefficients.

8. Vibration of Beams. The Critical Speed of a Rotating Shaft.—We now take up the problem of harmonic vibration of a beam. We restrict ourselves to beams of uniform cross section. Then Eq. (6.4) becomes

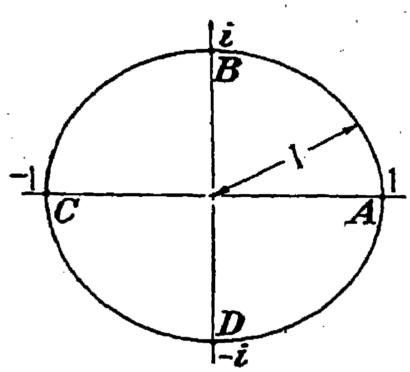

FIG. 8.1.—The four values of $\sqrt[4]{1}$.

$$EI\,\frac{d^4w}{dx^4} - \rho\omega^2 w = 0 \qquad (8.1)$$

Using the notation $\beta^4 = \rho\omega^2/EI$, we have

$$\frac{d^4w}{dx^4} - \beta^4 w = 0 \qquad (8.2)$$

The characteristic equation of Eq. (8.2) is $\lambda^4 - \beta^4 = 0$, and, therefore,

$$\lambda = \beta\sqrt[4]{1}$$

The fourth roots of unity are located at the points A, B, C, and D of the unit circle as shown in Fig. (8.1), and the general solution of Eq. (8.2) is

$$w = C_1e^{\beta x} + C_2e^{-\beta x} + C_3e^{i\beta x} + C_4e^{-i\beta x} \qquad (8.3)$$

or, in real form,

$$w = A\cosh\beta x + B\sinh\beta x + C\cos\beta x + D\sin\beta x \qquad (8.4)$$

We consider various boundary conditions.

a. Let us assume that a vibrating beam is hinged (simply supported) at both ends $x = 0$ and $x = l$. Then we have

$w = 0$ and $d^2w/dx^2 = 0$ for $x = 0$ and $x = l$. From the conditions at $x = 0$ it follows that $A = C = 0$, and we are left with the conditions

$$\begin{aligned} B \sinh \beta l + D \sin \beta l &= 0 \\ B \sinh \beta l - D \sin \beta l &= 0 \end{aligned} \tag{8.5}$$

Equations (8.5) are satisfied if $B = D = 0$ or if $\sin \beta l \sinh \beta l = 0$. The last condition leads to $\beta l = k\pi$ where k is an integer. It

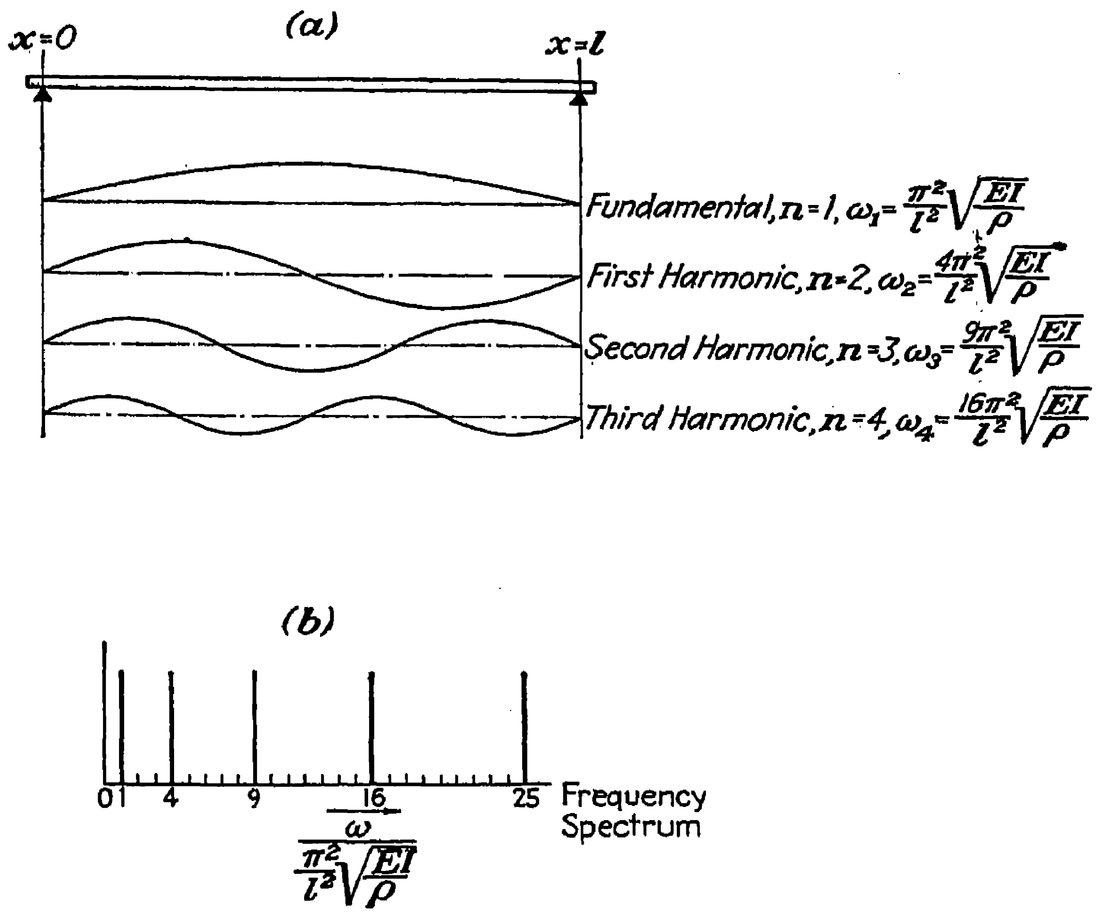

FIG. 8.2.—The modes of oscillation and the frequency spectrum of a uniform simply supported beam.

follows from (8.5) that in this case $B = 0$ and D is undetermined. There are an infinite number of possible vibrations of sinusoidal shape $w_k = D_k \sin \frac{k\pi x}{l}$. Their angular frequencies are given by the formula

$$\omega_k = \frac{k^2\pi^2}{l^2}\sqrt{\frac{EI}{\rho}} \tag{8.6}$$

It is seen that the modes of vibration of a hinged beam (Fig. 8.2a) are identical with those of a vibrating string. However,

comparing expressions (7.3) and (8.6), we find an essential difference in the frequencies: The frequencies of the beam increase as the squares of successive integers (Fig. 8.2b), whereas the frequencies of the string are proportional to the integers themselves.

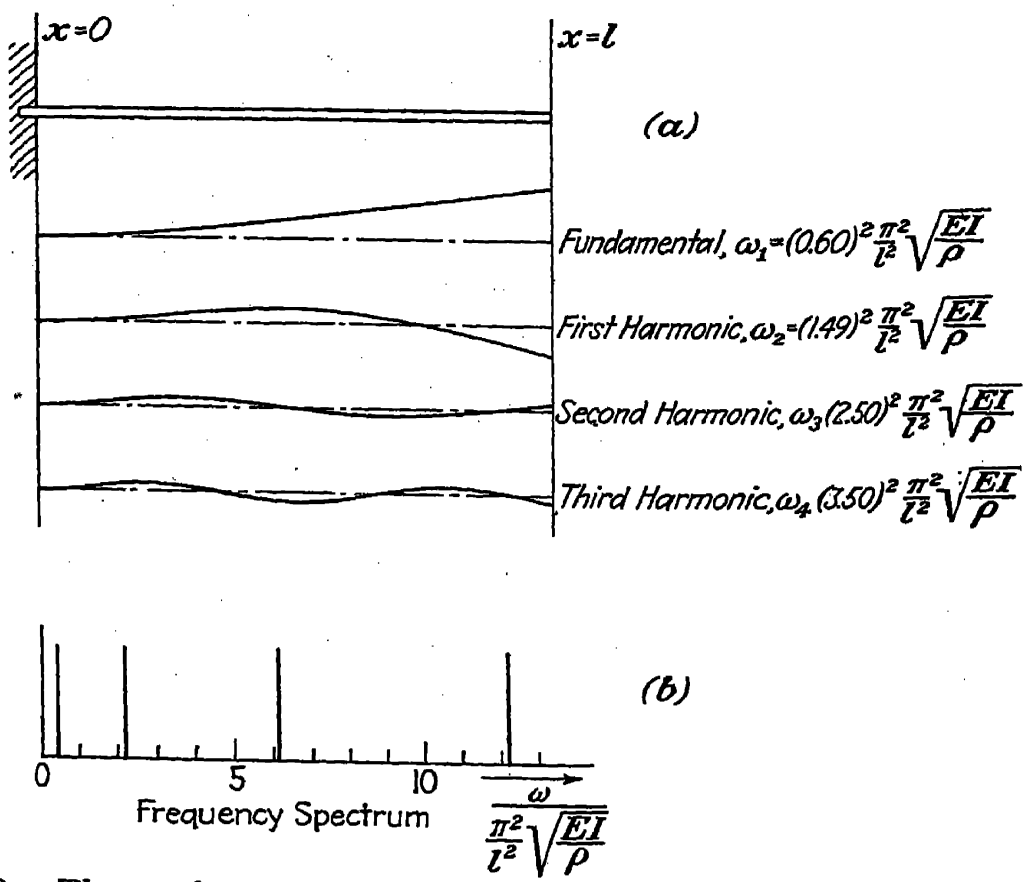

FIG. 8.3.—The modes of oscillation and the frequency spectrum of a uniform cantilever beam.

b. Let us consider a cantilever beam of the length l clamped at $x = 0$ (Fig. 8.3). We must satisfy the boundary conditions $w = dw/dx = 0$ for $x = 0$ and $d^2w/dx^2 = d^3w/dx^3 = 0$ at the free end $x = l$. From the boundary conditions at $x = 0$ it follows that $A + C = B + D = 0$, and, therefore,

$$w = A(\cosh \beta x - \cos \beta x) + B(\sinh \beta x - \sin \beta x)$$

The conditions for $x = l$ lead to the following relations:

$$\left(\frac{d^2w}{dx^2}\right)_{x=l} = A\beta^2(\cosh \beta l + \cos \beta l) + B\beta^2(\sinh \beta l + \sin \beta l) = 0 \quad (8.7)$$

$$\left(\frac{d^3w}{dx^3}\right)_{x=l} = A\beta^3(\sinh \beta l - \sin \beta l) + B\beta^3(\cosh \beta l + \cos \beta l) = 0 \quad (8.8)$$

Eliminating A and B between (8.7) and (8.8), we find the following condition for the characteristic values of βl:

$$1 + \cosh \beta l \cdot \cos \beta l = 0 \tag{8.9}$$

or

$$\cos \beta l = -\frac{1}{\cosh \beta l} \tag{8.10}$$

There are an infinite number of values of βl satisfying this equation. These values correspond to the abscissas of the points

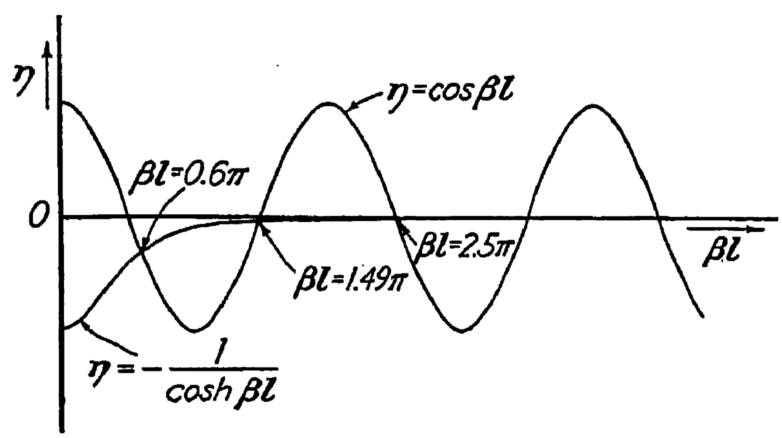

FIG. 8.4.—Graphical construction for finding the natural frequencies of a cantilever beam.

of intersection of the curve $\eta_1 = -\dfrac{1}{\cosh \beta l}$ and the curve $\eta_2 = \cos \beta l$ (Fig. 8.4).

The three first roots of the equation are $\beta l = 0.600\pi$, 1.49π, 2.50π. Since $1/\cosh \beta l$ becomes infinitely small for large values of βl, the higher roots are given with satisfactory accuracy by the equation

$$\cos \beta l = 0 \tag{8.11}$$

or

$$\beta l = (k - \tfrac{1}{2})\pi \tag{8.12}$$

The second root $\beta_2 l = 1.49\pi$ already differs only very slightly from the value 1.50π, which corresponds to Eq. (8.12). The angular frequencies are given by $\omega_k = \beta_k^2 \sqrt{EI/\rho}$ and, therefore, the higher frequencies are

$$\omega_k = \frac{(k - \frac{1}{2})^2 \pi^2}{l^2} \sqrt{\frac{EI}{\rho}} \tag{8.13}$$

Equation (8.13) shows that the frequencies of the higher harmonics increase as the squares of the successive integers. But this rule is not true for the lowest frequencies.

The modes of vibration are obtained from Eq. (8.9) and either (8.7) or (8.8). We find

$$w_k = C_k\left[\frac{\cosh \beta_k x - \cos \beta_k x}{\cosh \beta_k l + \cos \beta_k l} - \frac{\sinh \beta_k x - \sin \beta_k x}{\sinh \beta_k l + \sin \beta_k l}\right] \tag{8.14}$$

where C_k is an arbitrary constant. Some of these modes and the spectrum are plotted in Fig. 8.3. It can be seen that for the

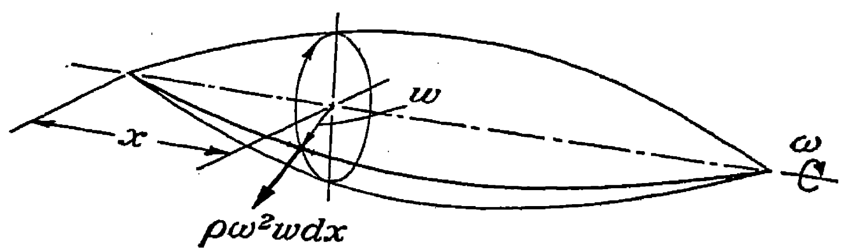

FIG. 8.5.—Whirling of a shaft.

higher harmonics the deflection curves approach the sinusoidal shape.

Let us now assume that a shaft of uniform cross section is rotating around its axis with the angular velocity ω. If, owing to lateral deflection of the shaft, the center of gravity of a cross section does not lie on the axis of rotation, a centrifugal force equal to $\rho\omega^2 w$ per unit length will act on the shaft as lateral load, where $w(x)$ is the deflection (Fig. 8.5). We obtain as the condition of equilibrium

$$EI\,\frac{d^4w}{dx^4} = \rho\omega^2 w \tag{8.15}$$

where EI is the flexural rigidity of the shaft. It is seen that the equation of equilibrium for a rotating shaft is identical with the equation for the modes of vibration of a beam. In general, the only equilibrium configuration will be $w = 0$, but there are certain values of the angular velocity ω for which equilibrium shapes different from $w = 0$ are possible. These angular velocities are known as the *critical speeds* of the shaft.

9. Vibration of a Beam Carrying a Concentrated Mass.—A uniform beam of length l resting on two supports at $x = 0$ and $x = l$ carries a mass m at its mid-point (Fig. 9.1). We shall

first consider the modes of vibration that are symmetrical on both sides of the concentrated mass. Then it is sufficient to compute w for one-half of the beam, for example, for $0 < x < l/2$. We use Eq. (8.4) and satisfy the boundary condition at the origin by putting $A = C = 0$. Thus,

$$w = B \sinh \beta x + D \sin \beta x \tag{9.1}$$

The boundary conditions at $x = l/2$ are:

a. $dw/dx = 0$, owing to the assumption of symmetrical modes of vibration.

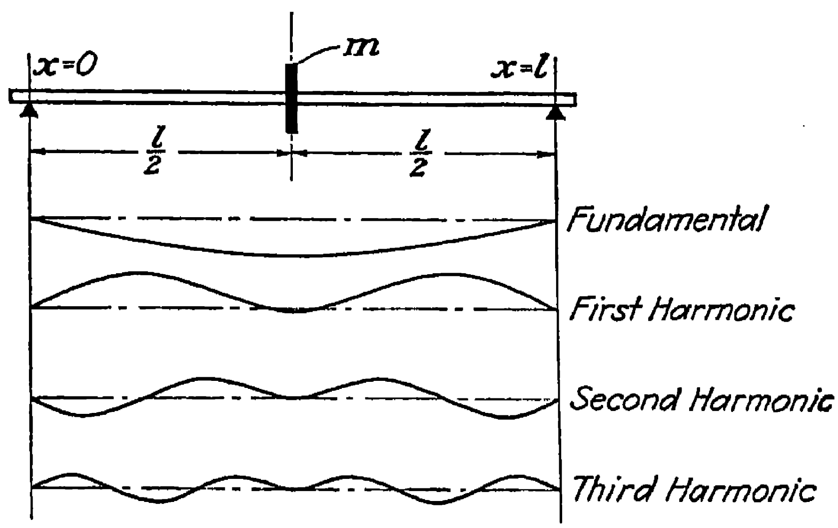

FIG. 9.1.—The modes of oscillation of a simply supported beam with a concentrated mass at the center.

b. The shearing force at $x = l/2$ must be equal to half the load, which is equal to the inertia force due to the concentrated mass m. The shearing force is equal to [cf. Eq. (3.5)]

$$S = -EI \frac{d^3w}{dx^3}$$

Hence,

$$EI \frac{d^3w}{dx^3} = -\frac{1}{2} m\omega^2 w \tag{9.2}$$

These two conditions lead to the following expressions:

$$B\beta \cosh \frac{\beta l}{2} + D\beta \cos \frac{\beta l}{2} = 0 \tag{9.3}$$

$$-EI\left(B\beta^3 \cosh \frac{\beta l}{2} - D\beta^3 \cos \frac{\beta l}{2}\right) = \frac{m\omega^2}{2}\left(B \sinh \frac{\beta l}{2} + D \sin \frac{\beta l}{2}\right) \tag{9.4}$$

Elimination of B and D between these two equations yields the following condition for β:

$$-4 = \frac{m\omega^2}{EI\beta^3}\left(\tanh\frac{\beta l}{2} - \tan\frac{\beta l}{2}\right) \tag{9.5}$$

Remembering that $\beta = \sqrt[4]{\rho\omega^2/EI}$ and denoting the total mass of the beam ρl by m_b, Eq. (9.5) becomes

$$2\frac{m_b}{m} = \frac{\beta l}{2}\left(\tan\frac{\beta l}{2} - \tanh\frac{\beta l}{2}\right) \tag{9.6}$$

Equation (9.6) is a transcendental equation for $\beta l/2$ if the mass ratio m_b/m is given. Since, according to (9.6),

$$\tan\frac{\beta l}{2} - \tanh\frac{\beta l}{2} = \frac{4}{\beta l}\frac{m_b}{m} \tag{9.7}$$

the left side must be small for large values of $\beta l/2$, and because $\tanh \beta l/2 \to 1$ for $\beta l/2 \to \infty$, we have for large roots approximately $\tan \beta l/2 = 1$, or

$$\frac{\beta l}{2} = k\pi + \frac{\pi}{4} \tag{9.8}$$

The frequencies of the higher symmetrical harmonics are, therefore,

$$\omega_k = 4\left(k + \frac{1}{4}\right)^2 \frac{\pi^2}{l^2}\sqrt{\frac{EI}{\rho}} \tag{9.9}$$

The shapes of vibration are found from either (9.3) or (9.4):

$$w_k = C_k\left(\frac{\sinh \beta_k x}{\cosh\dfrac{\beta_k l}{2}} - \frac{\sin \beta_k x}{\cos\dfrac{\beta_k l}{2}}\right) \tag{9.10}$$

where C_k is an arbitrary constant. For higher modes the mass m remains practically stationary (cf. Fig. 9.1); we have

$$(w_k)_{x=\frac{l}{2}} = C_k\left(\tanh\frac{\beta_k l}{2} - \tan\frac{\beta_k l}{2}\right)$$

The term in parentheses converges to zero for $k \to \infty$, according to Eq. (9.7), and, therefore, the deflection at $x = l/2$ tends to zero. As far as the higher symmetrical modes are concerned

the mass m has the same effect as if the beam were clamped at its mid-point.

If the mass m is very small and $\beta l/2$ is not very large, the right side of Eq. (9.7) becomes large. Then we have approximately $\tan(\beta l/2) = \infty$, or $\beta_k l = (2k - 1)\pi$. These are the roots already found for the symmetric modes of a freely supported beam [Eq. (8.6)]; in this case the influence of the mass at the mid-point is negligible.

Let us now assume that the mass of the beam m_b is small compared to the mass m. In this case the right side of Eq. (9.6) must be small. It is seen that a small value of $\beta l/2$ can satisfy this condition. This will give us the lowest, so-called *fundamental frequency* of the system. To find $\beta l/2$ we expand the right side of Eq. (9.6) in a power series in $\xi = \beta l/2$. Substituting the power series

$$\tan \xi = \xi + \frac{1}{3}\xi^3 + \frac{2}{15}\xi^5 + \frac{17}{315}\xi^7 + \frac{62}{2{,}835}\xi^9 + \frac{1{,}382}{155{,}925}\xi^{11} + \cdots$$

$$\tanh \xi = \xi - \frac{1}{3}\xi^3 + \frac{2}{15}\xi^5 - \frac{17}{315}\xi^7 + \frac{62}{2{,}835}\xi^9 - \frac{1{,}382}{155{,}925}\xi^{11} + \cdots$$

in Eq. (9.6), we obtain

$$3\frac{m_b}{m} = \xi^4 + \frac{17}{105}\xi^8 + \frac{1{,}382}{51{,}975}\xi^{12} + \cdots \tag{9.11}$$

We want to solve this equation with respect to ξ^4, *i.e.*, we want to *invert the series* and express ξ^4 as function of m_b/m. We put $3m_b/m = \eta$ and write

$$\xi^4 = \eta + a\eta^2 + b\eta^3 + \cdots \tag{9.12}$$

Introducing this expression into Eq. (9.11), we find

$$\eta = \eta + \left(a + \frac{17}{105}\right)\eta^2 + \left(b + \frac{34}{105}a + \frac{1{,}382}{51{,}975}\right)\eta^3 + \cdots$$

By equating the coefficients of the same powers of η we obtain the following relations for the coefficients a, b, etc.:

$$a = -\frac{17}{105}$$

$$b = -\frac{34}{105}a - \frac{1{,}382}{51{,}975} = \frac{376}{14{,}553}$$

Hence,

$$\xi^4 = \frac{\beta^4 l^4}{16} = 3\frac{m_b}{m}\left[1 - \frac{17}{35}\left(\frac{m_b}{m}\right) + \frac{376}{1{,}617}\left(\frac{m_b}{m}\right)^2 \cdots\right] \tag{9.13}$$

Let us denote the fundamental angular frequency of the system by ω_1. Remembering that $\omega = \beta^2\sqrt{EI/\rho}$ and $m_b = \rho l$, we obtain from Eq. (9.13)

$$\omega_1 = \sqrt{\frac{48EI}{ml^3}}\left[1 - \frac{17}{35}\left(\frac{m_b}{m}\right) + \frac{376}{1,617}\left(\frac{m_b}{m}\right)^2 \cdots \right]^{1/2} \tag{9.14}$$

If the mass of the beam is zero, the frequency of the system is

$$\omega_1 = \sqrt{\frac{48EI}{ml^3}} \tag{9.15}$$

This last result could have been found directly. The deflection of the beam under a concentrated force P is equal to $f = Pl^3/48EI$. Hence, the *spring constant*, *i.e.*, the force that produces a unit deflection, is $k = 48EI/l^3$. If we neglect the mass m_b of the beam, we may consider m as a mass attached to a spring of stiffness k. Then, as shown in Chapter IV, section 2, the angular frequency of the oscillating mass m is equal to

$$\omega_1 = \sqrt{\frac{k}{m}} = \sqrt{\frac{48EI}{ml^3}}$$

in accordance with (9.15). Equation (9.14) indicates that in order to take into account the mass of the beam approximately, we have to increase m by the factor

$$\frac{1}{1 - \frac{17}{35}\frac{m_b}{m} + \frac{376}{1,617}\left(\frac{m_b}{m}\right)^2} \cong 1 + 0.486\left(\frac{m_b}{m}\right) + 0.003\left(\frac{m_b}{m}\right)^2 + \cdots$$

This gives the practical rule that if m_b is small compared to m, one-half the mass of the beam should be added to the oscillating mass.

10. Forced Vibration of a Uniform Cantilever Beam.—A vertical beam of uniform cross section is clamped in a horizontal base. This base is given a horizontal harmonic displacement of the amplitude a_0 and of the angular frequency ω.

According to the theory of relative motion explained in Chapter III, section 5, if we use the equations of motion relative to a moving coordinate system, we have to introduce additional forces equal to the negative products of the masses and the acceleration of the system of reference. The acceleration of the base is equal to $-a_0\omega^2 \sin \omega t$. Hence, we must assume an additional load per unit length equal to $\rho_0 a_0 \omega^2 \sin \omega t$. If we assume that the deflection of the beam relative to the base is

$\zeta = w \sin \omega t$, we must modify Eq. (8.1) by writing

$$EI \frac{d^4w}{dx^4} - \rho\omega^2 w = \rho\omega^2 a_0 \quad , \tag{10.1}$$

The general solution of this equation is (with $\beta^4 = \rho\omega^2/EI$)

$$w = A \cosh \beta x + B \sinh \beta x + C \cos \beta x + D \sin \beta x - a_0 \tag{10.2}$$

The boundary conditions are $w = dw/dx = 0$ at the clamped end, $x = 0$; $d^2w/dx^2 = d^3w/dx^3 = 0$ at the free end, $x = l$. These conditions are satisfied by $A + C - a_0 = 0$, $B + D = 0$ and

$$\begin{aligned} \left(\frac{d^2w}{dx^2}\right)_{x=l} &= A\beta^2(\cosh \beta l + \cos \beta l) + B\beta^2(\sinh \beta l + \sin \beta l) \\ &\qquad - a_0\beta^2 \cos \beta l = 0 \\ \left(\frac{d^3w}{dx^3}\right)_{x=l} &= A\beta^3(\sinh \beta l - \sin \beta l) + B\beta^3(\cosh \beta l + \cos \beta l) \\ &\qquad + a_0\beta^3 \sin \beta l = 0 \end{aligned} \tag{10.3}$$

The determinant of these two linear equations for A and B is proportional to the expression $1 + \cosh \beta l \cos \beta l$ of Eq. (8.9), whose roots give the frequencies of the free vibration of the beam. The values of the coefficients A and B are found by solving the two equations (10.3) thus:

$$\begin{aligned} A &= \frac{a_0}{2} \frac{\cosh \beta l \cos \beta l + \sinh \beta l \sin \beta l + 1}{\cosh \beta l \cos \beta l + 1} \\ B &= -\frac{a_0}{2} \frac{\cosh \beta l \sin \beta l + \sinh \beta l \cos \beta l}{\cosh \beta l \cos \beta l + 1} \end{aligned} \tag{10.4}$$

A, B, and, therefore, w become infinite if ω is one of the roots of Eq. (8.9), *i.e.*, if ω is equal to one of the frequencies of free vibration. This result is analogous to the results obtained for systems with a finite number of degrees of freedom. The beam develops *resonance* when the frequency of the exciting force coincides with one of the frequencies of its free vibration. The difference between the beam and the systems treated in Chapter V is that the number of such frequencies is infinite in the case of of beam.

11. Buckling of a Uniform Column under Axial Load.—We consider in this and in the following sections columns loaded by axial forces and assume that the axial forces are so large that their influence on the bending must be taken into account.

We assume (Fig. 11.1) that a column is hinged at a fixed point $x = l$ and has a support at $x = 0$ which prevents lateral deflection but allows free rotation and deflection of the column in the direction of the axis. The column is under the action of an axial load P, which is considered positive if it produces compression. If the normal deflection at an arbitrary point is $w(x)$,

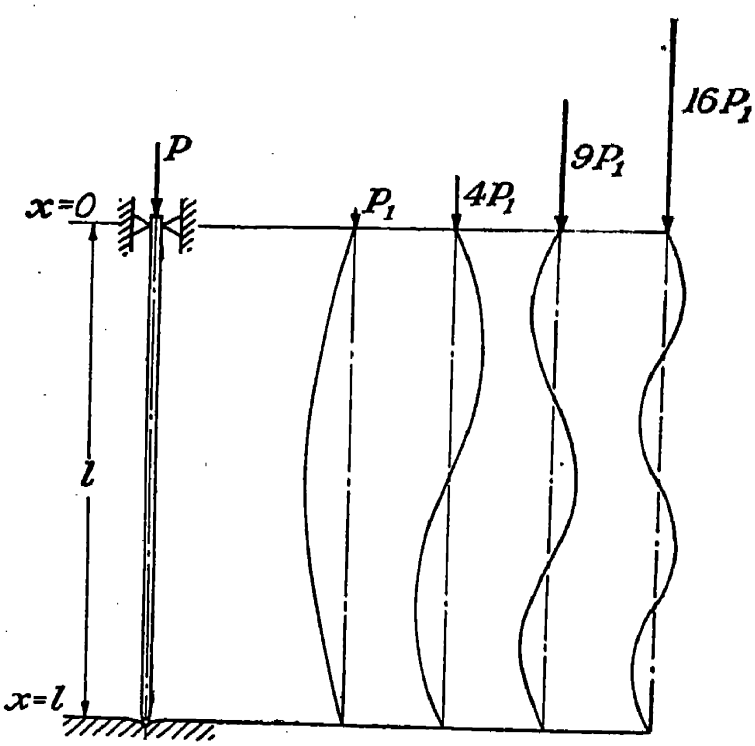

FIG. 11.1.—Modes of buckling of an axially loaded beam. Only the first mode can be produced without introducing additional restraint.

the bending moment M due to the force P is Pw. Applying Eq. (3.4), we obtain

$$EI\frac{d^2w}{dx^2} + Pw = 0 \tag{11.1}$$

We are thus led to a second-order equation mathematically identical with the equation of the vibrating string [cf. Eq. (6.2)]. Moreover, the boundary conditions are identical with those of the vibrating string. The general solution of Eq. (11.1) is

$$w = C_1 \cos\sqrt{\frac{P}{EI}}x + C_2 \sin\sqrt{\frac{P}{EI}}x \tag{11.2}$$

In order that $w = 0$ at $x = 0$, we must have $C_1 = 0$, and the condition $w = 0$ for $x = l$ implies

$$\sqrt{\frac{P}{EI}}l = k\pi \tag{11.3}$$

where k is an integer.

The corresponding deflection shapes known as the *modes of buckling* are (Fig. 11.1)

$$w_k = C_k \sin \frac{k\pi x}{l} \tag{11.4}$$

where C_k is undetermined. Each mode corresponds to a load

$$P_k = k^2\pi^2\frac{EI}{l^2} \tag{11.5}$$

Since for these values of the load the corresponding mode of buckling represents an equilibrium position with arbitrary amplitude, we say that under these loads the column is in *neutral equilibrium.* The corresponding values of the load are called *critical loads.* However, we must notice that the fact that the amplitude of the deflection curve is undetermined is due to the disregard of higher order terms in the equation of the elastic equilibrium, especially in the expression for the curvature. The undetermined character of the deflection is eliminated by a more exact theory.

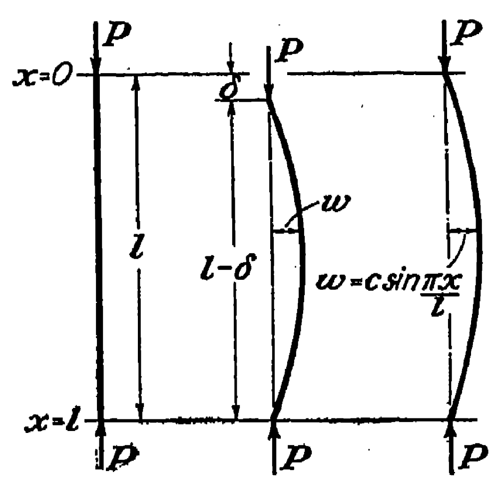

FIG. 11.2.—Axial and lateral deflections of a column under axial load.

From the engineering point of view, the first critical load is of special importance, because it is the upper limit for the stability of the undeflected equilibrium position of the column.

We can show that if the axial thrust P reaches its lowest critical value corresponding to $k = 1$,

$$P_1 = \pi^2\frac{EI}{l^2} \tag{11.6}$$

the work done by the axial thrust is equal to the work required for bending the beam into the corresponding buckling mode

$$w = C_1 \sin \frac{\pi x}{l} \tag{11.7}$$

The work done by the axial thrust P is equal to the product of P and the axial deflection δ of the end point $x = 0$ of the column (Fig. 11.2). Now δ is equal to the length of the straight column minus the chord of the bent column. Strictly speaking, we should take a deflection curve whose length of arc is equal to l and whose chord is $l - \delta$. However, we can take instead the difference between the arc and the chord corresponding to the curve (11.7); the error thus made is negligible. Hence, we put

$$\delta = \int_0^l \sqrt{1 + \left(\frac{dw}{dx}\right)^2}\, dx - l \cong \frac{1}{2}\int_0^l \left(\frac{dw}{dx}\right)^2 dx$$

Substituting w from Eq. (11.7), we obtain

$$\delta \cong \frac{1}{2} C_1^2 \frac{\pi^2}{l^2} \int_0^l \cos^2 \frac{\pi x}{l}\, dx = \frac{1}{4} C_1^2 \frac{\pi^2}{l}$$

Hence, the work W_1 done by the axial thrust is equal to

$$W_1 = \frac{P}{4} C_1^2 \frac{\pi^2}{l} \tag{11.8}$$

The work necessary for bending the column into the shape (11.7) is given by [cf. Eq. (3.7)]

$$W_2 = \frac{EI}{2}\int_0^l \left(\frac{d^2w}{dx^2}\right)^2 dx = \frac{EI}{2} C_1^2 \frac{\pi^4}{l^4} \int_0^l \sin^2 \frac{\pi x}{l}\, dx = \frac{EI}{4} C_1^2 \frac{\pi^4}{l^3} \tag{11.9}$$

It is seen that

$$W_1 < W_2 \qquad \text{if} \qquad P < EI\frac{\pi^2}{l^2} \tag{11.10}$$

and

$$W_1 > W_2 \qquad \text{if} \qquad P > EI\frac{\pi^2}{l^2} \tag{11.11}$$

This result shows that if $P > P_1$, which is the first critical load, then in the bent position the total potential energy $W_2 - W_1$ of the system is smaller than in the straight position. Hence, for $P > P_1$ the column is certainly unstable in the straight equilibrium position. [To prove that it is stable for $P < P_1$, it would be necessary to show that $W_1 < W_2$, not only for the particular deflection curve (11.7), but for arbitrary variations of the straight shape.]

It can be shown in a similar way that the column is in neutral equilibrium position, *i.e.*, $W_1 = W_2$ for all critical values P_k. However, the higher modes of buckling can be realized only by maintaining the straight shape beyond the lowest critical value by additional restraint.

Equation (11.6), which gives the lowest critical load, is known as *Euler's formula*.

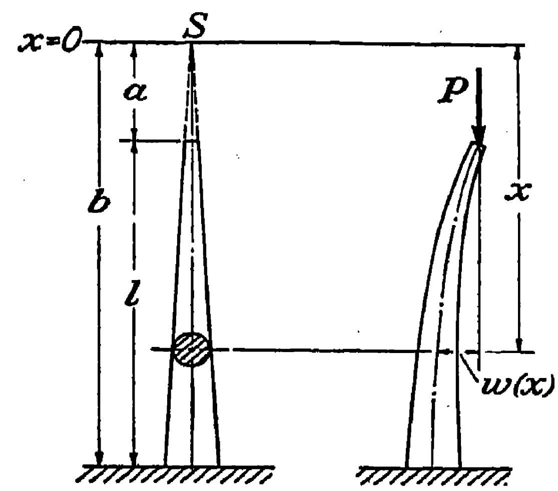

FIG. 12.1.—Buckling of a tapered column.

12. Buckling of a Tapered Column. Buckling of a Column under Its Own Weight.—Let an axial load P be applied to a column of circular cross section with linear taper (Fig. 12.1). We take the origin of the abscissa x at the fictitious vertex S of the cone and assume that the column is clamped at its base $x = b$. The law of variation of the moment of inertia of the cross section is

$$I(x) = I_b\left(\frac{x}{b}\right)^4 \tag{12.1}$$

where I_b is the moment of inertia of the cross section at the base $x = b$. The horizontal distance between the deformed beam axis and the line of action of P is denoted by w. Then the equation for the deflection of the beam becomes

$$EI_b\left(\frac{x}{b}\right)^4 \frac{d^2w}{dx^2} + Pw = 0$$

or

$$\frac{d^2w}{dx^2} + \frac{Pb^4}{EI_b}\frac{w}{x^4} = 0 \tag{12.2}$$

This differential equation belongs to the class represented by Eq. (7.8) of Chapter II, where $m = 0$ and $n = -4$.

The solution of Eq. (12.2) becomes [cf. Eq. (7.13) of Chapter II], with $C = Pb^4/EI_b$,

$$w = x^{1/2}\, Z_{-1/2}\left(-\sqrt{\frac{Pb^4}{EI_b}}x^{-1}\right) \tag{12.3}$$

Now, according to section 4, Chapter II [Eqs. (4.6) and (4.8)], $J_{-\frac{1}{2}}(z)$ and $Y_{-\frac{1}{2}}(z)$ have the form const. $\times \cos z/\sqrt{z}$ and const. $\times \sin z/\sqrt{z}$, respectively. Therefore, $w(x)$ can be written in the form:

$$w = x\left(A \cos \frac{V}{x} + B \sin \frac{V}{x}\right) \tag{12.4}$$

where $V^2 = Pb^4/EI_b$.

The boundary conditions are $w = 0$ at $x = a$ and $dw/dx = 0$ at $x = b$. Hence, we have

$$A \cos \frac{V}{a} + B \sin \frac{V}{a} = 0 \tag{12.5}$$

$$A\left(\cos \frac{V}{b} + \frac{V}{b} \sin \frac{V}{b}\right) + B\left(\sin \frac{V}{b} - \frac{V}{b} \cos \frac{V}{b}\right) = 0$$

Elimination of A and B between these two equations yields the following condition for V:

$$\tan\left(\frac{V}{a} - \frac{V}{b}\right) = -\frac{V}{b} \tag{12.6}$$

With the notations $b - a = l$ and $Vl/ab = \alpha$, we obtain the following transcendental equation for α:

$$\tan \alpha = -\frac{a}{l}\alpha \tag{12.7}$$

This equation has an infinite number of roots α_k (see Fig. 12.2) given by the intersection of the curves $\eta_1 = \tan \alpha$ and $\eta_2 = -\alpha\frac{a}{l}$. The corresponding critical loads are

$$P_k = \alpha_n^2\left(\frac{a}{b}\right)^2 \frac{EI_b}{l^2} \tag{12.8}$$

The modes of buckling are given by

$$w_k(x) = A_k x \sin\left[\alpha_k \frac{b}{l}\left(1 - \frac{a}{x}\right)\right] \tag{12.9}$$

Let us assume that we remove the vertex of the cone to infinity but keep the length of the column and the cross section at the

base fixed, then the ratio $a/b \to 1$, and the column takes on a uniform cross section. Since $a/l \to \infty$, it is seen in Fig. 12.2 that the first intersection of the curves occurs when $\alpha_1 \to \pi/2$, and thus the lowest critical load is found to be

$$P_1 = \frac{\pi^2}{4}\frac{EI_b}{l^2}$$

This result checks with the results obtained in the last section, for we may consider a column with one clamped and one free end as the upper half of a column with two freely supported ends.

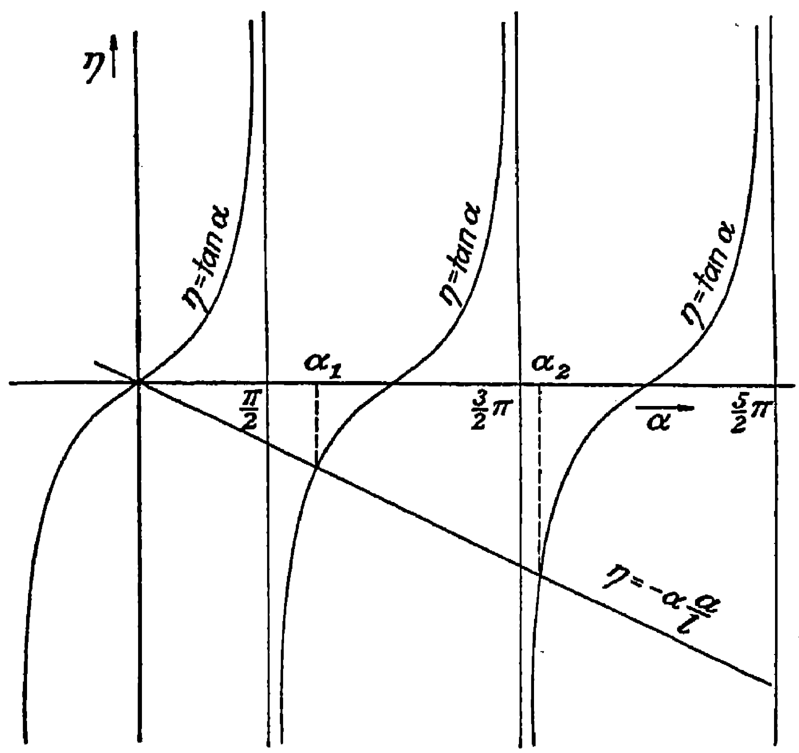

FIG. 12.2.—Graphical construction for determining the critical buckling loads of a tapered column.

Hence, the column of length l with one clamped and one free end buckles under the same load as a column of the length $2l$ with two freely supported edges. If, on the other hand, the shape is a complete cone, then $a = 0$, $\alpha_1 = \pi$, and $P_1 = 0$. Hence, according to this theory a cone would buckle at an arbitrarily small load. However, in this case we are applying the theory beyond its range of validity because the underlying assumptions of the theory of bending are not satisfied near the vertex.

Another interesting buckling problem is the stability of a vertical column loaded by its own weight. It occurs as a practical problem in the manufacture of very thin tungsten filaments for incandescent lamps.

Let us denote the cross section of the column by A and the specific weight of its material by γ. In the buckled position (Fig. 12.3) the weight of the portion of the beam between the top and the cross section at x is in equilibrium with the resultant of the axial stresses and the shear force acting at x. Therefore, if the inclination of the axis of the column to the vertical is θ, the shear force S is equal to $S = \gamma A x \theta$. The bending moment is $M = -EI \dfrac{d\theta}{dx}$, where EI is the flexural rigidity of the column. Therefore, the shear force $S = \dfrac{dM}{dx} = -EI \dfrac{d^2\theta}{dx^2}$. Obviously we obtain the differential equation for θ:

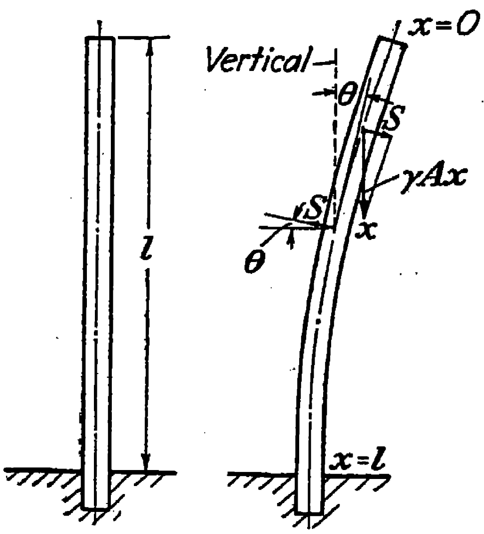

FIG. 12.3.—Buckling of a uniform column under its own weight.

$$EI \frac{d^2\theta}{dx^2} = -\gamma A x \theta$$

or

$$\frac{d^2\theta}{dx^2} + \frac{\gamma A}{EI} x\theta = 0 \qquad (12.10)$$

If we put $x\sqrt[3]{\gamma A/EI} = \xi$, the differential equation (12.10) becomes

$$\frac{d^2\theta}{d\xi^2} + \xi\theta = 0 \qquad (12.11)$$

This equation is identical with Eq. (7.14), Chapter II, and its general solution is [Eq. (7.15), Chapter II]

$$\theta = \xi^{1/2} Z_{1/3}\left(\tfrac{2}{3}\xi^{3/2}\right) \qquad (12.12)$$

where the symbol $Z_{1/3}$ means the general solution of Bessel's differential equation of one-third order. The general solution (12.12) is a linear combination of two particular solutions; one is of the form [(Eq. 7.17), Chapter II]:

$$\theta_1 = \xi(a_0 + a_1\xi^3 + a_2\xi^6 + \cdots) \qquad (12.13)$$

and the other of the form [Eq. (7.18), Chapter II]:

$$\theta_2 = b_0 + b_1\xi^3 + b_2\xi^6 + \cdots \qquad (12.14)$$

We have the boundary conditions $d\theta/dx = d\theta/d\xi = 0$ for $\xi = 0$ since the bending moment is zero at the top cross section and $\theta = 0$ for $x = l$ or $\xi = l\sqrt[3]{\gamma A/EI}$, if l is the height of the column and we assume that the column is clamped normal to a horizontal base. It follows from the first condition that $a_0 = 0$; therefore [cf. Eq. (7.20), Chapter II], $\theta_1 = 0$, and we are left with

$$\theta = b_0 + b_1\xi^3 + b_2\xi^6 + \cdots \tag{12.15}$$

where [cf. Eq. (7.19), Chapter II]

$$b_1 = -\frac{b_0}{2\cdot 3}, \qquad b_2 = -\frac{b_1}{5\cdot 6}, \qquad \cdots$$

and, therefore,

$$\theta_1 = b_0\left(1 - \frac{\xi^3}{6} + \frac{\xi^6}{180} - \cdots\right) \tag{12.16}$$

The second boundary condition is satisfied if the expression in the parentheses vanishes for $\xi = l\sqrt[3]{\gamma A/EI}$. Therefore, we have to calculate the roots of the equation

$$1 - \frac{\xi^3}{6} + \frac{\xi^6}{180} - \cdots = 0 \tag{12.17}$$

The first approximation is $\xi^3 = 6$ or $\xi = \sqrt[3]{6} = 1.817$. In order to calculate the next approximations, we use Graeffe's method (Chapter V, section 8). Let us cut off (12.17) after the term with ξ^6 and substitute $\xi^3 = 1/z$. Then we have, after dividing (12.17) by ξ^6,

$$z^2 - \frac{z}{6} + \frac{1}{180} = 0 \tag{12.18}$$

Then multiplying Eq. (12.18) by $z^2 + \frac{z}{6} + \frac{1}{180}$, we obtain

$$z^4 - z^2\left(\frac{1}{36} - \frac{1}{90}\right) + \frac{1}{180^2} = 0$$

Hence, the largest root z (corresponding to the smallest root ξ) is approximately

$$z = \sqrt{\tfrac{1}{60}}$$

or

$$\xi = \sqrt[6]{60} = 1.98$$

(The exact value of the root is 1.986.) Hence, the critical length of the column is

$$l_{cr} = 1.98\sqrt[3]{\frac{EI}{\gamma A}} \tag{12.19}$$

For a filament of circular cross section of radius r we have $I/A = r^2/4$, and, therefore,

$$l_{cr} = 1.98\frac{1}{\sqrt[3]{4}}r^{2/3}\left(\frac{E}{\gamma}\right)^{1/3} \tag{12.20}$$

The length $L = E/\gamma$ is the length of a filament which by its own weight would produce a tensile stress equal to E; obviously, L is a characteristic length of the material. Then it is seen that

$$l_{cr} \sim r^{2/3}L^{1/3} \tag{12.21}$$

By repeated multiplication of the two series,

$$g(\xi) = 1 - \frac{\xi^3}{6} + \frac{\xi^6}{180} - \cdots$$
$$g(-\xi) = 1 + \frac{\xi^3}{6} + \frac{\xi^6}{180} + \cdots$$

we can obtain approximations to the roots $\xi_1, \xi_2, \cdots$ of $g(\xi) = 0$ in ascending order. These roots would give the loads corresponding to higher modes of buckling.

13. Buckling of an Elastically Supported Beam.—The differential equation for the deflection of an elastically supported uniform beam was given in section 4:

$$EI\frac{d^4w}{dx^4} + kw = p(x) \tag{13.1}$$

To investigate the buckling of such a beam under action of an axial force P, we replace the latter by an equivalent lateral load. We have seen that the moment of an axial force P is equal to Pw; according to (3.3) the load corresponding to a moment distribution $M(x)$ is equal to $p(x) = -d^2M/dx^2$; hence the

transverse load which would produce a moment $M(x) = Pw$ is equal to $p(x) = -P \frac{d^2w}{dx^2}$. Therefore, Eq. (13.1) becomes

$$EI \frac{d^4w}{dx^4} + kw = -P \frac{d^2w}{dx^2} \tag{13.2}$$

or

$$\frac{d^4w}{dx^4} + \frac{P}{EI}\frac{d^2w}{dx^2} + \frac{k}{EI}w = 0 \tag{13.3}$$

We can also deduce this equation by considering the equilibrium of the beam in slightly curved shape. Denoting the radius of curvature by R and assuming an axial force P acting on two cross sections a unit distance apart, we obtain a resultant force normal to the axis of the beam which is equal to P/R or, with the approximation used in the beam theory, to $-P \frac{d^2w}{dx^2}$.

The characteristic equation of the differential Eq. (13.3) is

$$\lambda^4 + \frac{P}{EI}\lambda^2 + \frac{k}{EI} = 0 \tag{13.4}$$

or

$$\lambda^2 = -\frac{P}{2EI} \pm \sqrt{\frac{1}{4}\left(\frac{P}{EI}\right)^2 - \frac{k}{EI}} \tag{13.5}$$

It is seen that the two values of λ^2 are real and negative when $P > 0$ and

$$P^2 > 4kEI \tag{13.6}$$

In this case all solutions are trigonometric functions.

We are especially interested in the buckling of a beam of infinite length; in this case we can limit ourselves to periodic solutions because any other solution would yield infinite deflection either at $x = \infty$ or at $x = -\infty$. Writing the periodic solution in the form

$$w = C \sin \frac{2\pi(x - a)}{l} \tag{13.7}$$

where a is an arbitrary constant and $\lambda = 2\pi i/l$, we obtain from Eq. (13.4)

$$\frac{16\pi^4}{l^4} - \frac{P}{EI}\frac{4\pi^2}{l^2} + \frac{k}{EI} = 0$$

or

$$P = \frac{4\pi^2}{l^2}EI + \frac{kl^2}{4\pi^2} \tag{13.8}$$

It is seen that in the case of the infinite beam we do not obtain distinct critical loads, but a certain range for the load P which is capable of holding the beam in a deflected shape. This critical range extends from a smallest value $P_{\min}$ to $P = \infty$. We obtain $P_{\min}$ by differentiation of the expression (13.8)

$$\frac{dP}{dl} = -\frac{8\pi^2}{l^3}EI + \frac{2kl}{4\pi^2} = 0 \tag{13.9}$$

The *critical wave length*, *i.e.*, the wave length produced by the smallest axial load which causes buckling, is given by

$$l_{cr} = 2\pi\sqrt[4]{\frac{EI}{k}} \tag{13.10}$$

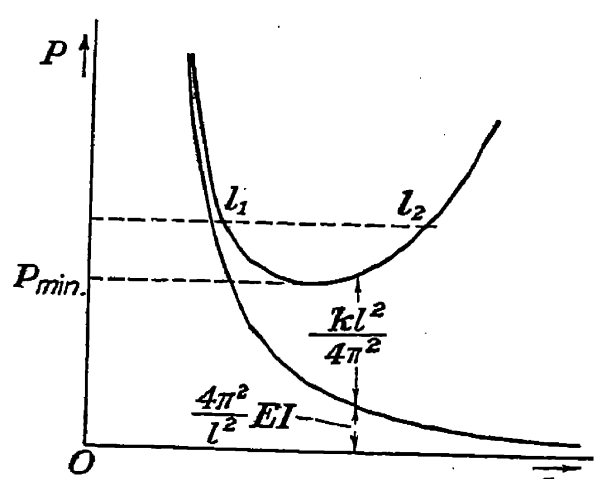

FIG. 13.1.—Diagram for the buckling load of an elastically supported beam.

Substituting (13.10) in (13.8), we obtain [see Eq. (13.6)]

$$P_{\min} = 2\sqrt{kEI} \tag{13.11}$$

For loads $P < P_{\min}$ the straight form is stable. For loads $P > P_{\min}$ we obtain two different wave lengths l_1 and l_2 for each value of the buckling load (Fig. 13.1). However, these configurations do not occur unless some additional restraint is introduced, as the beam buckles when P reaches the value $P_{\min}$.

14. Combined Axial and Lateral Load Acting on the Spar of an Airplane Wing.—Figure 14.1 represents schematically the spar of an airplane wing hinged at the fuselage at $A(x = 0)$ and supported by a hinged strut at $B(x = l)$. The horizontal component of the strut force exerts an axial compression upon the spar, which is subjected to an additional direct bending load p. We denote the bending moment due to p by M_p; then the differential equation for the deflection w becomes*

$$EI\frac{d^2w}{dx^2} = -Pw - M_p$$

* The load p and the deflection w are here measured positive upward. The bending moment on the left is positive counterclockwise.

Differentiating twice, we obtain

$$EI\,\frac{d^4w}{dx^4} + P\,\frac{d^2w}{dx^2} = p \tag{14.1}$$

The total bending moment at the cross section x is equal to $M = -EI\,\dfrac{d^2w}{dx^2}$. Introducing M as the unknown variable, we obtain

$$\frac{d^2M}{dx^2} + \frac{P}{EI}M = -p \tag{14.2}$$

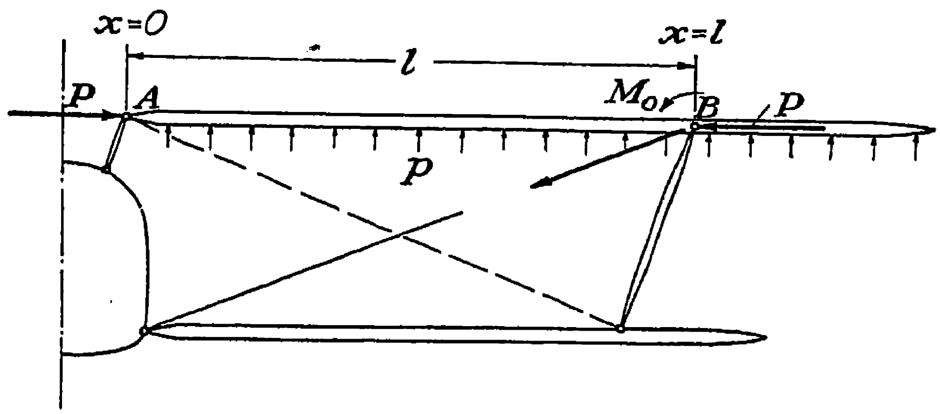

FIG. 14.1.—Wing spar of a biplane under combined axial and lateral loading.

We assume that p is constant; then the general solution of equation (14.2) is

$$M = C_1 \cos\sqrt{\frac{P}{EI}}x + C_2 \sin\sqrt{\frac{P}{EI}}x - \frac{pEI}{P}$$

The boundary conditions at the hinges are $M = 0$ at $x = 0$ and $M = -M_0$ at $x = l$, where l is the distance between the hinges and $-M_0$ is the bending moment about the hinge point B of the lift forces acting on the cantilever part of the wing. The values of the constants C_1 and C_2 determined by the boundary conditions are

$$C_1 = \frac{pEI}{P}$$

$$C_2 = p\frac{EI}{P}\,\frac{1 - \cos\sqrt{(P/EI)}l - (M_0P/pEI)}{\sin\sqrt{(P/EI)}l}$$

Hence, the total bending moment acting at an arbitrary section of the spar is

$$M = p\frac{EI}{P}\left[\cos\sqrt{\frac{P}{EI}}x + \left(\frac{1 - \cos\sqrt{(P/EI)}l - (M_0P/pEI)}{\sin\sqrt{(P/EI)}l}\right)\sin\sqrt{\frac{P}{EI}}x - 1\right] \quad (14.3)$$

It is seen that when the axial load P reaches the value $P = \pi^2\frac{EI}{l^2}$, so that $\sin\sqrt{\frac{P}{EI}}l = \sin\pi = 0$, the bending moment becomes infinite. Thus the deflection tends to infinity when the axial load approaches a critical value, as, similarly, the amplitude of a beam subjected to periodic forces tends to infinity in the case of resonance. In both cases infinite amplitudes are reached for certain critical values of a parameter (load or frequency) occurring in the equation. These values of the parameter are identical with the characteristic values of the associated homogeneous equation; *i.e.*, the values for which this equation has solutions different from zero. For instance,

$$\frac{d^2M}{dx^2} + \frac{P}{EI}M = 0 \quad (14.4)$$

has solutions of the form:

$$M_k = C_k \sin\sqrt{\frac{P_k}{EI}}x$$

for certain given values for P_k. The smallest of these values is the buckling load

$$P_1 = \frac{\pi^2EI}{l^2}$$

For the same value of the axial thrust, the bending moment, according to (14.3), becomes infinite if the beam is subjected to a lateral load in addition to the axial force.

If, instead of compression, the beam were under tension, the equation would differ only by the sign of P

$$\frac{d^2M}{dx^2} - \frac{P}{EI}M = -p \quad (14.5)$$

In this case, however, the general solution involves exponential functions, and the value of M will never become infinite. The reader will remember that an equation of this type was encoun-

tered in the theory of the suspension bridge (section 5). In fact, the load condition of the bridge truss could be described as a combination of a lateral load and an axial tension.

15. Graphical Representation of the Bending Moment.—We put

$$\tan\varphi = \frac{1 - \cos\sqrt{(P/EI)}l - (M_0/p)(P/EI)}{\sin\sqrt{(P/EI)}l} \tag{15.1}$$

Then Eq. (14.3) becomes

$$M = \frac{pEI}{P\cos\varphi}\left[\cos\left(\sqrt{\frac{P}{EI}}x - \varphi\right) - \cos\varphi\right]$$

or

$$\frac{MP\cos\varphi}{pEI} = \cos\left(\sqrt{\frac{P}{EI}}x - \varphi\right) - \cos\varphi \tag{15.2}$$

We draw a circle of unit diameter (Fig. 15.1) and draw the radius vector $\overline{OA}$ so that the angle $AOy = \varphi$. Then if we draw a circle

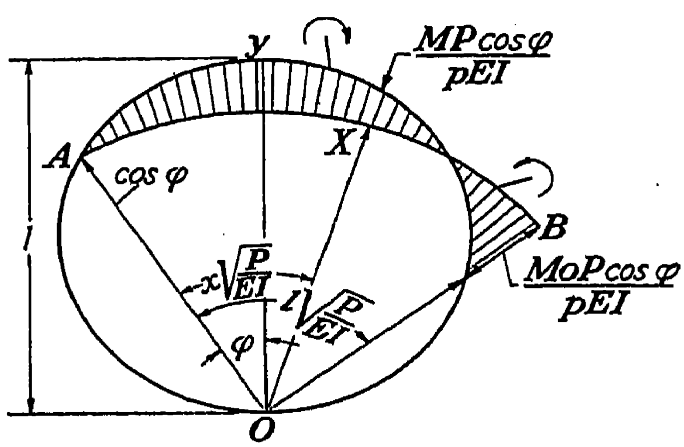

FIG. 15.1.—Graphical construction for determining the bending-moment distribution for an axially and laterally loaded beam.

through A with O as center, the line segments between the two circles are equal to the right side of Eq. (15.2), provided the angle AOX represents $x\sqrt{\frac{P}{EI}}$. It is seen that if $P \to \pi^2\frac{EI}{l^2}$, $\varphi \to \frac{\pi}{2}$ and $M \to \infty$.

16. General Discussion of the Boundary Problems Encountered in This Chapter.—We have dealt in this chapter with differential equations of second and fourth orders. Let us consider as two representative examples of second-order equations

the equation for the deflection of a loaded string with elastic restraint $k > 0$, *viz.*,

$$F \frac{d^2w}{dx^2} - kw = -p(x) \tag{16.1}$$

and the equation for the harmonic vibration of a string under action of an external load [mathematically identical with Eq. (14.2)], *viz.*,

$$F \frac{d^2w}{dx^2} + \rho\omega^2 w = -p(x) \tag{16.2}$$

We treated similar equations in Chapter IV; however, the physical problems with which we were concerned in that chapter involved *initial conditions* which specify the value of the unknown function and its derivative for one value of the independent variable. We found that, in general, such initial conditions determine a *unique* solution of the problem.

In the problems of this chapter the conditions that we call *boundary conditions* have bearing on the values of w or its derivatives for at least two values of the independent variable. For example, w was given at two points, $x = 0$ and $x = l$. We find that in this case Eqs. (16.1) and (16.2) behave in a very different manner. Before we classify the different cases, let us introduce the following terminology: If a *boundary condition* is satisfied by $Cw(x)$, provided that it is satisfied by $w(x)$, where C is an arbitrary constant, we call the boundary condition *homogeneous*. For example, $w = 0$ and $dw/dx = 0$ are *homogeneous boundary conditions*, whereas $w = 1$ is a *nonhomogeneous boundary condition*. We call the problem of finding a solution of a homogeneous equation for homogeneous boundary conditions a *homogeneous boundary problem*. If either the equation or at least one boundary condition is nonhomogeneous, we call the problem a *nonhomogeneous boundary problem*. If we replace the right side of a nonhomogeneous equation and the nonhomogeneous boundary conditions by zero, we call the homogeneous problem obtained in this way *the associated homogeneous problem*. Then we find the following results:

a. Equation (16.1) has one, and only one, solution if *two* boundary conditions are to be satisfied. The only solution of the homogeneous problem is $w = 0$.

b. In the case of Eq. (16.2), $w = 0$ is the only solution of the homogeneous problem, except when the parameter ω^2 is equal to one of the infinite number of *characteristic values* $\omega_1^2, \omega_2^2, \ldots$. If ω^2 coincides with one of these values, the homogeneous problem has solutions different from zero; these are determined only to an arbitrary multiplicative constant. A nonhomogeneous problem has one and only one solution if the associated homogeneous problem is solved by $w = 0$ only. If the parameter ω^2 is equal to one of the characteristic values of the associated homogeneous problem, the nonhomogeneous problem has no finite solution. We encounter the homogeneous problem in problems of free vibration and buckling. The characteristic values represent the natural frequencies and the critical loads, respectively. The problems of forced vibration (resonance) and the problem of combined buckling and direct load lead to nonhomogeneous equations or to nonhomogeneous boundary conditions.

The mathematical reason for the different behavior of the two equations was indicated earlier in this chapter. The solution of the homogeneous equation associated with Eq. (16.2) has oscillatory character, whereas the homogeneous equation associated with (16.1) has no more than one zero point. Hence, it is important to find criteria that determine when a differential equation has oscillatory solutions.

In general, *e.g.*, for nonlinear differential equations and linear differential equations of higher order than the second, this is a very difficult mathematical problem. In the case of a linear differential equation of second order, we can state a simple theorem which gives a criterion for a wide class of such differential equations.

A homogeneous linear differential equation of second order has the general form:

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = 0 \tag{16.3}$$

where $a(x)$ and $b(x)$ are given functions of x. Equation (16.3) can always be reduced to the form:

$$\frac{d^2z}{dx^2} + g(x)z = 0 \tag{16.4}$$

In fact, if we substitute in Eq. (16.3) $y = e^{-\int^x \frac{a(x)}{2} dx} z$, the term with dz/dx drops out, and we obtain the form (16.4).

Now, if $g(x) > 0$ and has a positive lower bound between $x = a$ and $x = \infty$, it can be proved that the solution $z(x)$ of Eq. (16.4) takes the value $z = 0$ an infinite number of times in the interval between $x = a$ and $x = \infty$. From Eq. (16.4) it follows that $d^2z/dx^2 = -g(x)z$. Since $g(x) > 0$, this means that d^2z/dx^2 and z have opposite signs, and, therefore, the curve $z = z(x)$ seen from the x-axis is always concave. All inflection points must lie on the x-axis. It is easily seen that if we start from an arbitrary point (x,z), the curve due to the sign and the bounded character of its curvature must intersect the x-axis; if we pass through this axis, the sign of the curvature changes, and, therefore, we obtain another intersection, and this process continues indefinitely.

Hence, if we assume that in the equation

$$\frac{d^2z}{dx^2} + [h(x) + \lambda g(x)]z = 0 \tag{16.5}$$

the function $h(x)$ is bounded and $g(x)$ is larger than zero and has a lower bound, the solution of Eq. (16.5) has oscillatory character, provided that the paramater λ is positive and sufficiently large, so that $h(x) + \lambda g(x) > 0$. In other words, Eq. (16.5) has always positive characteristic values.

If $g(x)$ changes its sign, it is very difficult to decide whether or not real characteristic values exist. Fortunately, this is an exceptional case in engineering problems.

As a representative example of an equation of fourth order, the equation for the deflection of beams was discussed. The most general form of this equation is the following:

$$\frac{d^2}{dx^2}\left[a(x)\frac{d^2w}{dx^2}\right] + \frac{d}{dx}\left[b(x)\frac{dw}{dx}\right] + c(x)w = p(x) \tag{16.6}$$

An equation of such form is called a *self-adjoint* linear differential equation. (A linear differential equation of second order can always be written in the analogous form $d/dx\,[a(x)\,dy/dx] + b(x)w = p(x)$ and, therefore, is always *self-adjoint*.) It would take us too far into the domain of the theory of differential equations to explain the origin of this term. It will be sufficient to state that all linear differential equations that are derived from a variation principle belong to this class. Hence, all equilibrium, buckling, and vibra-

tion problems that refer to conservative systems, *i.e.*, to systems that have potential energy and are governed by linear equations, lead to equations of the self-adjoint class. The homogeneous equations of this class have the property that, if one particular solution has more than one zero point, it crosses the x-axis an infinite number of times.

It is difficult to find simple conditions that indicate whether or not the homogeneous equation associated with Eq. (16.6) has solutions of the oscillatory type, and the engineer will resort to physical reasoning.

17. Determination of Characteristic Values by the Iteration Method.—For the *characteristic values* of differential equations with constant coefficients we obtained transcendental equations in which trigonometric or exponential functions were involved. If the coefficients of the differential equation are variable, the determination of the characteristic values (natural frequencies, buckling loads) can cause considerable difficulty. However, in many cases the iteration method can be employed with success.

Let us assume that the differential equation has the form:

$$\frac{d}{dx}\left[a(x)\,\frac{dw}{dx}\right] + \lambda b(x)w = 0 \tag{17.1}$$

We shall determine a value λ for which $w(x)$ satisfies certain homogeneous boundary conditions.

We choose a function $w_1(x)$ in such a way that it satisfies the boundary conditions set for the solution $w(x)$, but is arbitrary otherwise. Then we can solve the equation

$$\frac{d}{dx}\left[a(x)\,\frac{dw}{dx}\right] = -b(x)w_1(x) \tag{17.2}$$

by direct integration. Assume that we obtain a function $w = w_2(x)$. If $w_1(x)$ were the exact solution corresponding to a characteristic value λ of Eq. (17.1), for example, one of the modes of oscillation or a buckling mode of the beam, etc., the function $w_2(x)$ would be proportional to $w_1(x)$, *viz.*,

$$w_2(x) = \frac{w_1(x)}{\lambda} \tag{17.3}$$

where λ is the characteristic value sought for. If the proportionality between $w_2(x)$ and $w_1(x)$ is not satisfactory, we multiply $w_2(x)$ by a constant so that for a suitably chosen value $x = x_0$

we make $w_2(x_0) = w_1(x_0)$ and substitute the function $w_2(x)$, reduced in this way, on the right side of Eq. (17.2):

$$\frac{d}{dx}\left[a(x)\frac{dw}{dx}\right] = -b(x)w_2(x) \tag{17.4}$$

We obtain in this way $w_3(x)$. If $w_3(x)$ and $w_2(x)$ are proportional to each other with satisfactory accuracy, we consider $w_3(x)$ as the solution corresponding to the characteristic value:

$$\lambda \cong \frac{w_2(x_0)}{w_3(x_0)} \tag{17.5}$$

It can be shown that by this procedure the limiting value

$$\lambda = \lim_{n\to\infty} \frac{w_{n-1}(x)}{w_n(x)} \tag{17.6}$$

gives us exactly the smallest characteristic value of the differential equation, *i.e.*, the fundamental frequency, the lowest buckling load, or the lowest critical speed. The function $w_n(x)$ converges to the particular solution corresponding to this characteristic value, *i.e.*, the first mode of vibration or buckling. The process can be also applied to equations of fourth order.

Let us consider, for example, a shaft of constant flexural rigidity EI which carries so many masses that the mass distribution can be replaced by a continuous function $\rho(x)$. The equation for the critical speed of the shaft is [cf. section 8, Eq. (8.15)]

$$EI\frac{d^4w}{dx^4} = \rho(x)\omega^2 w \tag{17.7}$$

If we assume an arbitrary deflection curve $w_1(x)$ and substitute $w_1(x)$ on the right side of the equation, putting $\omega^2 = 1$,

$$EI\frac{d^4w}{dx^4} = \rho(x)w_1 \tag{17.8}$$

this means that we load the shaft by centrifugal forces corresponding to the shape of deflection determined by $w_1(x)$ and to unit rotational speed. If $w_1(x)$ were the correct shape of deflection and ω the critical speed, the centrifugal force $\rho(x)w_1\omega^2$ would produce a deflection equal to w_1; hence, from Eq. (17.8) we would obtain

$$w_2(x) = \frac{w_1(x)}{\omega^2} \tag{17.9}$$

in accordance with Eq. (17.3).

It is seen that the method of iteration or successive approximations consists of correcting the deflection curve until the deflection produced by the centrifugal forces is exactly proportional to the assumed deflection.

Let us demonstrate this method on the example of the free oscillations of a circular membrane held under a uniform tension T. We shall restrict ourselves to oscillations of axial symmetry, *i.e.*, we assume that the deflection $w(x)$ is only a function of the distance x from the center. We may consider a sector of the membrane as a string of variable cross section. Take, for example, a sector whose central angle is $\Delta\alpha$. The vertical component of the tension is equal to $Tx\Delta\alpha dw/dx$, and, therefore, the equation for the equilibrium of the membrane under a vertical load p per unit area is given by

$$\frac{d}{dx}\left(xT\Delta\alpha\frac{dw}{dx}\right) = -px\Delta\alpha \tag{17.10}$$

or

$$T\frac{d}{dx}\left(x\frac{dw}{dx}\right) = -px \tag{17.11}$$

In the case of harmonic oscillations we must assume a load per unit area equal to $\rho w\omega^2$ where ρ is the mass of the membrane per unit area. Hence, we have

$$\frac{d}{dx}\left(x\frac{dw}{dx}\right) = -\left(\frac{\rho}{T}\omega^2\right)wx \tag{17.12}$$

If the membrane is supported by a circular frame of radius a, we have the boundary conditions $dw/dx = 0$ for $x = 0$ and $w = 0$ for $x = a$. The solution

$$w = \text{const. } J_0\left(\sqrt{\frac{\rho}{T}}\omega x\right) \tag{17.13}$$

satisfies Eq. (17.12) and the boundary condition for $x = 0$. The second particular solution of (17.12) is infinite for $x = 0$. The boundary condition for $x = a$ is satisfied if

$$\sqrt{\frac{\rho}{T}}\omega a = \alpha_i \tag{17}$$

where α_i is one of the roots of $J_0(\alpha) = 0$.

Let us determine α_1 by the iteration method. We choose

$$w_1 = 1 - \left(\frac{x}{a}\right)^2 \tag{17}$$

and write

$$\frac{d}{dx}\left(x\frac{dw_2}{dx}\right) = -k^2\left(1 - \frac{x^2}{a^2}\right)x$$

where $k^2 = \rho/T$.

Then we obtain

$$x\frac{dw_2}{dx} = -k^2\left(\frac{x^2}{2} - \frac{x^4}{4a^2}\right) + \text{const.} \tag{17.}$$

Since $dw/dx = 0$ for $x = 0$, the constant in (17.16) is zero and we have

$$w_2 = k^2a^2\left(C - \frac{x^2}{4a^2} + \frac{x^4}{16a^4}\right)$$

For $x = a$, we must have $w = 0$; therefore

$$w_2 = k^2a^2\left(\frac{3}{16} - \frac{1}{4}\frac{x^2}{a^2} + \frac{1}{16}\frac{x^4}{a^4}\right) \tag{17.}$$

We now calculate

$$\frac{w_1(x)}{w_2(x)} = \frac{1 - (x/a)^2}{\frac{3}{16}[1 - \frac{4}{3}(x/a)^2 + \frac{1}{3}(x/a)^4]k^2a^2}$$

e.g., for $x/a = 0$,

$$\lambda \cong \frac{w_1(0)}{w_2(0)} = \frac{16}{3}\frac{1}{k^2a^2} = 5.333\frac{1}{k^2a^2} \tag{17.1}$$

Hence, the first approximation for ω^2 is

$$\omega^2 = \frac{5.333}{k^2a^2} = 5.333\frac{T}{\rho a^2}$$

If we continue the above procedure, we obtain as successive approximations

$$\frac{\omega^2\rho a^2}{T} = 5.333,\ 5.526,\ 5.756$$

The exact value of $\alpha_1 = 2.405$; therefore, from Eq. (17.14)

$$\frac{\omega^2 \rho a^2}{T} = (2.405)^2 = 5.784$$

As was mentioned on p. 314, the iteration process leads to the fundamental frequency.

The iteration method sketched in this section is known in engineering practice as *Stodola's* or *Vianello's method*. The reader will notice that in principle it is analogous to the matrix method that we used in Chapter V for solving oscillation problems involving systems with a finite number of degrees of freedom.

Problems

1. A string of length l which extends between $x = 0$ and $x = l$ is held under a tension F and is loaded by a uniformly distributed load p between the points $x = l/3$ and $x = l/2$. Find the location and magnitude of the maximum deflection.

2. A semi-infinite uniform beam which extends between $x = 0$ and $x = \infty$ and rests on an elastic foundation carries a concentrated load at its end $x = 0$. Find the distribution of the bending moment.

3. Find the influence function $g(x,\xi)$ which gives the deflection at the point x for a load applied at ξ to a beam of uniform cross section clamped at $x = 0$ and hinged at $x = l$. Verify that $g(x,\xi) = g(\xi,x)$.

Hint: Use the general solutions of Eq. (4.1) for $p = 0$ with different coefficients for $x < \xi$ and $x > \xi$. Remember that g, $\frac{\partial g}{\partial x}$, and $\frac{\partial^2 g}{\partial x^2}$ are continuous at $x = \xi$.

4. Two equal loads a constant distance a apart travel along a beam of span l. Find the maximum bending moment at a given point $x = \xi$ along the beam. Show that the problem is mathematically the same as that of finding the maximum deflection at a certain point of a string of length l under tension when the same loads travel on it.

5. A cantilever beam of rectangular cross section is loaded by a concentrated force P at its free end. The beam has constant width; however, its height varies along the length in such a way that the maximum bending stress is constant (beam of uniform strength). Find the shape of the deflection curve.

Hint: The maximum stress is $\sigma = \frac{6M}{bh^2}$, where b is the width and h the height.

6. Find the distribution of the bending moment for the simply supported truss of a suspension bridge of span l. The dead load is q per unit length. A live load p per unit length is uniformly distributed over the half span $0 \leqq x \leqq l/2$. Use the method of the influence functions:

a. Calculate the moment and the deflection corresponding to a concentrated unit load applied at an arbitrary point ξ of a beam under axial tension.

b. Calculate the moment distribution for the load distributed between $x = 0$ and $x = l/2$ by the method of superposition.
c. Calculate the increase of the cable tension neglecting the expansion of the cable.

Solution: First determine the solution of the equation

$$EI\frac{d^4w_1}{dx^4} - (H + h)\frac{d^2w_1}{dx^2} = p(x)$$

for a beam under the axial tension $(H + h)$ and a distributed load $p(x)$. We find

$$w_1 = \int_0^l g(x,\xi)p(\xi)\,d\xi$$

where

$$g(x,\xi) = \begin{cases} \dfrac{(l-\xi)x}{EIl\mu^2} - \dfrac{\sinh\mu(l-\xi)\sinh\mu x}{EI\mu^3\sinh\mu l} & \text{for} \quad x \leqq \xi \\[2ex] \dfrac{\xi(l-x)}{EIl\mu^2} - \dfrac{\sinh\mu\xi\sinh\mu(l-x)}{EI\mu^3\sinh\mu l} & \text{for} \quad x \geqq \xi \end{cases}$$

with $\mu = \sqrt{\dfrac{H+h}{EI}}$ and $g(x,\xi) = g(\xi,x)$. Then we calculate the solution of

$$EI\frac{d^4w_2}{dx^4} - (H + h)\frac{d^2w_2}{dx^2} = 1$$

This solution is

$$w_2 = \frac{1}{H+h}\left[\frac{x(l-x)}{2} + \frac{1}{\mu^2}\left(\frac{\cosh\mu\left(x - \dfrac{l}{2}\right)}{\cosh\mu l/2} - 1\right)\right]$$

The equation for the deflection of the suspension bridge under a load $p(x)$ is $EI\dfrac{d^4w}{dx^4} - (H + h)\dfrac{d^2w}{dx^2} = p(x) - \dfrac{h}{H}q$, and, therefore, by superposition $w = w_1 - \dfrac{h}{H}qw_2$. The horizontal tension increment is still unknown. We calculate it by the condition $\int_0^l w\,dx = 0$, or $\int_0^l w_1\,dx = hq/H\int_0^l w_2\,dx$. This relation may be simplified as follows: We write

$$\int_0^l w_1\,dx = \int_0^l dx\int_0^l g(x,\xi)p(\xi)\,d\xi$$

Because $g(x,\xi) = g(\xi,x)$, we have

$$\int_0^l w_1\,dx = \int_0^l p(x)\,dx\int_0^l g(x,\xi)\,d\xi$$

Now $\int_0^l g(x,\xi)\,d\xi$ is the value w_1 for $p(x) = 1$. Hence, it is equal to w_2 and, therefore, $\int_0^l w_1\,dx = \int_0^l w_2 p\,dx$. Finally $\frac{h}{H} = \frac{1}{q}\frac{\int_0^l pw_2\,dx}{\int_0^l w_2\,dx}$. If we put $h = 0$ in the integrals, this is a first approximation for h, which will be sufficiently accurate for practical purposes.

In our particular problem p is uniformly distributed over the half span. The above formulas must be applied by carrying the integration over the half span. Because of the symmetry of the function w_2 we have rigorously $h/H = p/2q$. The deflection of the bridge is $w = p\int_0^{l/2} g(x,\xi)\,d\xi - \frac{p}{2}w_2(x)$ in which $\mu = \sqrt{\frac{H}{EI}\left(1 + \frac{p}{2q}\right)}$. The bending moment is $M = -EI d^2w/dx^2$.

7. Calculate the deflection of the truss and the decrease of the cable tension for a suspension bridge loaded by a dead load q uniformly distributed over the span under the assumption that the cable temperature is raised by 55°F. and the coefficient of the linear expansion of the material is 0.0000065/°F. Neglect the elasticity of the cable.

The characteristic constants of the bridge are $H/ql = 2.5$ and $Hl^2/EI = 16$.

8. Find the natural frequencies and the modes of vibration for a uniform beam hinged at $x = 0$ and elastically supported at the end $x = l$. The spring constant of the elastic support is equal to k. Discuss the limiting cases $k = 0$ and $k = \infty$. Solve the same problem for the case where the beam is clamped at $x = 0$.

Hint: The frequency ω appears in the dimensionless quantity

$$l\sqrt[4]{\rho\omega^2/EI} = \beta l$$

(cf. section 8 of this chapter). The equation for the frequencies will be found with βl as unknown and the quantity kl^3/EI as a dimensionless parameter. Solve the frequency equation graphically for various values of kl^3/EI.

9. A cantilever shaft of length l carries at its free end $x = l$ a disk of mass m and moment of inertia C. Find the equation for the critical speeds, taking into account the gyroscopic effect of the disk. If the slope of the shaft at $x = l$ is equal to θ_l, the gyroscopic moment represents a restoring moment of the magnitude $C\omega^2\theta_l$. Discuss the transition to the limiting case in which the mass of the beam is negligible.

10. Find the lowest critical speed of a uniform shaft running in two bearings that are restrained elastically so that an angular deflection θ of the shaft produces a restoring moment at the bearing equal to $k\theta$.

11. A column is tested for buckling in a testing machine between knife edges (Fig. P.11). The distance between the knife edges is l. The ends of the column are clamped in shoes of the length a, so that the center portion of the column of the length $l - 2a$ is free, whereas a section of the length a

at each end may be considered as perfectly rigid. Find the correction for Euler's formula (11.6).

Solution: Taking $x = 0$ at the center of the column, the deflection may be represented by $w \sim \cos \sqrt{\frac{P}{EI}}\, x$ and the boundary condition is

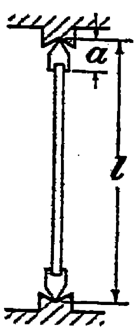

FIG. P. 11.

$$EI \frac{d^2w}{dx^2} = Pa \frac{dw}{dx}$$

at $x = \frac{l}{2} - a$. This yields for the critical load P the equation

$$\tan\left(\frac{\pi}{2} - k + k\alpha\right) = k\alpha$$

where $k = l/2\sqrt{P/EI}$ and $\alpha = 2a/l$. When $\alpha = 0$, the solution is $k = \pi/2$. We write, therefore, $k = \frac{\pi}{2} + \epsilon$. Hence,

$$\tan\left[-\epsilon + \left(\frac{\pi}{2} + \epsilon\right)\alpha\right] = \left(\frac{\pi}{2} + \epsilon\right)\alpha$$

Since ϵ and α are small, this equation is approximately

$$-\epsilon + \frac{1}{3}\left(\frac{\pi}{2}\alpha + \epsilon(\alpha - 1)\right)^3 = 0$$

In the first approximation, $\epsilon = \frac{1}{3}\left(\frac{\pi}{2}\alpha\right)^3$. The corrected critical load is

$$P = 4k^2\frac{EI}{l^2} = \frac{\pi^2 EI}{l^2}\left(1 + \frac{\pi^2}{6}\alpha^3\right)$$

12. A cantilever beam carries a mass at its free end. The mass is equal to half the total mass of the beam. Find the natural frequencies of the system.

13. A cylindrical steel vessel of length l, radius r, and wall thickness t is subjected to an internal pulsating pressure $p_0 + p \sin \omega t$. Find the frequency for which resonance occurs between the pressure pulsation and the fundamental vibration of the walls of the vessel. Assume that the cylindrical shell is clamped at both ends to circular rigid plates. Calculate the effect of the average pressure p_0 on the lowest natural frequency.

Method: A longitudinal strip of unit width of the cylindrical shell behaves like a beam under an axial tension $p_0\pi r^2/2\pi r = p_0 r/2$ and resting on an elastic support $k = Et/r^2$. The equation for the oscillations of this beam is $\frac{Et^3}{12}\frac{d^4w}{dx^4} - \frac{p_0 r}{2}\frac{d^2w}{dx^2} + \left(\frac{Et}{r^2} - \rho\omega^2\right)w = 0$ where ρ is the mass of the shell

per unit area. With the origin $x = 0$ at the center, the solution for the fundamental mode must be of the form $w = A \cos \mu x + B \cosh \lambda x$ with the boundary conditions $w = dw/dx = 0$ for $x = l/2$ and with real values for μ and λ. Putting $\alpha = \mu^2 l^2$ and $\beta = \lambda^2 l^2$, we find $\alpha = -\frac{a}{2} + \sqrt{\frac{a^2}{4} + b}$, $\beta = \frac{a}{2} + \sqrt{\frac{a^2}{4} + b}$ where $a = \frac{6p_0 r l^2}{Et^3}$ and $b = \frac{12\rho\omega^2 l^4}{Et^3} - \frac{12l^4}{r^2t^2}$. From the boundary conditions we find $-\sqrt{\alpha}\tan\sqrt{\alpha}/2 = \sqrt{\beta}\tanh\sqrt{\beta}/2$ (1). If a is considered as a parameter, this is an equation for b. The lowest root b yields the fundamental frequency.

To determine b as function of a, plot the left and right sides of (1). Points on the two curves that have equal ordinates and abscissas differing by a determine α and β as functions of a. Then $b = \alpha\beta$.

14. The temperature distribution along a cooling fin of unit width is governed by the differential equation

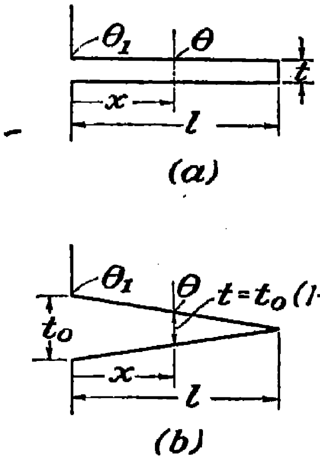

FIG. P.14.

$$\frac{d}{dx}\left(k\frac{d\theta}{dx}t\right) = \alpha(\theta - \theta_0)$$

where θ is the temperature of the fin at the cross section x, t the thickness of the fin, k the coefficient of the heat conduction of the material, α the coefficient of heat transfer between the fin and the surrounding air, and θ_0 the temperature of the air. Find the temperature distribution and the total heat transferred:

a. For a fin of length l and a constant thickness t (Fig. P.14*a*).

b. For a fin of triangular shape, whose thickness is given by

$$t = t_0\left(1 - \frac{x}{l}\right)$$

(Fig. P.14*b*), provided in both cases the temperature at the root $x = 0$ is equal to θ_1 and we assume $d\theta/dx = 0$ at $x = l$.

Hint: Take as unknown $\theta - \theta_0 = u$ and as independent variable $\xi = l - x$.

15. A cylindrical vessel of an inner radius $r = 12$ in. and outer radius 18 in. contains molten zinc of 900°F. temperature. Calculate the amount of heat transferred from the inner to the outer wall per unit area of the outer surface if the temperature of the outer wall is held at 300°F., and the coefficient of thermal conductivity of the material is equal to

$$k = 26 \text{ B.t.u./(°F.)(hr.) (ft.)}$$

16. A heat exchanger between two fluids is made of two coaxial pipes. A fluid of specific heat c_1 per unit volume flows with an average volume rate of flow v_1 through the inner pipe; another fluid of specific heat c_2 flows in

opposite direction with the volume rate of flow v_2 between the inner and the outer pipe. The total length of the heat exchanger is l. Find the temperature distribution of the two fluids along the exchanger, assuming a uniform temperature distribution in a cross section. The first fluid enters the pipe at the temperature θ_2' (for $x = 0$), and the temperature of the second fluid at the entrance section ($x = l$) is θ_1' where $\theta_2' < \theta_1'$. The external surface of the exchanger is perfectly insulated. The coefficient of heat transfer between the two fluids is α per unit length of the exchanger.

Hint: If we set up the equations of heat balance for a section of the exchanger of length dx, we find

$$\begin{aligned} v_1 c_1 \, d\theta_1 &= \alpha(\theta_2 - \theta_1) \, dx \\ v_2 c_2 \, d\theta_2 &= \alpha(\theta_2 - \theta_1) \, dx \end{aligned}$$

17. Find the fundamental frequency of a beam hinged at its two ends and loaded by an axial thrust P. Discuss the influence of the sign and magnitude of P on the frequency. Calculate the frequency for the case that P approaches the critical buckling load.

18. A homogeneous circular column of uniform diameter and length l supports a weight W. The column is rigidly clamped in the ground. Find the lowest natural frequency of the system. Neglect the mass of the column. Plot the frequency as a function of the weight.

Solution: Calculate the deflection w_0 at the top of the column ($x = l$) produced by a horizontal force F applied at that point by means of the equation $EI \dfrac{d^2w}{dx^2} - W(w_0 - w) = F(l - x)$ and the boundary conditions $w = dw/dx = 0$ at $x = 0$. We find $w_0 = F/k$ with $k = \dfrac{W}{l} \dfrac{1}{(1/\alpha) \tan \alpha - 1}$ and $\alpha = l\sqrt{W/EI}$. The frequency is given by $\omega = \sqrt{gk/W}$. Plot $\omega\sqrt{l/g}$ as function of α. Compare with Prob. 17.

19. Find the lowest natural frequency of the cantilever beam of uniform strength, described in Prob. 5, by the iteration method.

References

1. References 3 and 4, Chapter IV; reference 3, Chapter V.
2. SOUTHWELL, R. V.: "An Introduction to the Theory of Elasticity for Engineers and Physicists," Oxford University Press, New York, 1936.
3. TIMOSHENKO, S.: "Strength of Materials," D. Van Nostrand Company, Inc., New York, 1932.
4. TIMOSHENKO, S.: "Theory of Elastic Stability," McGraw-Hill Book Company, Inc., New York, 1936.

CHAPTER VIII

FOURIER SERIES APPLIED TO STRUCTURAL PROBLEMS

> Fourier's Theorem is not only one of the most beautiful results of modern analysis but it may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics.
>
> —Lord Kelvin and Peter Guthrie Tait, "Treatise on Natural Philosophy."

Introduction.—In this chapter the notion of the infinite trigonometric series is introduced and the methods for the expansion of arbitrary functions in such a series are given. The trigonometric series are used for finding particular solutions of nonhomogeneous differential equations with constant coefficients. In this way the Fourier series serve in structural problems for the computation of deflection and moment curves under arbitrary static and periodically alternating load distributions. Finally, the application of trigonometric series in the Rayleigh-Ritz method (energy method) is demonstrated by a few examples. Section 6 deals with Fourier integrals and presents the Fourier integral theorem.

1. Solution of the Differential Equation of a Beam with Elastic Support by Trigonometric Series.—Let us consider the problem of the uniform beam with elastic support treated in Chapter VII, section 4. We found that the deflection of the beam under a given external load $p(x)$ is given by the equation of that section

$$EI\frac{d^4w}{dx^4} + kw = p(x) \tag{1.1}$$

The general solution of the homogeneous equation was easily found and was used for the determination of the deflection caused by a concentrated load. If the load is constant or the load distribution function is a linear or a quadratic function of the coordinate x, we also easily find a particular solution of (1.1). For instance, if $p(x) = p_0$, where p_0 is a constant, $w = p_0/k$ is such a particular solution. If $p(x)$ is a polynomial in x, there will always

be a polynomial in x of the same degree which represents a particular solution. However, if $p(x)$ is a more complicated function or is given graphically, it is impossible to "guess" the particular solution needed. In this section we give a method, which can be used for an arbitrary function $p(x)$ and which leads to a good approximation in a simple way.

We notice that if w is a trigonometric function of x, *i.e.*, of the form $w = A \sin \lambda x$ or $w = B \cos \lambda x$, where A, B, and λ are constants, the fourth derivative will also be proportional to $\sin \lambda x$ or $\cos \lambda x$, respectively. Hence, if $p(x)$ is a trigonometric function, say $p = p_1 \sin \lambda x$, where p_1 is a constant, it is easy to find a particular solution of (1.1). In fact, by substituting $w = A_1 \sin \lambda x$ in Eq. (1.1), we find

$$A_1(EI\lambda^4 + k) \sin \lambda x = p_1 \sin \lambda x$$

and, therefore,

$$A_1 = \frac{p_1}{EI\lambda^4 + k} \tag{1.2}$$

Hence, if we approximate $p(x)$ by a series composed of trigonometric functions, we obtain a particular solution of (1.1) approximated likewise by a trigonometric series.

In order to simplify the calculations, we try to find directly the particular solution which satisfies the boundary conditions of the particular problem considered. This can be done by proper choice of the constant λ. Let us assume as an example that the beam is supported by hinges at the points $x = 0$ and $x = l$. Then the boundary conditions are

$$w = 0 \qquad \text{and} \qquad \frac{d^2w}{dx^2} = 0$$

for $x = 0$ and $x = l$. It is seen that every term of the form $A_n \sin \frac{nx\pi}{l}$ satisfies these boundary conditions.

We write $p(x) = \sum_{n=1}^{m} p_n \sin \frac{nx\pi}{l}$, *i.e.*, we approximate the load distribution function by a *trigonometric series*. Then substituting for w in Eq. (1.1) the expression $w = \sum_{n=1}^{m} A_n \sin \frac{nx\pi}{l}$ with

undetermined coefficients A_n, it is seen that the coefficients are determined by the relation

$$A_n\left(EI\,\frac{n^4\pi^4}{l^4} + k\right) = p_n \tag{1.3}$$

and the approximate solution of the differential equation is given by

$$w = \sum_{n=1}^{m} \frac{p_n}{EI(n^4\pi^4/l^4) + k} \sin\frac{nx\pi}{l} \tag{1.4}$$

The problem is, therefore, reduced to the determination of the coefficients p_n for the approximation of the load distribution function. For this purpose, we shall consider in the next section the development of an arbitrary function in an infinite trigonometric series, in a so-called *Fourier series*. The coefficients p_n are then given by the so-called *Fourier coefficients* of the function $p(x)$.

2. Fourier Series and Fourier Coefficients.—Let us consider a function $f(z)$ of a real variable in the interval $0 < z < 2\pi$. We assume that $f(z)$ is finite in this interval. It might be continuous or discontinuous at certain points; however, we assume that the number of points of discontinuity is finite. Then it is possible to determine an infinite series of the form

$$\sum_{n=1}^{\infty} a_n \sin nz + \sum_{n=0}^{\infty} b_n \cos nz$$

such that the sum of this series converges to the value of the function $f(z)$ at every point of the interval with the exception of the points of discontinuity and with the possible exception of the end points of the interval. The mathematical proof of this statement will be discussed later.

Hence, we write

$$f(z) = \sum_{n=1}^{\infty} a_n \sin nz + \sum_{n=0}^{\infty} b_n \cos nz \tag{2.1}$$

We call the series on the right side of (2.1) the *Fourier expansion* of the function $f(z)$ and call the coefficients a_n and b_n the *Fourier coefficients* of the function $f(z)$.

In order to determine the coefficients a_n and b_n, we use the relations

$$\int_0^{2\pi} \sin nz \sin mz\, dz = 0 \quad \text{if} \quad n \neq m \qquad (2.2)$$
$$= \pi \quad \text{if} \quad n = m$$

$$\int_0^{2\pi} \sin nz \cos mz\, dz = 0 \qquad (2.3)$$

and

$$\int_0^{2\pi} \cos nz \cos mz\, dz = 0 \quad \text{if} \quad n \neq m \qquad (2.4)$$
$$= \pi \quad \text{if} \quad n = m$$

These relations can be easily proved by means of the addition formulas for trigonometric functions. For instance, in order to prove (2.3), we substitute

$$\sin nz \cos mz = \tfrac{1}{2} \sin (n + m)z + \tfrac{1}{2} \sin (n - m)z$$

and obtain, provided $n \neq m$,

$$\int_0^{2\pi} \sin nz \cos mz\, dz = \frac{1}{2}\left[\frac{-\cos (n + m)z}{n + m}\right]_0^{2\pi} + \frac{1}{2}\left[\frac{-\cos (n - m)z}{n - m}\right]_0^{2\pi}$$

The expressions on the right side vanish because of the periodical character of the cosine function. If $n = m$, $\sin nz \cos nz = \frac{1}{2} \sin 2nz$, and the integral between the limits 0 and 2π vanishes.

Keeping the relations (2.2), (2.3), and (2.4) in mind, we multiply both sides of (2.1) by $\sin mz$ and integrate between the limits 0 and 2π. Then,

$$\int_0^{2\pi} f(z) \sin mz\, dz = a_m \int_0^{2\pi} \sin^2 mz\, dz = a_m\pi$$

because all other terms on the right side vanish. Hence,

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(z) \sin mz\, dz \qquad (2.5)$$

If we multiply both sides of (2.1) by $\cos mz$, assuming that $m \neq 0$ we obtain, as above,

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(z) \cos mz\, dz \qquad (2.6)$$

Finally, integrating both sides of (2.1), we easily obtain

$$\int_0^{2\pi} f(z)\,dz = b_0 \int_0^{2\pi} dz = 2\pi b_0$$

or

$$b_0 = \frac{1}{2\pi}\int_0^{2\pi} f(z)\,dz \tag{2.7}$$

Let us apply the results (2.5), (2.6), and (2.7) to certain simple cases:

a. A function $f(z)$ shall be given in the interval $0 < z < 2\pi$ by the following rule: $f(z)$ takes only two values, $+1$ and -1 (Fig. 2.1), so that $f(z) = 1$ for $0 < z < \pi/2$ and $3\pi/2 < z < 2\pi$; whereas $f(z) = -1$ for $\pi/2 < z < 3\pi/2$. The function $f(z)$ is, therefore, discontinuous at the points $z = \pi/2$ and $z = 3\pi/2$.

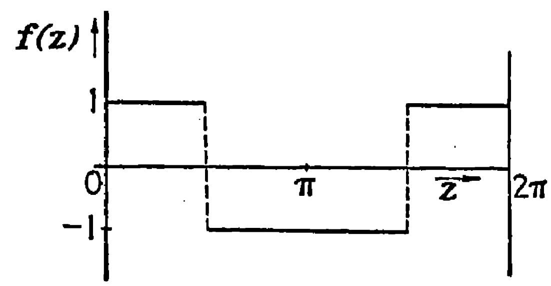

FIG. 2.1.—A discontinuous function having only two values, *viz.*, +1 and −1.

Computing the Fourier coefficients for this function, we find that $a_m = 0$ for an arbitrary value of m because $f(z) = f(2\pi - z)$ and $\sin mz = -\sin m(2\pi - z)$ and, therefore,

$$\int_0^{\pi} f(z) \sin mz\,dz = -\int_{\pi}^{2\pi} f(z) \sin mz\,dz$$

Thus, the integral on the right side of (2.5) vanishes.

In the calculation of the coefficients of the cosine terms, we must discriminate between odd and even values of m. For all values of m we have $\cos mz = \cos m(2\pi - z)$, hence,

$$\int_0^{\pi} f(z) \cos mz\,dz = \int_{\pi}^{2\pi} f(z) \cos mz\,dz$$

and, therefore, from (2.6)

$$b_m = 2/\pi \int_0^{\pi} f(z) \cos mz\,dz$$

Now for *even values of m*, we have $\cos mz = \cos m(\pi - z)$, and, therefore, since $f(z) = -f(\pi - z)$, the coefficient b_m is equal to zero. For *odd values of m*, $\cos mz = -\cos m(\pi - z)$ and $\int_0^{\pi/2} f(z) \cos mz\,dz = \int_{\pi/2}^{\pi} f(z) \cos mz\,dz$. Hence,

$$b_m = \frac{4}{\pi}\int_0^{\pi/2} f(z) \cos mz\,dz = \frac{4}{\pi}\int_0^{\pi/2} \cos mz\,dz$$

or

$$b_m = \frac{4}{\pi}\left[\frac{\sin mz}{m}\right]_0^{\frac{\pi}{2}} = \frac{4}{\pi m}\sin\frac{m\pi}{2}$$

Taking into account that for odd values of m, $\sin\frac{m\pi}{2} = (-1)^{\frac{m-1}{2}}$, we obtain

$$b_m = (-1)^{\frac{m-1}{2}}\frac{4}{\pi m} \tag{2.8}$$

where $m = 1, 3, 5, \cdots$. Thus the Fourier series for the function $f(z)$ is given by

$$f(z) = \frac{4}{\pi}\left(\cos z - \frac{1}{3}\cos 3z + \frac{1}{5}\cos 5z - \cdots\right) \tag{2.9}$$

or

$$f(z) = \frac{4}{\pi}\sum_{n=1,3}^{\infty}(-1)^{\frac{n-1}{2}}\frac{\cos nz}{n} \tag{2.10}$$

Substituting $z = 0$ in (2.9), it follows that

$$1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

or

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

For $z = \pi/2$ and $z = 3\pi/2$ the series on the right side of (2.10) has the value zero. This is equal to the mean value between $+1$ and -1.

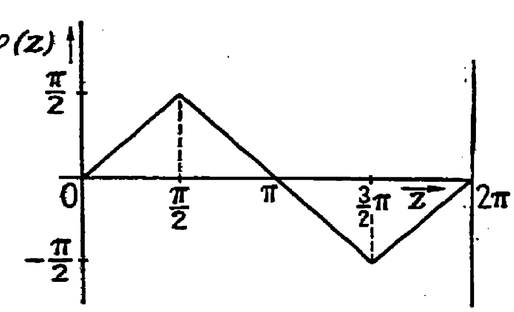

FIG. 2.2.—Example of a function, whose derivative has discontinuities.

b. Let us now assume a function $\phi(z)$ having the shape of the broken line shown in Fig. 2.2. We take, for example, $\phi(z) = z$ for $0 < z < \pi/2$, $\phi(z) = \pi - z$ for $\pi/2 < z < 3\pi/2$, and $\phi(z) = z - 2\pi$ for $3/2 < z < 2\pi$.

In this case $\phi(z) = -\phi(2\pi - z)$; hence, the coefficients b_m all vanish, and we are left with

$$a_m = \frac{1}{\pi}\int_0^{2\pi}\phi(z)\sin mz\,dz = \frac{2}{\pi}\int_0^{\pi}\phi(z)\sin mz\,dz$$

Now $\phi(z) = \phi(\pi - z)$ and $\sin mz = \sin m(\pi - z)$ for *odd values of* m, and $\sin mz = -\sin m(\pi - z)$ for *even values of* m. Therefore, $a_m = 0$ for even values of m and

$$a_m = \frac{4}{\pi}\int_0^{\pi/2} \phi(z) \sin mz \, dz = \frac{4}{\pi}\int_0^{\pi/2} z \sin mz \, dz$$

for odd values of m. Carrying out the integration, we find

$$a_m = \frac{4}{\pi m^2} \sin \frac{m\pi}{2} = 4\frac{(-1)^{\frac{m-1}{2}}}{\pi m^2} \tag{2.11}$$

The Fourier series for $\phi(z)$ is, therefore,

$$\phi(z) = \frac{4}{\pi}\left(\sin z - \frac{1}{3^2} \sin 3z + \frac{1}{5^2} \sin 5z - \cdots\right) \tag{2.12}$$

Substituting $z = \pi/2$ into (2.12), we have

$$\frac{\pi}{2} = \frac{4}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right)$$

or

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

Comparing the two functions $f(z)$ and $\phi(z)$, we easily find that $f(z)$ is the derivative of $\phi(z)$. We also find that the series (2.9) for $f(z)$ can be obtained by differentiating the series for $\phi(z)$ (2.12) term by term.

c. Finally, let us define a function $\psi(z)$ by the equation $\psi(z) = z(\pi - z)$ for $0 < z < \pi$ and by the relation $\psi(\pi + z) = -\psi(z)$ for $\pi < z < 2\pi$. The shape of the function is shown in Fig. 2.3. The derivative of $\psi(z)$ is represented graphically by the dotted line, which can be expressed by the function $\phi(z)$. We find that $\psi'(z) = 2\phi\left(\frac{\pi}{2} + z\right)$ and, therefore,

$$\psi(z) = 2\int_0^z \phi\left(\frac{\pi}{2} + z\right) dz$$

Replacing in equation (2.12) z by $\frac{\pi}{2} + z$, we obtain

$$\phi\left(\frac{\pi}{2} + z\right) = \frac{4}{\pi}\left(\cos z + \frac{1}{3^2} \cos 3z + \frac{1}{5^2} \cos 5z + \cdots\right)$$

and carrying out the integration term by term, we have

$$\psi(z) = \frac{8}{\pi}\left(\sin z + \frac{1}{3^3}\sin 3z + \frac{1}{5^3}\sin 5z + \cdots\right) \qquad (2$$

or

$$\psi(z) = \frac{8}{\pi}\sum_{n=1,3}^{\infty}\frac{\sin nz}{n^3} \qquad (2$$

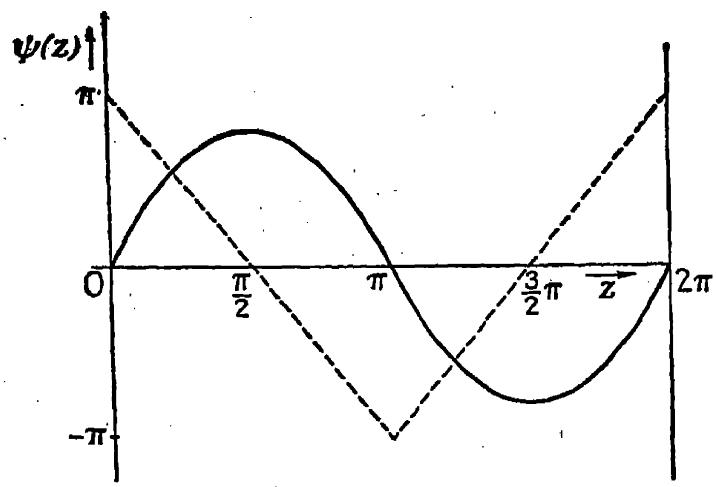

FIG. 2.3.—Example of a function composed of parabolic arcs; its second de-tive is discontinuous.

Substituting $z = \pi/2$ in (2.13), we find

$$\frac{\pi^2}{4} = \frac{8}{\pi}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right)$$

or

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots$$

It is interesting to note that the Fourier expansions for three functions $f(z)$, $\phi(z)$, and $\psi(z)$ furnish three diffe infinite series that can be used for the computation of number π.

3. Approximation of Arbitrary Functions by Fourier Serie Let us investigate the following problem. Given an arbit function of the real variable z with the only restriction that is finite and integrable in the interval $0 < z < 2\pi$. We cons a finite trigonometric series of the form:

$$S_n(z) = \sum_{1}^{n} a_k \sin kz + \sum_{0}^{n} b_k \cos kz \qquad ($$

with undetermined coefficients a_k and b_k. Our problem will consist of the determination of these coefficients in such a way that the integral

$$E = \frac{1}{2\pi}\int_0^{2\pi} [f(z) - S_n(z)]^2 \, dz \tag{3.2}$$

is a minimum. The expression E is the measure of the deviation between the function $f(z)$ and the function defined by the finite series $S_n(z)$. If we consider $f(z) - S_n(z)$ the *error* of the approximation, E gives the mean square of the error averaged over the interval between 0 and 2π.

The quantity E is a function of the parameters $a_1, \ldots,$ a_n and $b_0, \ldots, b_n$. Hence we obtain the best approximation if these coefficients satisfy the conditions

$$\frac{\partial E}{\partial a_1} = \frac{\partial E}{\partial a_2} = \cdots = \frac{\partial E}{\partial b_n} = 0$$

Differentiating the right side of Eq. (3.2) with respect to a certain coefficient a_m, we obtain

$$\frac{\partial E}{\partial a_m} = -\frac{1}{\pi}\int_0^{2\pi} [f(z) - S_n(z)] \frac{\partial S_n}{\partial a_m} \, dz = 0$$

or substituting $S_n(z)$ from Eq. (3.1)

$$-\frac{1}{\pi}\int_0^{2\pi} \left[f(z) - \sum_{k=1}^{n} a_k \sin kz - \sum_{k=0}^{n} b_k \cos kz \right] \sin mz \, dz = 0 \tag{3.3}$$

Carrying out the integrations, we find that because of the relations (2.2), (2.3), and (2.4), Eq. (3.3) is reduced to

$$-\frac{1}{\pi}\int_0^{2\pi} [f(z) - a_m \sin mz] \sin mz \, dz = 0$$

or

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(z) \sin mz \, dz$$

in accordance with Eq. (2.5).

Similarly, we obtain for the coefficients of the cosine terms the values given by Eqs. (2.6) and (2.7).

The meaning of this result is the following: If we cut off the infinite series obtained in the last section after the terms containing $\sin nz$ and $\cos nz$, the finite series obtained in this way is the best approximation possible for the function $f(z)$ by any trigonometric series with the same number of terms. Furthermore, if we increase the number of the terms in the series, the coefficients calculated for the previous approximation enter unchanged into the new approximation.

Because of this behavior of the successive approximations, we easily prove that the least value of the mean square of the error, given by $\frac{1}{2\pi}\int_0^{2\pi} [f(z) - S_n(z)]^2\, dz$, decreases with every new term added to the approximation. Let us call E_n the mean square of the error corresponding to the approximation S_n and E_{n+1} the error corresponding to S_{n+1}. Then, we have

$$E_{n+1} = \frac{1}{2\pi}\int_0^{2\pi} [f(z) - S_n(z) - a_{n+1}\sin (n+1)z - b_{n+1}\cos (n+1)z]^2\, dz$$

or

$$E_{n+1} = E_n - \frac{1}{\pi}\int_0^{2\pi} [f(z) - S_n(z)]\,[a_{n+1}\sin (n+1)z + b_{n+1}\cos (n+1)z]\, dz + \frac{1}{2\pi}\int_0^{2\pi} [a_{n+1}\sin (n+1)z + b_{n+1}\cos (n+1)z]^2\, dz$$

Carrying out the integration and taking into account the relations (2.2), (2.3), and (2.4), we obtain

$$E_{n+1} = E_n - \frac{a_{n+1}^2}{2} - \frac{b_{n+1}^2}{2} \tag{3.4}$$

Hence, $E_{n+1} = E_n$ if $a_{n+1} = b_{n+1} = 0$, and $E_{n+1} < E_n$ if a_{n+1} and b_{n+1} are different from zero.

Let us call $S_0 = b_0$ the *zero approximation* where b_0 is equal to the mean value of the function $f(z)$. Then the mean square of the error of the zero approximation will be:

$$E_0 = \frac{1}{2\pi}\int_0^{2\pi} [f(z) - b_0]^2\, dz = \frac{1}{2\pi}\int_0^{2\pi} f(z)^2\, dz - b_0^2$$

Therefore, if we use Eq. (3.4), the mean square of the error of the *first approximation* will be

$$E_1 = \frac{1}{2\pi}\int_0^{2\pi} f(z)^2\,dz - b_0^2 - \frac{a_1^2 + b_1^2}{2}$$

Continuing this procedure, we obtain for the *nth approximation*

$$E_n = \frac{1}{2\pi}\int_0^{2\pi} f(z)^2\,dz - b_0^2 - \frac{1}{2}\sum_{k=1}^{n}(a_k^2 + b_k^2) \tag{3.5}$$

We remember that E_n is an integral of a squared quantity and, therefore, is always positive. Hence, we obtain the following inequality for the sum of the squares of the coefficients:

$$b_0^2 + \tfrac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2) \leqq \int_0^{2\pi} f(z)^2\,dz \tag{3.6}$$

From relation (3.6) it follows that the sum on the left side cannot increase beyond an upper limit if $n \to \infty$. It can be shown that if the function $f(z)$ is continuous and $f(0) = f(2\pi)$, the left side of (3.6) converges to the value of the integral on the right side, *i.e.*, the mean error E_n converges to zero.

We calculated in the previous section the Fourier coefficients for a discontinuous function (the so-called *meander function*) and found that the series is convergent. Hence, these questions arise: what class of function can be developed in convergent Fourier series, and if a function is discontinuous at a certain point, what value is approximated by the Fourier series?

Fourier's first paper dealing with the series that carries his name was presented to the French Academy in 1807. His most famous contributions on the subject of the conduction of heat in solid bodies were written in 1811 and printed in 1824 to 1826. Poisson and Cauchy further developed the theory of Fourier series. The fundamental investigation concerning the conditions under which an arbitrary function can be expanded in a convergent series was carried out by Dirichlet in 1829 to 1837. Since those classic times an extended scientific literature has developed around the central subject of the properties and applications of trigonometric series and the properties of functions that can be expanded in such series.

It was shown by Dirichlet that if an arbitrary function is finite in a given interval and has a finite number of maxima and minima, the Fourier series associated with $f(z)$, *i.e.*, the trigonometric series whose coefficients are given by the Eqs. (2.5), (2.6), and (2.7), is convergent, and its sum is equal to the value $f(z)$ if $f(z)$ is continuous at the point z. If the function $f(z)$ is discontinuous at the point z and has two different values $f(z - 0)$ and $f(z + 0)$, the sum of the Fourier series converges to the mean value $\frac{1}{2}[f(z - 0) + f(z + 0)]$. We have observed this behavior of the Fourier series in the special examples given in section 2.

Dirichlet also investigated functions that are infinite for a finite number of values of x. He found that if the function satisfies the above condition, with the exception of an arbitrarily close neighborhood of the points of infinity, and if, in addition to that, the integral $\int|f(z)|\,dz$ taken over the whole interval is finite, the Fourier series is still convergent and gives the value of $f(z)$ at every point of the interval with the exception of the points of infinite discontinuity.

It is seen that these conditions cover a very wide range of functions, and, therefore, one will seldom encounter in engineering problems functions that cannot be expanded in a convergent Fourier series. The restriction that the function in question must have a finite number of maxima and minima in a finite interval is of little importance for practical applications since functions with an infinite number of maxima and minima will hardly occur in engineering problems. A well-known example of such a function is $\sin 1/z$, which has an infinite number of oscillations in an arbitrarily small interval including $z = 0$. If a function behaves at $z = 0$ as $1/z$, the associated Fourier series will not converge because the integral $\int|f(z)|\,dz$ is divergent. If at a point $z = a$, the function $f(z) \to \infty$ like $\log(z - a)$, the Fourier series might be convergent, but it probably will be of very little use since, in general, in the neighborhood of such irregular points a very large number of terms will be needed for a fair approximation.

If the function $f(z)$ is continuous in the interval considered and also $f(a) = f(b)$, the convergence of the Fourier series is *uniform*. This means that if we fix a certain amount E, as the *allowed error*, we are able to determine a certain number n, such that if we stop at the nth term of the Fourier series, at no point of the interval will the error exceed the fixed amount E. The coeffi-

cients of the Fourier series will converge to zero at least as fast as $1/n^2$ if $f(z)$ is continuous everywhere, and as fast as $1/n$ if $f(z)$ has discontinuous points.

There is a peculiarity of the Fourier series of discontinuous functions that is interesting enough to be mentioned. It was discovered in a special case by the famous American physicist, Josiah Willard Gibbs, in 1899 and is called *Gibbs's phenomenon;* its general theory is due to M. Bôcher. It has been mentioned that if the function $f(z)$ has a finite *jump* at a point z, so that $f(z - 0) \neq f(z + 0)$, the sum of the Fourier series converges to the mean value $\frac{1}{2}[f(z + 0) + f(z - 0)]$. However, if we plot the curve obtained as the nth approximation, *i.e.*, the sum of the first n terms of the series, and proceed to $n \to \infty$, the curve, instead of approximating the line consisting of the discontinuous curve $f(z)$ and the vertical line that connects the points $f(z + 0)$ and $f(z - 0)$, *overshoots* the jump at both ends by a length proportional to the jump. In Fig. 3.1 the jump occurs between B and C; the Fourier approximation converges to the line $ABB'C'CD$, where the portions BB' and CC' of the vertical line are equal to $\frac{1}{\pi}[f(z + 0) - f(z - 0)]\int_{\pi}^{\infty} \frac{\sin t}{t}\,dt$. Since the value of this definite integral is $\int_{\pi}^{\infty} \frac{\sin t}{t}\,dt = -0.2808$, the segments BB' and CC' amount to about 9 per cent of the jump. Owing to Gibbs's phenomenon, the approximation of discontinuous functions by Fourier series immediately near the discontinuity will not be very satisfactory.

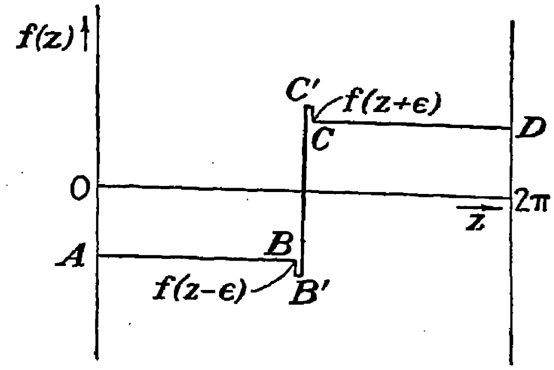

FIG. 3.1.—Gibbs's phenomenon.

4. Numerical, Graphical, and Mechanical Methods for the Determination of Fourier Coefficients. Harmonic Analysis.—The determination of the Fourier coefficients of a given function is often referred to as the *harmonic analysis* of the function. If we extend the function $f(z)$ beyond the interval between 0 and 2π, by the assumption $f(z) = f(z + 2k\pi)$, where k is an integer, we obtain a periodic function of z. Then the Fourier expansion approximates this function by the superposition of a trigonometric function with the fundamental period 2π and of an infinite number of *harmonics*. The coefficients determine the amplitudes

and phases of the harmonics. For this reason the expression *harmonic analysis* is used for the evaluation of the Fourier coefficients.

a. Numerical Methods.—Assume that the function $f(z)$ is given by graph or by tabulation. Then the simplest method for the computation of the first few Fourier coefficients is to choose a certain number of equidistant values of the independent variable z and to determine a finite trigonometric series in such a way that the mean square of the differences between the value of the function and the value of the trigonometric series at these points is a minimum.

Let us assume that we divide the interval between 0 and 2π in r equal parts, the values of the coordinate z of these points being $0, z_1, \ldots, z_{r-1}, z_r = 2\pi$. Let the corresponding values of the function be denoted by $f_0, f_1, \ldots, f_r$. (We assume that $f_0 = f_r$. If this is not the case, we deduce a linear function from $f(z)$ and calculate the coefficients for this linear function separately, using the method shown in section 2.) Then we consider as the mean square of the error the sum $1/r\sum_{k=1}^{r}(f_k - S_k)^2$ where S_k is the sum of the trigonometric series obtained by substituting $z = z_k$.* If we determine the condition that $1/r\sum_{k=1}^{r}(f_k - S_k)^2$ is a minimum, we obtain for the coefficients of the series

$$S = a_1 \sin z + a_2 \sin 2z + \cdots + a_n \sin nz + b_0 + b_1 \cos z + b_2 \cos 2z + \cdots + b_n \cos nz \tag{4.1}$$

the values

$$\begin{aligned} a_m &= \frac{2}{r}\sum_{k=1}^{r} f_k \sin mz_k \\ b_m &= \frac{2}{r}\sum_{k=1}^{r} f_k \cos mz_k \end{aligned} \tag{4.2}$$

* The sum $1/r\sum_{k=1}^{r}(f_k - S_k)^2$ may be considered as an approximation of the integral in Eq. (3.2). We take the sum since we use only discrete values of the function $f(z)$.

where $m = 1, 2, \ldots, n$ and

$$b_0 = \frac{1}{r}\sum_{k=1}^{r} f_k \tag{4.3}$$

The expressions (4.2) and (4.3) replace the expressions (2.5), (2.6), and (2.7).

The minimum problem is sensible only when the number of coefficients in S is not greater than the number r. If the number of the coefficients is equal to r, it is evident that we can determine them so that the approximation passes through the points taken, and, therefore, the error is zero. If the number of coefficients is larger than r, we can even make the error disappear in an infinite number of ways by suitable choice of the coefficients.

We shall assume that r is an even number. It is an advantage to choose r a multiple of 4, since in this case the number of different values of $\sin mz_k$ and $\cos mz_k$ is greatly reduced.

The following scheme is laid out for $r = 12$. In this case only four different values of the trigonometric functions occur in the sums to be calculated, *viz.*, $\sin 0° = 0$, $\sin 30° = \frac{1}{2}$, $\sin 60° = \sqrt{3}/2 = 0.866$, and $\sin 90° = 1$.

1. First Step. Write down the values of the given function $f(z)$ according to the following scheme and build sums and differences:

$$\begin{array}{lccccccc}
 & & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\
 & f_{12} & f_{11} & f_{10} & f_9 & f_8 & f_7 \\
\hline
\text{Sums} & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\
\text{Differences} & & d_1 & d_2 & d_3 & d_4 & d_5 &
\end{array}$$

2. Second Step. Write down the sums s and differences d according to the following scheme and build sums and differences again:

$$\begin{array}{lcccclccc}
 & s_0 & s_1 & s_2 & s_3 & \quad & d_1 & d_2 & d_3 \\
 & s_6 & s_5 & s_4 & & & d_5 & d_4 & \\
\hline
\text{Sums} & \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 & & \bar{\sigma}_1 & \bar{\sigma}_2 & \bar{\sigma}_3 \\
\text{Differences} & \delta_0 & \delta_1 & \delta_2 & & & \bar{\delta}_1 & \bar{\delta}_2 &
\end{array}$$

3. Third Step. The quantities σ_0, σ_1, σ_2, σ_3, δ_0, δ_1, δ_2 give the coefficients of the cosine terms, the quantities $\bar{\sigma}_1$, $\bar{\sigma}_2$, $\bar{\sigma}_3$, $\bar{\delta}_1$, $\bar{\delta}_2$ the coefficients of the sine terms, by multiplication by $\sin 30°$, $\sin 60°$, or $\sin 90°$ and addition according to the following computing form:

Multiply by	Sine terms					
0.500	$\bar{\sigma}_1$					
0.866		$\bar{\sigma}_2$	δ_1	δ_2		
1.000	$\bar{\sigma}_3$				$\bar{\sigma}_1$	$\bar{\sigma}_3$
Sum	I	II	I	II	I	II
I + II	$6a_1$		$6a_2$			
I − II	$6a_5$		$6a_4$		$6a_3$	

Multiply by	Cosine terms							
0.500			δ_2		$-\sigma_2$	σ_1		
0.866				δ_1				
1.000	$\sigma_0 + \sigma_2$	$\sigma_1 + \sigma_3$	δ_0		σ_0	$-\sigma_3$	δ_0	δ_2
Sum	I	II	I	II	I	II	I	II
I + II	$12b_0$		$6b_1$		$6b_2$			
I − II	$12b_6$		$6b_5$		$6b_4$		$6b_3$	

A number of similar schemes for different special cases are found in "Graphical and Mechanical Computation," Vol. II, by T. Lipka.

b. Graphical Methods.—Graphical methods are not very often used for harmonic analysis. Figure 4.1 shows the application

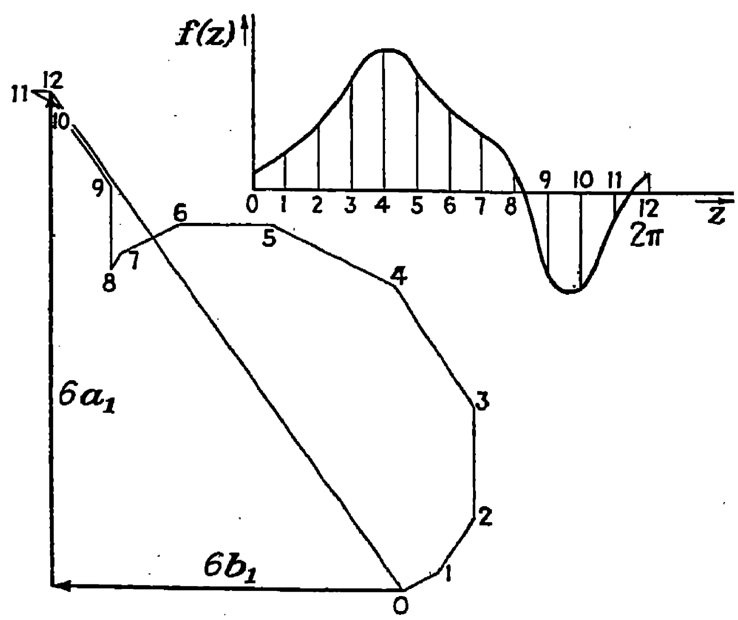

FIG. 4.1.—The Ashworth-Harrison method for harmonic analysis.

of the so-called *Ashworth-Harrison method.* This method is based on the vectorial addition of the values of the function to be analyzed. Assuming again a 12-ordinate scheme, we have

$$6b_m = f_1 \cos\frac{m\pi}{6} + f_2 \cos\frac{2m\pi}{6} + \cdots + f_{12}\cos 2m\pi$$

$$6a_m = f_1 \sin\frac{m\pi}{6} + f_2 \sin\frac{2m\pi}{6} + \cdots + f_{12}\sin 2m\pi$$

Hence, if we consider the quantities $f_1, f_2, f_3, \ldots, f_{12}$ as vectors in a plane, inclined to the horizontal axis at an angle of $m\pi/6$, $2m\pi/6$, etc., the quantities $6b_m$ and $6a_m$ will be equal to the components of their vectorial sum. In Fig. 4.1 the construction of $6a_1$ and $6b_1$ is shown. In order to obtain $6a_2$ and $6b_2$,

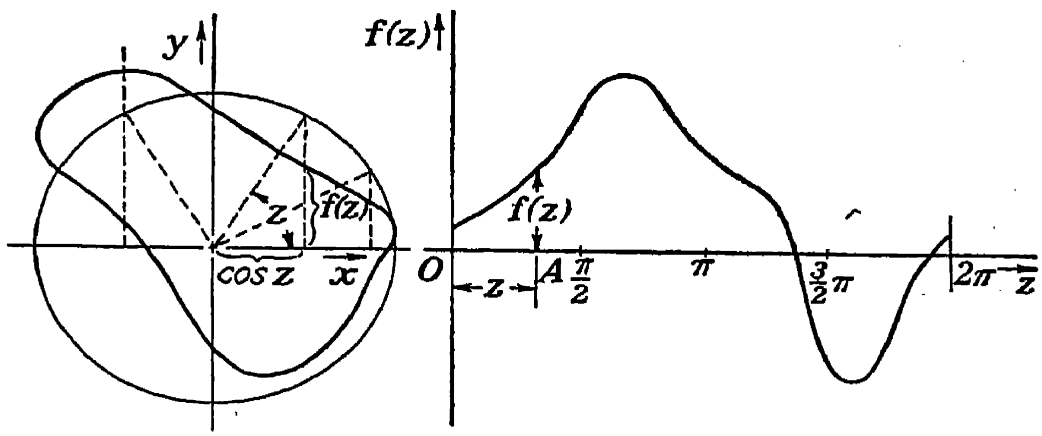

FIG. 4.2.—Mixed graphical-mechanical method for harmonic analysis.

the angle between two consecutive vectors will be equal to 60°, and so on.

c. Mechanical Methods.—For mechanical evaluation of the coefficients we may use either the common *planimeter* or the *integraph*, after doing some intermediary drawing work. Also, special mechanical *harmonic analyzers* are available.

For the computation of the first, or the first few, coefficients the following mixed graphical-mechanical method is very useful (Fig. 4.2). Let us consider the coefficient of the first sine term, which is given by

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(z)\sin z\,dz \qquad \text{or} \qquad a_1 = -\frac{1}{\pi}\int_0^{2\pi} f(z)\,d(\cos z)$$

If we draw a unit circle in an xy plane and put the angle $\tan^{-1} y/x$ equal to z, then evidently for the points of the x-axis between -1 and $+1$, $x = \cos z$ and $dx = d(\cos z)$. Consequently, if we draw a curve $y = \phi(x)$ whose ordinate y at the point x is equal to $f(z)$, the area enclosed by this curve gives us the value of the

integral $\int_{z=0}^{2\pi} f(z)\, d(\cos z)$. In order to carry out the computation, we choose a number of values $z_1, z_2, \ldots, z_r$ and mark on the x-axis the corresponding values of $\cos z$. We obtain in this way the points $x_1, x_2, \ldots, x_r$. Then, we choose the corresponding ordinates $y_1, y_2, \ldots, y_r$ equal to $f_1, f_2, \ldots, f_r$ and draw a smooth curve through the points obtained. The area enclosed by this curve can be determined by the common planimeter.

Obviously, b_0 can be determined by a quadrature of the given function. For b_1 and for the higher coefficients, the above method can be easily modified. However, for drawing the curves for coefficients of high order, we need a relatively large number of ordinates. The method is, therefore, practical only when the orders of the coefficients to be determined are small in comparison with the number of ordinates used.

A great number of mechanical harmonic analyzers have been designed, and quite a few types are available commercially.

5. The Problem of a Uniform Beam with Elastic Support (Continued).—Let us return to the problem treated in section 1 and to the solution of Eq. (1.1).

We must express the load distribution function $p(x)$ in the form:

$$p(x) = \sum_{n=1}^{\infty} p_n \sin \frac{nx\pi}{l} \tag{5.1}$$

This is easily done in the following way: We introduce as the new variable $z = x\pi/l$ and extend the function $p(x) = f(z)$ to the interval $0 < x < 2l$, which corresponds to $0 < z < 2\pi$. For $x > l$, we assume $p(2l - x) = -p(x)$, *i.e.*, $f(2\pi - z) = -f(z)$. Then developing $f(z)$ in a Fourier series, we find that the coefficients of the cosine terms all vanish, so that the sine terms give the development of $p(x)$ in Eq. (5.1), where

$$p_n = \frac{2}{l}\int_0^l p(x) \sin \frac{nx\pi}{l}\, dx \tag{5.2}$$

Substituting $w = \sum_{n=1}^{\infty} w_n \sin \frac{nx\pi}{l}$ in the equation

$$EI\frac{d^4w}{dx^4} + kw = \sum_{n=1}^{\infty} p_n \sin\frac{nx\pi}{l} \tag{5.3}$$

we obtain

$$w_n = \frac{p_n}{EI\dfrac{n^4\pi^4}{l^4} + k} \tag{5.4}$$

Let us now assume that we are solving the equation for the same beam without elastic support ($k = 0$) in an analogous way. If this solution is given by $\bar{w} = \sum_{n=1}^{\infty} \bar{w}_0 \sin\frac{nx\pi}{l}$, we obtain for $\bar{w}_n$ the formula

$$\bar{w}_n = \frac{p_n}{EI(n^4\pi^4/l^4)} \tag{5.5}$$

Let us now compute the difference between w and $\bar{w}$. Using (5.4) and (5.5), we find

$$w - \bar{w} = \sum_{n=1}^{\infty} p_n\left(\frac{1}{EI(n^4\pi^4/l^4) + k} - \frac{1}{EI(n^4\pi^4/l^4)}\right)\sin\frac{nx\pi}{l}$$

or

$$\begin{aligned} w - \bar{w} &= -\sum_{n=1}^{\infty} \frac{p_n}{EI(n^4\pi^4/l^4)}\,\frac{\sin(nx\pi/l)}{EI(n^4\pi^4/kl^4) + 1} \\ w - \bar{w} &= -\sum_{n=1}^{\infty} \frac{\bar{w}_n \sin(nx\pi/l)}{EI(n^4\pi^4/kl^4) + 1} \end{aligned} \tag{5.6}$$

Equation (5.6) allows an easy computation of the influence of the elastic support if the deflection of the beam without such support is known. If the dimensionless quantity $EI\frac{\pi^4}{kl^4}$ is not very small, the denominators of the terms on the right side increase very fast; in most cases it will be sufficiently accurate to retain one or two terms in Eq. (5.6). Then the problem is reduced to an ordinary deflection problem, which requires only

the evaluation of quadratures in the usual numerical or graphical way and the computation of one or two correction terms corresponding to (5.6).

The bending moment is equal to $M = -EI\, d^2w/dx^2$. If we denote the bending moment computed without elastic support by $\bar{M}$, it follows from Eq. (5.6) that

$$M = \bar{M} - \sum_{n=1}^{\infty} \frac{\bar{M}_n \sin (nx\pi/l)}{EI(n^4\pi^4/kl^4) + 1} \tag{5.7}$$

where $\bar{M}_n$ are the Fourier coefficients of the function $\bar{M}(x)$, *i.e.*,

$$\bar{M}(x) = \sum_{n=1}^{\infty} \bar{M}_n \sin \frac{nx\pi}{l} \tag{5.8}$$

Let us apply these results to the case of a beam loaded by a concentrated load P at its center ($x = l/2$). The moment function $\bar{M}(x)$ is in this case a *triangular function*, the maximum moment $\bar{M}(l/2)$ being equal to $Pl/4$. We have

$$\bar{M} = \frac{Px}{2} \qquad \text{for} \qquad 0 < x < \frac{l}{2}$$

and

$$\bar{M} = \frac{P(l - x)}{2} \qquad \text{for} \qquad \frac{l}{2} < x < l \tag{5.9}$$

Using $z = x\pi/l$, as independent variable and the function $\phi(z)$ as defined in section 2 of this chapter, the moment $\bar{M}$ can be written in the form:

$$\bar{M} = \frac{Pl}{2\pi}\phi(z) \tag{5.10}$$

Now according to Eq. (2.12),

$$\phi(z) = \frac{4}{\pi}\left(\sin z - \frac{1}{3^2} \sin 3z + \frac{1}{5^2} \sin 5z - \cdots \right)$$

consequently,

$$\bar{M}_1 = \frac{2Pl}{\pi^2}, \qquad \bar{M}_3 = -\frac{2Pl}{(3\pi)^2}, \qquad \bar{M}_5 = \frac{2Pl}{(5\pi)^2} \tag{5.11}$$

Substituting $\bar{M}$ from Eq. (5.9) and $\bar{M}_1$, $\bar{M}_3$, etc., from Eq. (5.11) in Eq. (5.7), we obtain

$$M = \frac{Px}{2} - \frac{2Pl}{\pi^2}\left[\frac{\sin (x\pi/l)}{1 + (EI\pi^4/kl^4)} - \frac{\frac{1}{9}\sin (3x\pi/l)}{1 + (81EI\pi^4/kl^4)} + \cdots\right]$$

for $x < l/2$. The same expression is valid for $x > l/2$, if we replace x by $l - x$.

In Chapter VII, section 4, we found that in the case of an infinite beam the distance of the first zero point of the deflection curve from the point of action of a concentrated load is equal to $3\pi/2\beta\sqrt{2}$. Denoting this length by l_0 and taking into account that $\beta^4 = k/EI$, we can write

$$\frac{EI\pi^4}{kl^4} = \left(\frac{\pi}{\beta l}\right)^4 = 4\left(\frac{2l_0}{3l}\right)^4$$

Let us assume, for example, that $l_0/l = \frac{3}{4}$. Then the maximum moment, which occurs at the point $x = l/2$, is equal to

$$M_{\max} = \frac{Pl}{4} - \frac{2Pl}{\pi^2}\left(\frac{1}{1 + \frac{1}{4}} + \frac{1}{1 + \frac{81}{4}} + \frac{1}{1 + \frac{625}{4}} + \cdots\right)$$

or

$$M_{\max} = \frac{Pl}{4}\left[1 - \frac{32}{\pi^2}\left(\frac{1}{5} + \frac{1}{85} + \frac{1}{629} + \cdots\right)\right]$$

It is seen that the three terms calculated from the Fourier series represent corrections to the value of $Pl/4$, amounting to 65, 3.8, and 0.5 per cent, respectively. It appears that for the accuracy usually required in such calculations, it is sufficient to retain the first two terms of the trigonometric series.

6. Beam of Infinite Span. Solution by Fourier Integral.— Let us now consider an elastically supported beam extending indefinitely from $x = -\infty$ to $x = +\infty$ with the only boundary condition that the deflection vanishes at infinity. We assume that the beam is subjected to distributed load $p(x)$; we also assume that at infinity $p = 0$ and that the total load $\int_{-\infty}^{\infty} p(x)\,dx$ is finite. The load distribution function might be continuous or discontinuous.

The question arises whether the method of the Fourier series applied to the beam of finite span can be extended to the case of an infinite span.

In order to investigate this question, let us assume an arbitrary function $f(x)$ defined between the limits $x = \pm l$ and try to find out what happens to the Fourier series when l is increased to infinity.

We introduce the variable $x\pi/l = z$ and identify the end points of the interval $x = \pm l$ with $z = \pm\pi$. The Fourier series for $f(x) = f(lz/\pi) = \bar{f}(z)$ will be

$$\bar{f}(z) = b_0 + \sum_{n=1}^{\infty} b_n \cos nz + \sum_{n=1}^{\infty} a_n \sin nz \tag{6.1}$$

where

$$\begin{aligned} b_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \bar{f}(z)\,dz = \frac{1}{2l}\int_{-l}^{l} f(\xi)\,d\xi \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \bar{f}(z)\cos nz\,dz = \frac{1}{l}\int_{-l}^{l} f(\xi)\cos\frac{n\xi}{l}\pi\,d\xi \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \bar{f}(z)\sin nz\,dz = \frac{1}{l}\int_{-l}^{l} f(\xi)\sin\frac{n\xi}{l}\pi\,d\xi \end{aligned} \tag{6.2}$$

The variable ξ is used in the definite integrals in (6.2) to avoid confusion with x.

Substituting the expressions (6.2) in Eq. (6.1), we obtain

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(\xi)\,d\xi + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} f(\xi)\left(\cos\frac{n\xi}{l}\pi\cos\frac{nx}{l}\pi + \sin\frac{n\xi}{l}\pi\sin\frac{nx}{l}\pi\right) d\xi$$

or

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(\xi)\,d\xi + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} f(\xi)\cos\frac{n\pi}{l}(x-\xi)\,d\xi \tag{6.3}$$

Let us denote $n\pi/l$ by λ_n and $\lambda_n - \lambda_{n-1} = \pi/l$ by $\Delta\lambda$. Then the second term in (6.3) takes the form

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\Delta\lambda\int_{-l}^{l} f(\xi)\cos\lambda(x-\xi)\,d\xi$$

If we approximate the sum by an integral and proceed to $l \to \infty$, *i.e.*, $\Delta\lambda \to 0$, we obtain in the limit

$$f(x) = \frac{1}{\pi}\int_0^\infty d\lambda \int_{-\infty}^\infty f(\xi)\cos\lambda(x - \xi)\,d\xi \tag{6.4}$$

We assumed in this deduction that $\int_{-\infty}^\infty f(\xi)\,d\xi$ is finite. Therefore, the first term on the right side of Eq. (6.3) vanishes in the limit.

The relation (6.4) is called the *Fourier integral theorem.** Written in the following form:

$$f(x) = \frac{1}{\pi}\int_0^\infty d\lambda\cos\lambda x\int_{-\infty}^\infty f(\xi)\cos\lambda\xi\,d\xi + \frac{1}{\pi}\int_0^\infty d\lambda\sin\lambda x\int_{-\infty}^\infty f(\xi)\sin\lambda\xi\,d\xi \tag{6.5}$$

it represents the generalization of the Fourier expansion for an infinite interval. The quantities

$$\frac{1}{\pi}\int_{-\infty}^\infty f(\xi)\cos\lambda\xi\,d\xi \qquad \text{and} \qquad \frac{1}{\pi}\int_{-\infty}^\infty f(\xi)\sin\lambda\xi\,d\xi$$

can be called again the Fourier coefficients $b(\lambda)$ and $a(\lambda)$. They are now, instead of sequences of discrete values, functions of a parameter λ. Then the expansion of the function $f(x)$ can be written in the form:

$$f(x) = \int_0^\infty d\lambda\,[b(\lambda)\cos\lambda x + a(\lambda)\sin\lambda x] \tag{6.6}$$

Let us apply the Fourier integral theorem to the problem of the elastically supported beam. The differential equation for the deflection w is

$$EI\frac{d^4w}{dx^4} + kw = p(x) \tag{6.7}$$

Let us write

$$p(x) = \frac{1}{\pi}\int_0^\infty d\lambda\cos\lambda x\int_{-\infty}^\infty p(\xi)\cos\lambda\xi\,d\xi + \frac{1}{\pi}\int_0^\infty d\lambda\sin\lambda x\int_{-\infty}^\infty p(\xi)\sin\lambda\xi\,d\xi \tag{6.8}$$

* For an exact proof of the Fourier integral theorem see Refs. 1 and 2 at the end of the chapter.

Furthermore, we use as an expansion for $w(x)$ a Fourier integral with undetermined functions $b(\lambda)$ and $a(\lambda)$

$$w(x) = \int_0^\infty d\lambda \cos \lambda x \, b(\lambda) + \int_0^\infty d\lambda \sin \lambda x \, a(\lambda) \tag{6.9}$$

Substituting (6.9) in the differential equation (6.7), we obtain the following relations:

$$\begin{aligned}(EI\lambda^4 + k)b(\lambda) &= \frac{1}{\pi}\int_{-\infty}^{\infty} p(\xi) \cos \lambda\xi \, d\xi \\ (EI\lambda^4 + k)a(\lambda) &= \frac{1}{\pi}\int_{-\infty}^{\infty} p(\xi) \sin \lambda\xi \, d\xi\end{aligned} \tag{6.10}$$

Hence, the solution of (6.7) is given by

$$w(x) = \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{EI\lambda^4 + k}\int_{-\infty}^{\infty} p(\xi) \cos \lambda(x - \xi) \, d\xi \tag{6.11}$$

Let us apply Eq. (6.11) to the simple case of a beam loaded by a uniform load p per unit length between $x = -a$ and $x = a$ with no load for $x > a$ and $x < -a$.
Then

$$\int_{-\infty}^{\infty} p(\xi) \sin \lambda\xi \, d\xi = 0$$

$$\int_{-\infty}^{\infty} p(\xi) \cos \lambda\xi \, d\xi = \frac{2p}{\lambda} \sin \lambda a$$

Substituting these expressions in Eq. (6.11), we obtain

$$w(x) = \frac{2p}{\pi}\int_0^\infty \frac{d\lambda \sin \lambda a}{\lambda(EI\lambda^4 + k)} \cos \lambda x \tag{6.12}$$

As the general solution of the differential equation for a beam on an elastic foundation consists of trigonometric and hyperbolic functions, it is obvious that the integral on the right side is equal to a combination of such functions. However, in many cases the integral representation of Eq. (6.12) will be easier to handle. For instance, the deflection at $x = 0$, *i.e.*, at the center of the load, is given by

$$w(0) = \frac{2p}{\pi k}\int_0^\infty \frac{d\lambda \sin \lambda a}{\lambda\,[(EI\lambda^4/k) + 1]} \tag{6.13}$$

This value can be computed easily by plotting the integrand and integrating by means of a planimeter. If $a \to 0$ and we must deal with a concentrated load $P = 2pa$, we put $\sin \lambda a \cong \lambda a$ and obtain

$$w(0) = \frac{P}{\pi k}\int_0^\infty \frac{d\lambda}{(EI\lambda^4/k) + 1} = \frac{P}{\pi k}\sqrt[4]{\frac{k}{EI}}\int^\infty \frac{d\zeta}{\zeta^4 + 1}$$

where $\zeta = \lambda\sqrt[4]{EI/k}$.

The definite integral $\displaystyle\int_0^\infty \frac{d\zeta}{\zeta^4 + 1} = \frac{\pi}{\sqrt{8}}$, so that we have

$$w(0) = \frac{1}{\sqrt{8}}\frac{P}{EI}\left(\frac{EI}{k}\right)^{3/4}$$

in accordance with Chapter VII, section 4 [cf. Eq. (4.7)].

7. Forced Vibration of a Beam under Harmonic Load.—Let us consider a uniform beam subjected to a load that is a harmonic function of the time. The motion of the beam in this case will consist of a free vibration and a forced vibration. The amplitude of the free vibration is determined by the initial conditions. However, if a small amount of damping exists, the free vibration practically disappears with the time and the forced vibration remains. We try to apply the method of the Fourier series to calculate the forced vibration.

The differential equation for the deflection ζ of a vibrating beam with uniform cross section, according to Eq. (6.3), Chapter VII, is given by

$$EI\frac{\partial^4\zeta}{\partial x^4} + \rho\frac{\partial^2\zeta}{\partial t^2} = 0$$

when there are no external forces. If the load per unit length is equal to $p(x)\sin \omega t$, the equation of motion will be

$$EI\frac{\partial^4\zeta}{\partial x^4} + \rho\frac{\partial^2\zeta}{\partial t^2} = p(x)\sin \omega t \tag{7.1}$$

It is seen that Eq. (7.1) admits a solution of the form

$$\zeta = w(x)\sin \omega t$$

Therefore, w must satisfy the equation

$$EI\frac{d^4w}{dx^4} - \rho\omega^2 w = p(x) \tag{7.2}$$

Our problem is to determine the function $w(x)$ in such a way tha the particular boundary conditions that depend on the type c support of the beam are satisfied.

a. Let us assume hinged ends at $x = 0$ and $x = l$. Then solu tions of the form $w = \text{const.} \sin \frac{nx\pi}{l}$ satisfy the boundary condi tions, and we can apply the method developed in the previou sections. We write

$$p(x) = \sum_{n=1}^{\infty} p_n \sin \frac{nx\pi}{l} \tag{7.3}$$

and

$$w(x) = \sum_{n=1}^{\infty} w_n \sin \frac{nx\pi}{l} \tag{7.4}$$

where the values of p_n are determined by harmonic analysis c the function $p(x)$ and the w_n are undetermined constants.

Substituting (7.3) and (7.4) in Eq. (7.2), we obtain

$$w_n = \frac{p_n}{(EIn^4\pi^4/l^4) - \rho\omega^2}$$

or

$$w_n = \frac{p_n l^4}{EIn^4\pi^4} \frac{1}{1 - (\rho\omega^2 l^4/EIn^4\pi^4)} \tag{7.5}$$

Now, according to Eq. (8.6), Chapter VII, the natural angula frequencies of the beam considered are given by the formula

$$\omega_n = \frac{n^2\pi^2}{l^2}\sqrt{\frac{EI}{\rho}} \tag{7.6}$$

Introducing ω_n into (7.5), we obtain

$$w_n = \frac{p_n l^4}{EIn^4\pi^4} \frac{1}{1 - (\omega^2/\omega_n^2)} \tag{7.7}$$

The physical meaning of this result can be explained in th following way:

Let us first solve the deflection problem of the beam for th static case, *i.e.*, under the stationary load $p(x) = \sum_{n=1}^{\infty} p_n \sin \frac{nx\pi}{l}$

The solution is given by

$$\bar{w} = \sum_{n=1}^{\infty} \bar{w}_n \sin \frac{nx\pi}{l} \tag{7.8}$$

where

$$\bar{w}_n = \frac{p_n l^4}{EIn^4\pi^4} \tag{7.9}$$

Then the deflection curve for the forced vibration can be obtained from (7.8) by multiplying the individual terms of the expansion by the so-called *resonance factors* $\frac{1}{1 - (\omega^2/\omega_n^2)}$.

It follows from (7.7) that if the frequency of the load coincides with one of the natural frequencies, the corresponding term in the series (7.9) becomes infinite. This holds, of course, only for infinitely small damping.

b. Let us now consider a cantilever beam whose end conditions are $w = dw/dx = 0$ for $x = 0$ and $d^2w/dx^2 = d^3w/dx^3 = 0$ for $x = l$.

It is seen that in this case the development of $p(x)$ in a Fourier series is not of great practical help because solutions of the form $w = \text{const.} \sin \frac{nx\pi}{l}$ do not satisfy the boundary conditions. Solutions that satisfy these conditions were found in section 8, Chapter VII. Equation (8.14) of that section gives the following modes of vibration:

$$w^{(n)}(x) = \text{const.}\left(\frac{\cosh \beta_n x - \cos \beta_n x}{\cosh \beta_n l + \cos \beta_n l} - \frac{\sinh \beta_n x - \sin \beta_n x}{\sinh \beta_n l + \sin \beta_n l}\right) \tag{7.10}$$

where the β_n's are roots of the characteristic equation

$$\cos \beta l \cosh \beta l = -1 \tag{7.11}$$

They are connected with the natural angular frequencies of the beam by the relations:

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho}} \tag{7.12}$$

Let us denote the function in parentheses on the right side of (7.10) by $\varphi_n(x)$. Then it is seen that if we choose a load distribu-

tion given by $p = p_n\varphi_n(x)$, a function of the form $w = w_n\varphi_n(x)$, where w_n is an undetermined constant, will satisfy both the differential equation (7.2) and the boundary conditions, since the function $\varphi_n(x)$ represents a mode of vibration, and therefore satisfies the equation

$$EI\,\frac{d^4w}{dx^4} - \rho\omega_n^2 w = 0 \tag{7.13}$$

where ω_n is the natural angular frequency of the beam corresponding to this particular mode. Consequently, we have

$$EI\,\frac{d^4\varphi_n}{dx^4} = \rho\omega_n^2\varphi_n \tag{7.14}$$

Substituting $w = w_n\varphi_n(x)$ in Eq. (7.2), we obtain

$$w_n\rho(\omega_n^2 - \omega^2)\varphi_n(x) = p_n\varphi_n(x)$$

and, therefore,

$$w_n = \frac{p_n}{\rho(\omega_n^2 - \omega^2)} \tag{7.15}$$

It is seen that this method of solution requires the expansion of the load distribution function $p(x)$ in a series of functions $\varphi_n(x)$. Each of them corresponds to a mode of vibration, and in this case they replace the simple trigonometric functions used for the Fourier expansion. If the load distribution is given in the form:

$$p(x) = \sum_{n=1}^{\infty} p_n\varphi_n(x) \tag{7.16}$$

then the solution for the deflection appears in a similar development of the form:

$$w(x) = \sum_{n=1}^{\infty} \frac{p_n}{\rho}\,\frac{\varphi_n(x)}{\omega_n^2 - \omega^2} \tag{7.17}$$

or

$$w(x) = \sum_{n=1}^{\infty} \frac{p_n}{\rho\omega_n^2}\varphi_n(x)\frac{1}{1 - (\omega^2/\omega_n^2)} \tag{7.18}$$

Equation (7.18) admits a physical interpretation similar to that given for Eq. (7.7) for the case of a beam supported at the ends. The equation for the static deflection is

$$EI\frac{d^4\bar{w}}{dx^4} = \sum_{n=1}^{\infty} p_n\varphi_n(x)$$

Substituting $\bar{w} = \sum_{n=1}^{\infty} \bar{w}_n\varphi_n(x)$ and making use of the relation (7.14), we have

$$\rho\omega_n^2\bar{w}_n = p_n$$

and, therefore,

$$\bar{w} = \sum_{n=1}^{\infty} \frac{p_n}{\rho\omega_n^2}\varphi_n(x) \tag{7.19}$$

Hence, we can write Eq. (7.18) in the form

$$w(x) = \sum_{n=1}^{\infty} \bar{w}_n(x)\frac{1}{1 - (\omega^2/\omega_n^2)} \tag{7.20}$$

In other words, we obtain the shape of the deflection curve for a forced vibration with the frequency ω by multiplying every term of the static deflection function given in Eq. (7.19) by the *resonance factor* $\dfrac{1}{1 - (\omega^2/\omega_n^2)}$.

This method of calculation of the forced vibrations is analogous to the method given in Chapter V for a system with a finite number of degrees of freedom. We remember that the calculation of the amplitude ratios became easy after we resolved the external forces into normal components, *i.e.*, into components that maintain normal modes of oscillations. This corresponds to the expansion of the load into a series of the form (7.16). The coefficients p_n of this expansion can be determined in a manner similar to that by which we determined the coefficients of a Fourier expansion. It can be shown that the functions $\varphi_n(x)$ satisfy relations similar to Eqs. (2.2), (2.3), and (2.4) for the trigonometric functions. For example, we have

$$\int_0^{2\pi} \varphi_n(x)\varphi_m(x)\,dx = 0$$

when $n \neq m$. These relations are analogous to the orthogonality relations which prevail between the coefficients of the normal modes of systems with a finite number of degrees of freedom [cf. Eq. (4.8), Chapter V]. Hence, such functions as $\varphi_n(x)$ which represent modes of oscillations of an elastic system, are known as *orthogonal* or *normal functions.*

8. Application of Trigonometric Series for the Determination of Characteristic Values. The Energy Method.—Up to this point the trigonometric series were used for the construction of particular solutions of differential equations with constant coefficients. However, they can be successfully applied to a much broader field of mechanical problems, *viz.*, for approximate solutions of equilibrium and of vibration and buckling problems which are not necessarily governed by equations with constant coefficients. The method is especially successful if the problem is formulated as a minimum problem or a problem of vanishing variation and is of great practical value for the determination of the characteristic values discussed in Chapter VII, section 16.

We have shown in Chapter III that the equations of equilibrium and of motion can often be formulated as statements concerning the vanishing of the variation of certain energy or work expressions. For example, the equilibrium of an elastic system is determined by the fact that in the equilibrium position the variation of the difference between external work and internal energy is zero. If we apply d'Alembert's principle to an oscillating elastic system, we can state that at every instant of the motion the variation of the elastic energy due to certain virtual displacements is equal to the work done by the inertia forces in the same virtual displacements. If we formulate the laws of equilibrium and of motion in this way, it is not necessary to consider the equilibrium of forces, and it is sufficient to vary the expressions for the energy and the work.

Now in the case of a continuous system, *e.g.*, a string or a beam, these expressions are integrals, and, therefore, we must vary the functions under the integral sign. Hence, the exact analysis of the ideas indicated above leads us to the *calculus of variations,* which does not fall within the scope of this volume. However, if we approximate the shape of the deflection curve or the mode of vibration of a beam by a finite trigonometric series, the expressions for energy and for work will be approximated by sums, and

will be functions of a finite number of coefficients, which we may consider as the coordinates of the system. Instead of the exact minimum of the potential energy we might then determine the smallest value that can be reached if the variation is restricted to that of a few parameters, *e.g.*, to the variation of the coefficients of a trigonometric expansion. Of course, the expansion must be chosen in such a way that every term complies with the boundary conditions, so that by variation of the coefficients the boundary conditions are not violated.

It is seen that by this method the variation of the energy of a continuum is reduced to the variation of the energy of a system determined by a finite number of parameters. The variation of an integral is replaced by the variation of a function of a finite number of variables. In the case of elastic systems with linear relation between force and displacement and, in general, in the case of small oscillations, the expressions to be varied are quadratic functions of the parameters; hence, the problem requires the solution of linear equations only. Therein lies the great merit of this method; all the rules and algorisms developed for systems of linear equations (*e.g.*, Chapter V) can be immediately applied.

The method will be clear when demonstrated by a few examples. It is sometimes called the *Rayleigh-Ritz* method, for it was extensively used by Lord Rayleigh in his theory of sound, and it was further developed and its convergence proved by Ritz. It is also referred to as the *energy* method. *It is not restricted at all to the use of trigonometric expansions;* series of any class of functions may be employed. However, the use of trigonometric functions often makes the calculations very simple, and, therefore, we explain the method in connection with trigonometric series.

Let us consider the problem of the critical speed of shafts. The critical speed is determined by the condition that the work of the centrifugal forces for an arbitrary virtual displacement, *i.e.*, for an arbitrary variation of the deflection curve satisfying the boundary conditions, is equal to the increase of the elastic energy, given by the work required for the bending of the shaft. The centrifugal force per unit length of the shaft is equal to $\rho(x)w(x)\omega^2$, where $\rho(x)$ is the mass per unit length, $w(x)$ is the deflection, and ω the speed of rotation. Hence, if we vary the deflection by δw, the work done by the centrifugal force is given

by $\rho w\omega^2\,\delta w$ and for the whole shaft by $\int_0^l \rho w\omega^2 \delta w\, dx$. It is seen that this work is equal to the variation of the expression

$$T = \tfrac{1}{2}\int_0^l \rho w^2\omega^2\, dx \tag{8.1}$$

T is equal to the kinetic energy of the shaft if we assume that the mass ρ is concentrated on the axis of the shaft and the axis whirls around the x-axis with an angular velocity ω (cf. Fig. 8.5, Chapter VII).

The elastic energy per unit length of the shaft is given by $\frac{1}{2}EI\,(d^2w/dx^2)^2$, where EI is the flexural rigidity of the shaft. Hence, the elastic energy of the whole shaft is equal to

$$U = \frac{1}{2}\int_0^l EI\left(\frac{d^2w}{dx^2}\right)^2 dx \tag{8.2}$$

The critical speed is given by the condition that the variation of the difference

$$L = T - U \tag{8.3}$$

vanishes for arbitrary variation of the deflection curve as long as the boundary conditions are complied with. The boundary conditions are the only geometrical restraints, and every displacement complying with them is a virtual displacement.

Let us assume now that the deflection can be approximated by a finite trigonometric series of the form

$$w(x) = A_1 \sin\frac{\pi x}{l} + A_2 \sin\frac{2\pi x}{l} + A_3 \sin\frac{3\pi x}{l} + \cdots + A_n \sin\frac{n\pi x}{l} \tag{8.4}$$

We notice that every term of the series satisfies the boundary conditions $w = d^2w/dx^2 = 0$ for $x = 0$ and $x = l$, where l is the distance between the two hinged supports. Therefore, we can vary $A_1, A_2, \ldots, A_n$ arbitrarily, and the critical speed is determined by

$$\frac{\partial L}{\partial A_1} = \frac{\partial L}{\partial A_2} = \frac{\partial L}{\partial A_3} = \cdots = \frac{\partial L}{\partial A_n} = 0 \tag{8.5}$$

Let us assume that the shaft has uniform cross section but carries heavy masses distributed uniformly over the section between $x = 0$ and $x = l/2$. Then the kinetic energy is given by

$$T = \frac{\omega^2 \rho_1}{2}\int_0^l w^2\,dx + \frac{\omega^2 \rho_2}{2}\int_0^{l/2} w^2\,dx \tag{8.6}$$

where ρ_1 is the mass per unit length of the shaft proper and ρ_2 is the mass per unit length due to the additional masses. Substituting expression (8.4), the first integral in (8.6) becomes

$$\int_0^l w^2\,dx = \frac{l}{2}(A_1^2 + A_2^2 + \cdots + A_n^2) \tag{8.7}$$

whereas the second integral is equal to

$$\begin{aligned}\int_0^{l/2} w^2\,dx &= \sum_{r=1}^{n}\sum_{s=1}^{n} A_r A_s \int_0^{l/2} \sin\frac{r\pi x}{l}\sin\frac{s\pi x}{l}\,dx \\ &= \sum_{r=1}^{n}\sum_{s=1}^{n}\frac{A_r A_s}{2}\left[\frac{\sin\,(r-s)\frac{\pi}{2}}{r-s} - \frac{\sin\,(r+s)\frac{\pi}{2}}{r+s}\right]\frac{l}{\pi}\end{aligned} \tag{8.8}$$

If $r = s$, the coefficient of A_r^2 is equal to $l/4$. The terms for which $r - s$ is an even number drop out. Thus, the expression for T becomes

$$\begin{aligned}T = \frac{\omega^2 l}{4}\Bigg[\left(\rho_1 + \frac{\rho_2}{2}\right)&(A_1^2 + A_2^2 + \cdots + A_n^2) \\ + \frac{2\rho_2}{\pi}\bigg(\frac{4}{3}A_1A_2 - \frac{8}{15}A_1A_4 &+ \frac{12}{35}A_1A_6 - \cdots + \frac{2}{15}A_2A_3 \\ &- \frac{4}{21}A_2A_5 + \cdots\bigg)\Bigg]\end{aligned} \tag{8.9}$$

The expression for the elastic energy of the bending is given by

$$U = \frac{EI}{4}\frac{\pi^4}{l^3}(A_1^2 + 16A_2^2 + 81A_3^2 + \cdots) \tag{8.10}$$

Let us first cut off the series (8.4) after the first term. Then we have

$$\frac{\partial T}{\partial A_1} = \frac{\omega^2 l}{2}\left(\rho_1 + \frac{\rho_2}{2}\right)A_1, \qquad \frac{\partial U}{\partial A_1} = \frac{EI}{2}\frac{\pi^4}{l^3}A_1 \tag{8.11}$$

and the condition

$$\frac{\partial T}{\partial A_1} - \frac{\partial U}{\partial A_1} = \left[\frac{\omega^2 l}{2}\left(\rho_1 + \frac{\rho_2}{2}\right) - \frac{EI}{2}\frac{\pi^4}{l^3}\right] A_1 = 0$$

gives

$$\omega^2 = \frac{EI}{\left(\rho_1 + \frac{\rho_2}{2}\right)l^4}\pi^4 \tag{8$$

We can obtain this first approximation also by putting $T =$ Hence, to obtain a first estimate for the critical speed we proceed in the following way: We substitute a function $w(x$ plausible shape into the expressions for the kinetic and elastic energies and determine the speed ω for which the energies are equal.

Taking two terms of the series (8.4), we have

$$T = \frac{\omega^2 l}{4}\left(\rho_1 + \frac{\rho_2}{2}\right)\left(A_1^2 + \frac{16\rho_2}{3\pi(2\rho_1 + \rho_2)}A_1A_2 + A_2^2\right) \tag{8.$$

and

$$U = \frac{EI}{4}\frac{\pi^4}{l^3}(A_1^2 + 16A_2^2) \tag{8.$$

Using the notation for the dimensionless quantity,

$$\lambda = \frac{EI}{\omega^2}\frac{\pi^4}{l^4}\frac{1}{\rho_1 + (\rho_2/2)} \tag{8.$$

we obtain by differentiation of $U - T$ the following two lin equations for A_1 and A_2:

$$\begin{aligned} (1 - \lambda)A_1 + \frac{8}{3\pi}\frac{\rho_2}{2\rho_1 + \rho_2}A_2 &= 0 \\ \frac{8}{3\pi}\frac{\rho_2}{2\rho_1 + \rho_2}A_1 + (1 - 16\lambda)A_2 &= 0 \end{aligned} \tag{8.$$

The characteristic equation becomes

$$(\lambda - 1)(16\lambda - 1) - \frac{64}{9\pi^2}\left(\frac{\rho_2}{2\rho_1 + \rho_2}\right)^2 = 0 \tag{8.$$

For $\rho_2 = 0$ we have $\lambda = 1$ and $\lambda = \frac{1}{16}$. The larger value o corresponds to the lowest critical speed. We can put $\lambda = 1 -$

and expand Eq. (8.17) in powers of η. We obtain, by neglecting terms of the order η^2,

$$15\eta \cong \frac{64}{9\pi^2}\left(\frac{\rho_2}{2\rho_1 + \rho_2}\right)^2$$

Hence, we have

$$\lambda \cong 1 + \frac{64}{135\pi^2}\left(\frac{\rho_2}{2\rho_1 + \rho_2}\right)^2 \tag{8.18}$$

or

$$\omega^2 \cong \frac{EI}{l^4} \frac{\pi^4}{\rho_1 + \frac{\rho_2}{2}} \frac{1}{1 + \frac{64}{135\pi^2}\left(\frac{\rho_2}{2\rho_1 + \rho_2}\right)^2} \tag{8.19}$$

It is seen that the approximation

$$\omega^2 = \frac{EI}{l^4} \frac{\pi^4}{\rho_1 + \frac{\rho_2}{2}} \tag{8.20}$$

is very satisfactory for small values of ρ_2/ρ_1. If $\rho_2 \gg \rho_1$, we obtain

$$\omega^2 = 1.91 \frac{EI}{l^4} \frac{\pi^4}{\rho_2} \tag{8.21}$$

The total mass carried by the shaft is in this case $m = \rho_2 l/2$. Introducing m into Eq. (8.21), we have

$$\omega^2 = 0.995 \frac{EI}{ml^3}\pi^4 \tag{8.22}$$

We remember (Chapter VII, section 9) that the critical speed of a shaft carrying a mass m concentrated at its center is equal to [cf. Eq. (9.15), Chapter VII]

$$\omega^2 = \frac{48EI}{ml^3} \tag{8.23}$$

Hence, if the mass is distributed over one-half of the length, the critical speed is increased by the factor 1.39.

To find good approximations for the higher critical speeds, it is necessary to take into account more and more terms of the expansion (8.4). The method applies without essential change to arbitrary oscillations of beams or to more complex elastic structures. It is important to notice that the influence of con-

centrated masses can be taken into account by adding their kinetic energy to the kinetic energy of the elastic system. The total energy appears also in this case as a quadratic form in the coefficients $A_1, A_2, \ldots, A_n$.

To find the critical loads in buckling problems (cf. Chapter VII, section 11) we must vary the difference $U - W$, where U is the elastic energy of the system (*i.e.*, the work required for the bending of the beam) and W is the work done by the external forces. The expansion for W must include the quadratic terms in $w(x)$ and its derivatives. For example, if a beam is loaded by an axial thrust P, the work done by P is equal to $P\delta$, where δ is the relative axial deflection of the two ends between which the thrust is applied. The deflection δ corresponds to the bending of the beam and is given by the difference of the integrals $\int_0^l \sqrt{1 + (dw/dx)^2}\, dx$ and $\int_0^l dx$. Hence,

$$\delta = \int_0^l \left[\sqrt{1 + \left(\frac{dw}{dx}\right)^2} - 1\right] dx = \frac{1}{2}\int_0^l \left(\frac{dw}{dx}\right)^2 dx + \text{higher terms}$$

Therefore, to find the buckling load, we must vary the expression

$$U - W = \frac{1}{2}\int_0^l EI \left(\frac{d^2w}{dx^2}\right)^2 dx - {}_2\int_0^l \left(\frac{dw}{dx}\right)^2 dx \qquad (8.24)$$

It is seen that, for example, an arbitrary distribution of the flexural rigidity over the length does not cause extraordinary difficulties for the Rayleigh-Ritz method, whereas the exact integration of the corresponding differential equation

$$\frac{d^2w}{dx^2} + \frac{P}{EI}w = 0 \qquad (8.25)$$

is feasible only if P/EI is a simple function of x. If we use as a first approximation for $w(x)$ the term

$$w(x) = A_1 \sin\frac{\pi x}{l} \qquad (8.26)$$

and substitute this expression in (8.24), we obtain

$$U - W = \frac{A_1^2}{2}\frac{\pi^4}{l^4}\int_0^l EI \sin^2\frac{\pi x}{l}\, dx - \frac{PA_1^2 l}{4}\frac{\pi^2}{l^2} \qquad (8.27)$$

The condition $\partial(U - W)/\partial A_1 = 0$ in this case is identical with $U - W = 0$ and yields the following expression for the critical value of P:

$$P_{cr} = \frac{2\pi^2}{l^3}\int_0^l EI \sin^2 \frac{\pi x}{l}\, dx \tag{8.28}$$

This result can be interpreted physically in the following way. If we substitute a function $w(x)$ of plausible shape into the expressions for the elastic energy, *i.e.*, the work required for the bending and the work done by the axial thrust, the value of the axial thrust that makes these two items equal gives a first estimate of the smallest buckling load.

The Rayleigh-Ritz method gives also an approximation for the modes of vibration and the modes of buckling. However, a fair approximation of these characteristic functions requires, in general, more terms than the approximation of the characteristic values.

9. The Rayleigh-Ritz Method Applied to the Equilibrium of a Loaded Membrane.—One of the merits of the Rayleigh-Ritz method is the easy way in which it can be applied to problems involving more than one independent variable without the need of solving partial differential equations.

Let us calculate, as an example, the equilibrium surface of a membrane or soap film held under tension over a rectangular frame and subjected to a uniform normal pressure p. It is known that this problem plays an important role in the theory of elasticity, because of the so-called *soap-film analogy* for the torsion of beams. The sides of the rectangular frame are denoted by a and b, the uniform tension of the membrane by T. The potential energy of the membrane or soap film is given by the product of the tension and the surface area. If the deflection is denoted by w, the surface area is given by the double integral

$$\int_0^a\int_0^b \sqrt{1 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2}\, dx\, dy$$

or with sufficient accuracy by

$$\int_0^a\int_0^b \left[1 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] dx\, dy$$

In the equilibrium position the variation of the potential energy is equal to the virtual work of the external pressure. Therefore, we have

$$\delta\left\{\frac{T}{2}\int_0^a\int_0^b\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]dx\,dy\right\}-\int_0^a\int_0^b p\,dx\,dy\,\delta w=0$$

or (9.1)

$$\delta\left\{\frac{T}{2}\int_0^a\int_0^b\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]dx\,dy-p\int_0^a\int_0^b w\,dx\,dy\right\}=0$$

Let us use the notations $I_1=\int_0^a\int_0^b\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]dx\,dy$ and $I_2=\int_0^a\int_0^b w\,dx\,dy$ and substitute for the deflection

$$w=\sum_{k=1,3}^{2n-1}\sum_{l=1,3}^{2n-1}c_{kl}\sin\frac{k\pi x}{a}\sin\frac{l\pi y}{b} \tag{9.2}$$

By this choice of the expansion the boundary conditions $w=0$ for $x=0$, $x=a$, $y=0$, and $y=b$ are satisfied for each term. Because of symmetry k and l can have only odd values. Substituting w from (9.2) into I_1 and I_2 and carrying out the integrations, we obtain

$$I_1=\sum_{k=1,3}^{2n-1}\sum_{l=1,3}^{2n-1}c_{kl}^2\left(\frac{k^2\pi^2}{a^2}+\frac{l^2\pi^2}{b^2}\right)\frac{ab}{4} \tag{9.3}$$

$$I_2=\sum_{k=1,3}^{2n-1}\sum_{l=1,3}^{2n-1}c_{kl}\frac{4}{kl}\frac{ab}{\pi^2} \tag{9.4}$$

The simplicity of the result is due to the relations (2.2) to (2.4) between the trigonometric functions.

According to (9.1),

$$\frac{T}{2}\,\delta I_1-p\,\delta I_2=0$$

Hence, differentiating $F=\frac{T}{2}I_1-pI_2$ with respect to c_{kl}, we obtain

$$\frac{\partial F}{\partial c_{kl}} = \left[T\frac{c_{kl}}{4}\left(\frac{k^2\pi^2}{a^2} + \frac{l^2\pi^2}{b^2}\right) - p\frac{4}{kl}\frac{1}{\pi^2}\right]ab = 0$$

or

$$c_{kl} = \frac{16p}{T}\frac{a^2b^2}{\pi^4(k^2b^2 + l^2a^2)kl} \tag{9.5}$$

Let us consider two examples: the case of a square frame $(a = b)$ and that of a rectangular frame of very large aspect ratio $(a/b \to \infty)$.

a. If $a = b$,

$$c_{kl} = \frac{16p}{T}\frac{a^2}{\pi^4}\frac{1}{kl(k^2 + l^2)}$$

Hence, the solution is given by

$$w(x,y) = \frac{8pa^2}{T\pi^4}\left(\sin\frac{\pi x}{a}\sin\frac{\pi y}{a} + \frac{1}{15}\sin\frac{3\pi x}{a}\sin\frac{\pi y}{a} + \frac{1}{15}\sin\frac{\pi x}{a}\sin\frac{3\pi y}{a} + \frac{1}{81}\sin\frac{3\pi x}{a}\sin\frac{3\pi y}{a} + \cdots\right)$$

For example, the deflection at the center $(x = y = a/2)$ is equal to

$$w\left(\frac{a}{2},\frac{a}{2}\right) = \frac{8pa^2}{T\pi^4}\left(1 - \frac{2}{15} + \frac{1}{81} - \cdots\right) \cong 0.071\frac{pa^2}{T}$$

The deflection at the center of a circular membrane of the diameter a is equal to $0.0625\,\frac{pa^2}{T}$.

b. If $a \gg b$,

$$c_{kl} = \frac{16p}{T}\frac{b^2}{\pi^4 kl^3}$$

and the double series for $w(x,y)$ can be written in the form of the product of two single series

$$w(x,y) = \frac{16pb^2}{T\pi^4}\left(\sum_{k=1,3}^{2n-1}\frac{\sin(k\pi x/a)}{k}\right)\left(\sum_{l=1,3}^{2n-1}\frac{\sin(l\pi y/b)}{l^3}\right) \tag{9.6}$$

The reader will verify without difficulty, using Eqs. (2.10) and (2.14), that for $n \to \infty$, except for $x = 0$ and $x = a$,

$$\sum_{k=1,3}^{\infty} \frac{\sin (k\pi x/a)}{k} = \frac{\pi}{4}$$

and that $\sum_{l=1,3}^{\infty} \frac{1}{l^3} \sin \frac{l\pi y}{b} = \frac{\pi^3}{8} \frac{y(b-y)}{b^2}$. Therefore,

$$w(x,y) = \frac{p}{2T} y(b-y) \tag{9.7}$$

Equation (9.7) can be obtained directly by assuming that w is a function of y only. Then every strip of the membrane parallel to the y-axis acts as a string of length b. We have, therefore, the differential equation

$$T \frac{d^2w}{dy^2} = -p \tag{9.8}$$

and (9.7) is the solution of (9.8) which satisfies the boundary conditions that $w = 0$ for $y = 0$ and $y = b$.

Problems

1. Determine the coefficients of the infinite trigonometric series,

$$a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{2\pi x}{l} + \cdots$$

such that the sum of the series is equal to unity for $0 < x < l$.

2. A simply supported uniform beam carries its own weight and is subjected to an axial compressive thrust. Find the maximum bending moment by using the method of expansion in Fourier series.

Hint: Use Eq. (14.2) of Chapter VII.

3. A suspension bridge of span l with a simply supported truss carries a live load p per unit length uniformly distributed over the half of the span between $x = 0$ and $x = l/2$. Find the Fourier expansion for the deflection. Assume the cable to be inextensible.

Solution: The increment of cable tension is found as in Chapter VII, Prob. 6. We find $h/H = p/2q$. The expansion of the live load in a Fourier series is $p(x) = \frac{2p}{\pi}\left[\left(\sin \frac{\pi x}{l} + \sin \frac{2\pi x}{l}\right) + \frac{1}{3}\left(\sin \frac{3\pi x}{l} + \sin \frac{6\pi x}{l}\right) + \cdots\right]$ while the expansion of the constant

$$-\frac{hq}{H} = -\frac{p}{2} \text{ is } -\frac{2p}{\pi}\left(\sin \frac{\pi x}{l} + \frac{1}{3} \sin \frac{3\pi x}{l} + \cdots\right).$$

A good approximation for the deflection will be

$$w = \frac{l}{2\pi^3}\frac{pl}{H}\frac{\sin(2\pi x/l)}{1 + (p/2q) + 4\pi^2(EI/Hl^2)}$$

4. A beam of infinite span and flexural stiffness EI rests on a continuous elastic foundation whose modulus is equal to k. A uniform load p is applied between $x = -a$ and $x = a$. Calculate the maximum bending moment by means of a Fourier integral. Plot the maximum bending moment as a function of the dimensionless parameter

$$a\sqrt[4]{\frac{k}{EI}} = \alpha$$

5. Solve the problem treated in Chapter VII, section 14, by expansion of the deflection in Fourier series. Calculate the bending moment.

Solution: The bending moment due to the uniform load p and the moment M_0 acting at the base of the cantilever is $\frac{px}{2}(l - x) - M_0\frac{x}{l}$.

The equation for the deflection when we add an axial compression P is $EI\frac{d^2w}{dx^2} + Pw = -\frac{px}{2}(l - x) + M_0\frac{x}{l}$. The right-hand side may be written approximately $-\frac{pl^2}{8}\sin\frac{\pi x}{l} + \frac{2}{\pi}M_0\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l}\right)$. The bending moment is

$$\frac{px}{2}(l - x) - \frac{M_0 x}{l} + Pw = \frac{px}{2}(l - x) - M_0\frac{x}{l} + \frac{P}{P_{cr}}\left[\left(\frac{pl^2}{8} - \frac{2}{\pi}M_0\right)\frac{\sin(\pi x/l)}{1 - \left(\frac{P}{P_{cr}}\right)} + \frac{2}{\pi}M_0\frac{\sin(2\pi x/l)}{8\left(1 - \frac{P}{4P_{cr}}\right)}\right]$$

with $P_{cr} = \pi^2 EI/l^2$.

6. Find the expression for the displacement of the piston in Chapter III, Prob. 14, as function of the angle θ of the crank with the horizontal. Find the Fourier expansion for the displacement as function of θ.

Hint: The elliptic integrals encountered can be most easily evaluated by expanding them in power series in r/l.

7. The vertical cylindrical water tank shown in Fig. P.7 is filled with water. The constant wall thickness is equal to t. Plot the bending moment, considering a section of unit width of the wall as an elastically restrained beam (cf. Chapter VII, section 4), assuming that the points A and B cannot move horizontally but that the inclination of the wall is not restrained. Use the method of Fourier expansion.

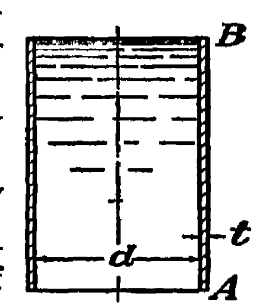

FIG. P.7.

8. Assume that the wall thickness of the water tank in Prob. 7 is a linear function of the height where the thickness at the bottom is t_0 and at the top $t_0/3$. Calculate approximately the bending moment using the Rayleigh-Ritz method. Solve the same problem assuming that the tank has a free edge at B, $(x = l)$, and that the wall is clamped at the bottom A, $(x = 0)$. Assume $l/t_0 = 6$, $l/d = \frac{1}{4}$.

Hint: In the case where the wall has a free edge at B and is clamped at A, we represent the deflection by the expression

$$w = a + b \cos \frac{\pi x}{2l} + c \cos \frac{3\pi x}{2l} + e \cos \frac{5\pi x}{2l}$$

This satisfies the condition $d^2w/dx^2 = 0$ at $x = l$ and $dw/dx = 0$ at $x = 0$. We must also satisfy the condition $w = 0$ at $x = 0$ and $d^3w/dx^3 = 0$ at $x = l$. Hence $a + b + c + e = 0$ and $b - 27c + 125e = 0$. This leaves two of the constants a, b, c, e still arbitrary. They will be determined by the condition of minimum potential energy.

9. Solve approximately Prob. 19 in Chapter VII, by the Rayleigh-Ritz method.

References

1. BYERLY, W. E.: "An Elementary Treatise on Fourier Series and Spherical, Cylindrical and Ellipsoidal Harmonics," Ginn and Company, Boston, 1893.
2. CARSLAW, H. S.: "Theory of Fourier Series and Integrals," Macmillan & Company, Ltd., London, 1921.
3. EAGLE, A.: "A Practical Treatise on Fourier's Theorem and Harmonic Analysis for Physicists and Engineers," Longmans, Green & Company, New York, 1925.
4. TEMPLE, G. F. J., and W. G. BICKLEY: "Rayleigh's Principle and Its Applications to Engineering," Oxford University Press, New York, 1933.

CHAPTER IX

COMPLEX REPRESENTATION OF PERIODIC PHENOMENA

> The imaginary number is a fine and wonderful recourse of the divine spirit, almost an amphibian between being and not being.
>
> —G. W. Leibnitz (1646–1716).

Introduction.—Whereas in Chapter VIII Fourier series were applied to structural problems, in this chapter they are applied to periodic processes occurring in mechanical and electrical systems. The main objective of the methods presented in this chapter is to find the amplitude and phase distribution of coupled systems in so-called *steady-state oscillations.* We make ample use of the complex representation of periodic phenomena and especially of the notion of the complex *impedance.* The methods discussed in this chapter are, in general, restricted to systems that are governed by linear differential or integral equations with constant coefficients.

1. Steady and Transient States.—Let us consider first the example of a single mass with elastic restraint and with damping under the action of a periodic force. This problem was treated in Chapter IV, section 7. The equation of motion for such a system is

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F_0 \sin \omega t \tag{1.1}$$

where m = mass, β = damping factor, k = spring constant, F_0 = amplitude of the external force. We found that the solution consisted of a damped free oscillation of the system and of a forced oscillation given by

$$x_1 = F_0\frac{(k - m\omega^2)\sin \omega t - \beta\omega \cos \omega t}{(k - m\omega^2)^2 + \beta^2\omega^2} \tag{1.2}$$

The free vibration is given by the solution of the homogeneous differential equation associated with Eq. (1.1). If for example,

$\beta^2 < 4km$, the expression for the free oscillation is

$$x_2 = Ce^{-\frac{\beta}{2m}t} \cos \sqrt{\frac{k}{m} - \frac{\beta^2}{4m^2}}t + De^{-\frac{\beta}{2m}t} \sin \sqrt{\frac{k}{m} - \frac{\beta^2}{4m^2}}t \qquad (1.3)$$

The constants of integration are determined by the initial conditions. Assume, for instance, that the mass is at rest for $t \leqq 0$ and the action of the force $F_0 \sin \omega t$ starts at $t = 0$. Then C and D are determined by the condition that at $t = 0$,

$$x = \frac{dx}{dt} = 0$$

As the time increases, $x_2 \to 0$ so that only x_1 remains. The period during which x_2 is not negligible is called the *transient state.* The remaining periodic motion x_1 is called the *steady-state motion.*

The equation for the steady-state motion can be written in the form:

$$x_1 = \frac{F_0 \sin (\omega t - \varphi)}{\sqrt{(k - m\omega^2)^2 + \beta^2\omega^2}} \qquad (1.4)$$

where φ is the phase angle, defined by $\tan \varphi = \dfrac{\beta\omega}{k - m\omega^2}$.

2. Vectorial Representation.—The expression (1.4) for the steady-state motion can be derived in a more elegant way by using *complex exponentials* as defined in Chapter I, section 8.

We consider the equation

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F_0e^{i\omega t} \qquad (2.1)$$

Since $F_0e^{i\omega t} = F_0 \cos \omega t + iF_0 \sin \omega t$, the real part of x will give the solution for a periodic force $F_0 \cos \omega t$, and the imaginary part of x that for $F_0 \sin \omega t$. Putting x proportional to $e^{i\omega t}$, we obtain

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = [m(i\omega)^2 + \beta i\omega + k]\, x$$

We shall denote the complex expression $m(i\omega)^2 + \beta i\omega + k$ by $Z(i\omega)$. Then we see that x can be written in the form:

$$x = \frac{F_0e^{i\omega t}}{Z(i\omega)} = \frac{F}{Z(i\omega)} \qquad (2.2)$$

where $F = F_0e^{i\omega t}$.

Hence, the complex quantity x appears as a ratio of two complex quantities F and Z.

Now complex quantities can be represented as vectors in a plane (Fig. 2.1). We take the horizontal axis as the real axis and the vertical axis as the imaginary axis. Then the length of the vector $a + bi$, called the *modulus*, is equal to $\sqrt{a^2 + b^2}$; the angle between the vector and the real axis, $\alpha = \tan^{-1} \dfrac{b}{a}$, is called the *argument* or the *phase* of the vector.

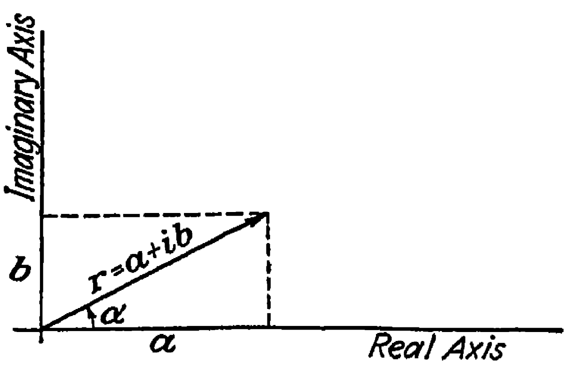

FIG. 2.1.—Geometrical representation of a complex quantity.

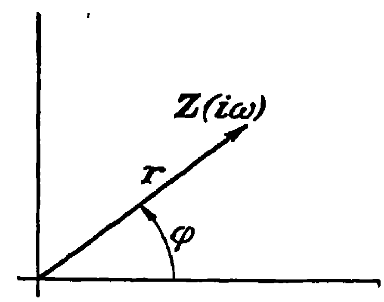

FIG. 2.2.—Vectorial representation of an impedance.

In this representation, F is a vector of constant modulus F_0, rotating with the angular velocity ω. The vector representing $Z(i\omega)$ has the constant modulus $r = \sqrt{(k - m\omega^2)^2 + \beta^2\omega^2}$ and its phase angle is given by $\varphi = \tan^{-1} \dfrac{\beta\omega}{k - m\omega^2}$ (Fig. 2.2). We divide two complex numbers by dividing their moduli and subtracting their arguments. Hence, the modulus of the ratio $x = F/Z$ is also constant and is equal to

$$|x| = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \beta^2\omega^2}}$$

Its phase angle is equal to $\omega t - \varphi$, *i.e.*,

$$x = \frac{F_0 e^{i(\omega t - \varphi)}}{\sqrt{(k - m\omega^2)^2 + \beta^2\omega^2}} \tag{2.3}$$

It is easily verified that the imaginary part of Eq. (2.3) is identical with Eq. (1.4). The vectors x and F constitute a rigid figure rotating with the angular velocity ω in the complex plane; the

ratio of their moduli $|F|/|x|$ and their phase difference φ are entirely defined by the complex expression $Z(i\omega)$ (Fig. 2.3). A different notation is sometimes used by putting $p = i\omega$ and writing $Z(p)$ instead of $Z(i\omega)$. Then Eq. (2.2) becomes $x = F/Z(p)$. We notice that this equation can be derived

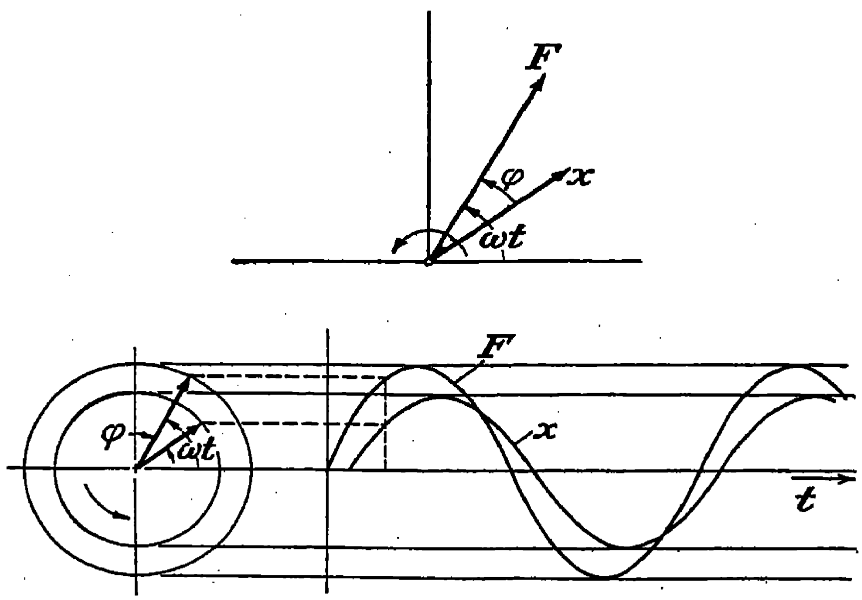

FIG. 2.3.—Vectorial representation of a periodic force and corresponding displacement.

directly from the equations of the problem by a formal process replacing the *operator* d/dt by a number p in the differential equation (2.1) thus:

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F$$

becomes

$$(mp^2 + \beta p + k)x = F$$

Solving this equation as a linear equation for x, we obtain

$$x = \frac{F}{mp^2 + \beta p + k}$$

or with $Z(p) = mp^2 + \beta p + k$,

$$x = \frac{F}{Z(p)} \tag{2.4}$$

Then we obtain Eq. (2.3) by substituting $F = F_0e^{i\omega t}$ and putting $p = i\omega$ in $Z(p)$.

Let us consider now the oscillations of an electric circuit (Fig. 2.4). In this case we have two simultaneous equations for the current I and for the charge Q (cf. Chapter VI, section 3):

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E$$

and (2.5)

$$\frac{dQ}{dt} = I$$

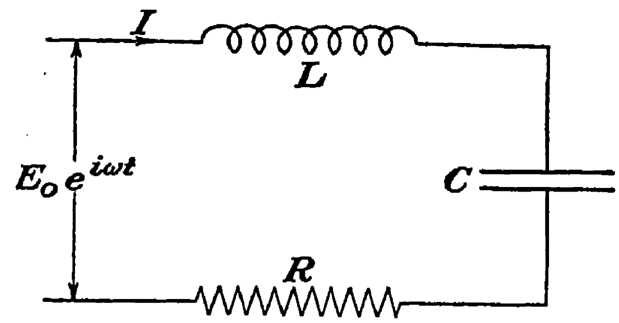

FIG. 2.4.—Schematic diagram for an electric circuit upon which an alternating electromotive force is impressed.

where L = inductance, R = resistance, C = capacity, and E = electromotive force. We assume that $E = E_0e^{i\omega t}$ and denote $i\omega$ by p. If we put I and Q proportional to e^{pt}, for example, $I = I_0e^{pt}$ and $Q = Q_0e^{pt}$, we find, from the second equation (2.5),

$$pQ = I \qquad \text{or} \qquad Q = \frac{I}{p}$$

Instead of the relation $dQ/dt = I$, some authors write $Q = \int^t I\,dt$. It is seen that if Q and I are proportional to e^{pt}, the integral sign can be replaced by the symbol $1/p$. Applying this rule, we obtain, from Eq. (2.5),

$$\left(Lp + R + \frac{1}{Cp}\right)I = E$$

Let us define the function $Z(p)$ by

$$Z(p) = Lp + R + \frac{1}{Cp} \tag{2.6}$$

Then we have

$$I = \frac{E}{Z(p)} \tag{2.7}$$

The same result can be obtained by differentiating the first Eq. (2.5). Thus we obtain the differential equation for I:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = i\omega E_0e^{i\omega t} \tag{2.8}$$

and replacing d/dt by p, Eq. (2.7) is immediately obtained.

We notice that the equation $Z(p) = 0$ is identical with the characteristic equation of the homogeneous equation associated with Eq. (2.1) or (2.8).

In the vectorial representation outlined in this section the alternating electromotive force and the alternating current are considered as vectors rotating with the angular velocity ω. In electrical engineering practice it is more usual to use a representation in which the rotation of the vectors is eliminated and only their magnitudes and phase differences are indicated. This method has great usefulness as it gives a graphical picture of the behavior of electrical systems when one or more of the parameters are variable. For example, for the case of an electric motor it is usual to draw the impressed electromotive force as a fixed vector along the imaginary axis and to plot the current, which is a function of the torque delivered by the motor, in the phase relationship corresponding to different values of the torque. The curve described by the end point of the variable current vector gives the "load diagram" of the motor.

3. The Concept of Impedance. *a. Electrical Systems.*—In the previous section we found that the current I in an electric circuit is expressed in the form (2.7):

$$I = \frac{E}{Z(p)} \tag{3.1}$$

The expression $Z(p)$, which is the complex ratio between the electromotive force and the current, is called the *impedance of the circuit.* In this sense the impedance of a coil of inductance L is equal to Lp, that of a resistor is equal to R, and that of a capacitor is equal to $1/Cp$. The expression $Z(p)$ given by Eq. (2.6) is then the resulting impedance of the coil, resistor, and capacitor in series.

If we have a network consisting of a number of circuits, the concept of impedance can be extended to express the ratio between an electromotive force applied to arbitrary points of the network and the current flowing through an arbitrary branch of the network.

Consider as an example the network shown in Fig. 3.1. We denote the currents flowing through the first and second meshes by

$$i_1 = I_1e^{pt} \qquad \text{and} \qquad i_2 = I_2e^{pt}$$

respectively, and assume that an electromotive force $e_1 = E_1e^{pt}$,

is applied to the terminals of the first mesh. Then we have the integrodifferential equations

$$
\begin{aligned}
L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int^t i_1\,dt + \frac{1}{C_{12}}\int^t (i_1 - i_2)\,dt &= e_1 \\
L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int^t i_2\,dt + \frac{1}{C_{12}}\int^t (i_2 - i_1)\,dt &= 0
\end{aligned}
\tag{3.2}
$$

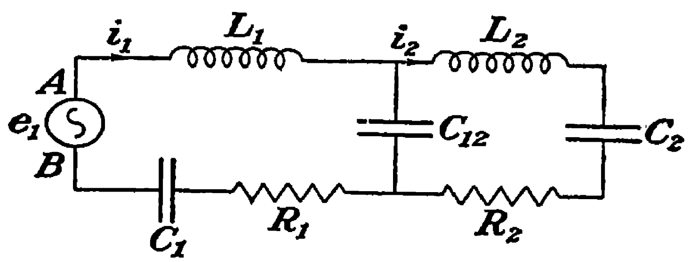

FIG. 3.1.—Double-mesh network upon which an alternating electromotive force is impressed.

Substituting the above values for i_1 and i_2, carrying out the differential and integral operations, and dividing all equations by the common factor e^{pt}, we find

$$
\begin{aligned}
\left(L_1 p + R_1 + \frac{1}{C_1 p}\right) I_1 + \frac{1}{C_{12} p}(I_1 - I_2) &= E_1 \\
\left(L_2 p + R_2 + \frac{1}{C_2 p}\right) I_2 + \frac{1}{C_{12} p}(I_2 - I_1) &= 0
\end{aligned}
\tag{3.3}
$$

We solve this system for I_1 and I_2. The determinant of the system is

$$
\Delta(p) = \begin{vmatrix}
L_1 p + R_1 + \left(\frac{1}{C_1} + \frac{1}{C_{12}}\right)\frac{1}{p} & -\frac{1}{C_{12} p} \\
-\frac{1}{C_{12} p} & L_2 p + R_2 + \left(\frac{1}{C_2} + \frac{1}{C_{12}}\right)\frac{1}{p}
\end{vmatrix}
\tag{3.4}
$$

and the expressions for I_1 and I_2 are

$$
\begin{aligned}
I_1 &= \frac{E_1}{\Delta(p)}\left[L_2 p + R_2 + \left(\frac{1}{C_2} + \frac{1}{C_{12}}\right)\frac{1}{p}\right] \\
I_2 &= \frac{E_1}{\Delta(p)}\,\frac{1}{C_{12} p}
\end{aligned}
\tag{3.5}
$$

We shall now define the impedance $Z_{11}(p)$ as the ratio between the electromotive force E_1 acting between the terminals A and B and the current I_1 which flows from A to B. $Z_{11}(p)$ is sometimes called the *total impedance* of the network or the *driving-*

point impedance at AB. We shall call the ratio between the electromotive force E_1 and the current I_2, which flows through the second branch of the network, the impedance Z_{12}. Since the electromotive force and the current refer to different branches of the network, $Z_{12}(p)$ is called a *transfer impedance*. Hence, we have

$$Z_{11}(p) = \frac{E_1}{I_1} = \frac{\Delta(p)}{L_2 p + R_2 + \left(\frac{1}{C_2} + \frac{1}{C_{12}}\right)\frac{1}{p}} \tag{3.6}$$

$$Z_{12}(p) = \frac{E_1}{I_2} = \Delta(p) C_{12} p$$

Besides considering electric networks in the proper sense, it is usual in electrical engineering practice to investigate the behavior of electric machines and devices, such as generators, motors, and transformers, by replacing them by *equivalent networks*. In the actual machines two or more separate circuits are electromagnetically coupled. However, if the electromagnetic coupling is a so-called *perfect coupling*, one can assume that a constant proportionality exists between the currents flowing through the electromagnetically coupled branches, for example, through the primary and secondary windings of a transformer. This proportionality makes possible the replacement of the system by a single network, where the voltage drops occurring in the coupled branches can be taken care of by inserting appropriate resistances and inductances in the network. A few examples of equivalent networks of motors and transformers are given in the problems at the end of this chapter. The reader interested in the subject may consult ref. 3, given at the end of the chapter.

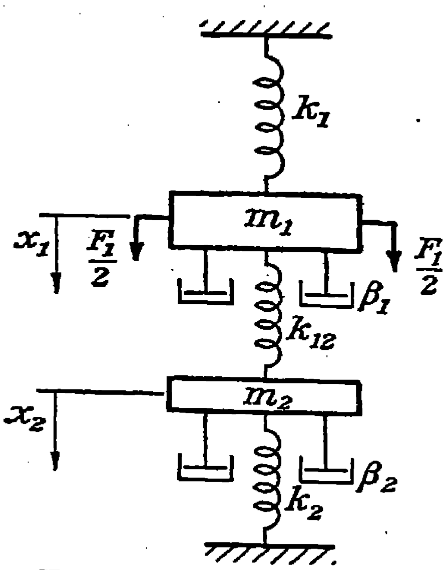

FIG. 3.2.—The mechanical equivalent of the double-mesh network in Fig. 3.1.

b. Mechanical Systems.—The concept of the impedance can be transferred to mechanical systems using the analogy which exists between electric networks and coupled mechanical systems (cf. Chapter VI). In this analogy inductance corresponds to inertia, resistance to damping, capacity to elastic resilience. The elements—*coils, resistors, capacitors*—which constitute a closed

circuit in the electric network—correspond to *masses*, *dashpots*, and *springs* acting in the mechanical system. Then it can be shown that if we replace the alternating electromotive forces by periodic external mechanical forces, the current flowing through a certain branch of a network and the velocity of the corresponding point of a mechanical system are governed by the equations of the same form.

Consider, for example, the mechanical system shown schematically in Fig. 3.2. The reader will verify easily that this system is in the above sense analogous to the electric network shown in Fig. 3.1. The equations of motion for the two masses m_1 and m_2 are

$$\begin{aligned} m_1 \frac{d\dot{x}_1}{dt} + \beta_1 \dot{x}_1 + k_1 x_1 + k_{12}(x_1 - x_2) &= F_1 \\ m_2 \frac{d\dot{x}_2}{dt} + \beta_2 \dot{x}_2 + k_2 x_2 + k_{12}(x_2 - x_1) &= 0 \end{aligned} \tag{3.7}$$

It is seen that Eqs. (3.7) become identical with (3.2) if we replace m_1 and m_2 by L_1 and L_2; β_1 and β_2 by R_1 and R_2; k_1, k_2, and k_{12} by $1/C_1$, $1/C_2$, and $1/C_{12}$; $\dot{x}_1$ and $\dot{x}_2$ by i_1 and i_2; x_1 and x_2 by $\int^t i_1\, dt$ and $\int^t i_2\, dt$; and finally F_1 by e_1.

Consequently, the solution of (3.7) will be, according to (3.5),

$$\begin{aligned} \dot{x}_1 &= \frac{F_1}{\Delta(p)} \left[m_2 p + \beta_2 + (k_2 + k_{12}) \frac{1}{p} \right] \\ \dot{x}_2 &= \frac{F_1}{\Delta(p)} \frac{k_{12}}{p} \end{aligned} \tag{3.8}$$

or

$$\begin{aligned} x_1 &= \frac{F_1}{\Delta(p)} \left[m_2 + \frac{\beta_2}{p} + (k_2 + k_{12}) \frac{1}{p^2} \right] \\ &= \frac{F_1}{\Delta(p) p^2} [m_2 p^2 + \beta_2 p + (k_2 + k_{12})] \\ x_2 &= \frac{F_1}{\Delta p} \frac{k_{12}}{p^2} = \frac{F_1}{\Delta(p) p^2} k_{12} \end{aligned} \tag{3.9}$$

We notice that $\Delta(p)$ has the form $G(p)/p^2$, where $G(p)$ is a polynomial of the 4th degree; hence, $\Delta(p)p^2$ is a polynomial of the 4th degree in p.

We can now call either the ratios $F_1/\dot{x}_1$ and $F_1/\dot{x}_2$ or the ratios F_1/x_1 and F_1/x_2 *mechanical impedances*. The first definition corresponds strictly to the electrical analogy and is used in acoustics in that manner. The ratios F_1/x_1 and F_1/x_2 can be called *force-displacement impedances;* in this book we shall use them preferably and denote them by the symbol $Z(p)$. For example $F_1/x_1 = Z_{11}(p)$ and $F_1/x_2 = Z_{12}(p)$.

The electrical impedance is a generalization of the concept of the electric resistance for an alternating current. The mechanical impedance as defined above is a generalization of the spring constant for a periodic motion. This will be clarified by a few examples in the next section.

The following table shows the impedances of the various electrical and mechanical elements:

Element	Electrical impedance	Element	Mechanical impedance	
			Force/velocity	Force/displacement
Coil.	Lp	Mass.	mp	mp^2
Resistor. . . .	R	Dashpot. . .	β	βp
Capacitor. . .	$1/Cp$	Spring.	k/p	k

Characteristic Equation and Characteristic Exponents.—The impedances defined in this section are obtained by solution of a system of linear equations, where the coefficients are functions of p. Hence, in all expressions that represent impedances, the determinant of the system $\Delta(p)$ appears in the numerator. The equation $\Delta(p) = 0$ is identical with the *characteristic equation* of the coupled system defined in Chapters V and VI. If we denote the roots of $\Delta(p) = 0$ by $p_k = \alpha_k + i\beta_k$, the solutions of the homogeneous equations associated with the problem, *i.e.*, the free oscillations of the system, will have the form

$$x = \sum_{k=1}^{n} A_k e^{p_k t} = \sum_{k=1}^{n} A_k e^{(\alpha_k + i\beta_k)t} \tag{3.10}$$

where the A_k's are undetermined complex constants. The p_k's are called the *characteristic exponents.* If the system is undamped all α_k's are equal to zero; if the system is damped, all α_k's must be negative.

4. Rules for Calculation of the Impedances of Electrical and Mechanical Systems. *a. Electrical Systems.*—If we are concerned with oscillations in a network composed of circuits, labor can often be saved by calculating the impedances by certain combination rules. In this way we can avoid the setting up of the differential equations. For example, the current flowing between the terminals on which the electromotive force is imposed, is given by the ratio of the electromotive force to the total impedance of the network. The main rules for the calculation are the following:

1. The resulting impedance of two impedances *in series* is equal to the sum of the impedances (Fig. 4.1). If the impedances

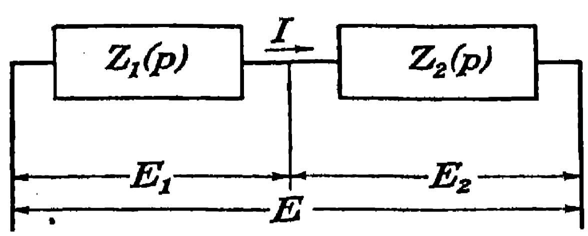

FIG. 4.1.—Two electrical impedances in series.

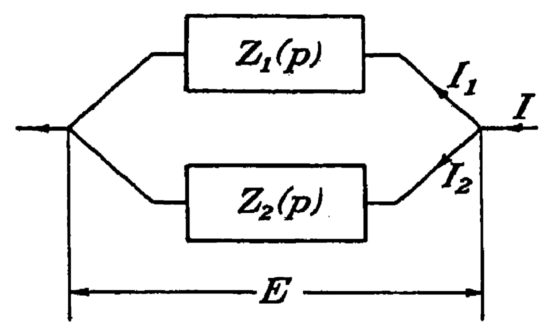

FIG. 4.2.—Two electrical impedances in parallel.

Z_1 and Z_2 are in series, the same current passes through both of them, and, therefore, the electromotive force acting at the terminals is equal to

$$E = I(Z_1 + Z_2) \tag{4.1}$$

2. The reciprocal of the resulting impedance of two impedances *in parallel* (Fig. 4.2) is equal to the sum of the reciprocals of the composing impedances. If Z_1 and Z_2 are in parallel, the electromotive force between the terminals of Z_1 and Z_2 is the same, and, therefore,

$$E = I_1Z_1 = I_2Z_2$$

or putting $(I_1 + I_2) = E/Z$,

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} \tag{4.2}$$

For example, if the impedance of a coil is Lp and the impedance of a capacitor is equal to $1/Cp$, their total impedance when

in parallel is

$$Z(p) = \frac{1}{(1/Lp) + Cp} \tag{4.3}$$

Applying this rule to the successive branches of the "ladder type" network (cf. Fig. 4.4*b*), we obtain the *continued fraction*

$$Z(p) = L_1p + \cfrac{1}{C_1p + \cfrac{1}{L_2p + \cfrac{1}{C_2p + \cfrac{1}{L_3p}}}} \tag{4.4}$$

b. Mechanical Systems.—If we have a mechanical system consisting of masses, springs, and damping devices, we can replace it by the equivalent electric network, calculate the

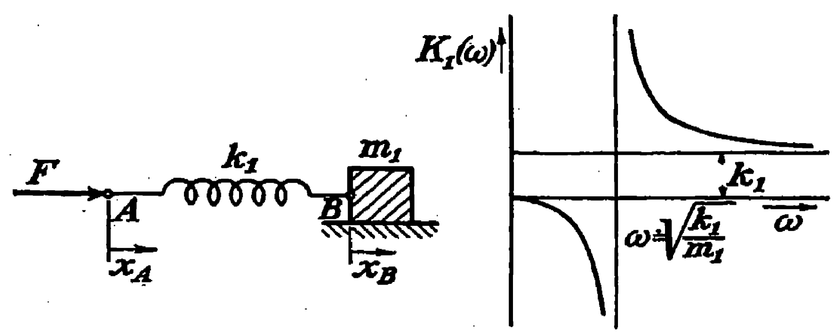

FIG. 4.3.—Schematic diagram illustrating the concept of mechanical impedance.

resulting impedance of the electric system, and translate the result back into the language of mechanics.

However, there are a few useful rules for the calculation of mechanical impedances which, without involving the electrical analogy, can often be employed with advantage.

Let us assume that an alternating force F is acting at the point A and that a mass m_1 is attached to the point A by a spring whose spring constant is k_1 (Fig. 4.3). Then the force acting at the end B of the spring is equal to $m_1p^2x_B$, where x_B is the displacement of B. The difference between the displacements at A and B is equal to $x_A - x_B = F/k_1$. Since the forces acting on the two ends of the springs are equal, $m_1p^2x_B = k_1(x_A - x_B)$ or $x_B = \dfrac{k_1x_A}{k_1 + m_1p^2}$; $x_A - x_B = \dfrac{m_1p^2}{k_1 + m_1p^2}x_A$ and therefore,

$$\frac{F}{x_A} = k_1\frac{1}{1 + (k_1/m_1p^2)}$$

We call $K_1 = k_1 \dfrac{1}{1 + (k_1/m_1p^2)}$ the *dynamic spring constant* of the spring-plus-mass combination. If $p = 0$, $K_1 = 0$, *i.e.*, the spring-plus-mass combination does not offer resistance to the force. If $p \to \infty$, $K_1 = k_1$, *i.e.*, the mass m acts practically as a fixed base. Taking into account that $p = i\omega$, where ω is the frequency, we see that $K_1 \to \infty$ if $\omega = \sqrt{k_1/m_1}$; in other words, an arbitrarily small force produces an arbitrarily large displacement (resonance). The dynamic spring constant is obviously the impedance of the spring-plus-mass combination. Hence, the mechanical impedance appears as a generalization of the spring constant, as was indicated in the previous section. Assume now that a second mass m_2 is attached to the end A of the spring. Then the force is increased by $x_A m_2 p^2$, and we have

$$F = m_2p^2x_A + K_1x_A \tag{4.5}$$

Hence, the resulting impedance of the combination consisting of the two masses and the spring is equal to

$$Z(p) = m_2p^2 + K_1 \tag{4.6}$$

It is verified easily that if the force F exerts its action on the mass m_2 through a second spring of the spring constant k_2, the resulting impedance of the system will be given by the relation

$$\frac{1}{Z(p)} = \frac{1}{m_2p^2 + K_1} + \frac{1}{k_2} \tag{4.7}$$

We may consider the impedance of this system as a dynamic spring constant K_2 and obtain

$$K_2 = k_2 \frac{1}{1 + k_2/(m_2p^2 + K_1)} \tag{4.8}$$

Equation (4.8) can be extended to the case of a chain consisting of an arbitrary number of masses. The K's can be expressed by a recurrence formula.

For systems undergoing torsional oscillations, *e.g.*, a shaft carrying disks, the displacements x of the masses are replaced by the angular displacements θ and their masses by the moments of inertia I of the disks. The spring constants k are in this case the moments required for unit relative angular displacements of

two neighboring disks. Assume that a shaft carries three disks (Fig. 4.4a) whose moments of inertia are I_1, I_2, and I_3. The spring constants for the two sections of the shaft are k_1 and k_2. An alternating torque T acts on the disk I_1. Then the ratio T/θ_1, where θ_1 is the angular displacement of the disk I_1, is equal to

$$Z(p) = I_1p^2 + \cfrac{1}{\cfrac{1}{k_1} + \cfrac{1}{I_2p^2 + \cfrac{1}{\cfrac{1}{k_2} + \cfrac{1}{I_3p^2}}}} \tag{4.9}$$

The equivalent electric circuit shown in Fig. 4.4b has been considered above.

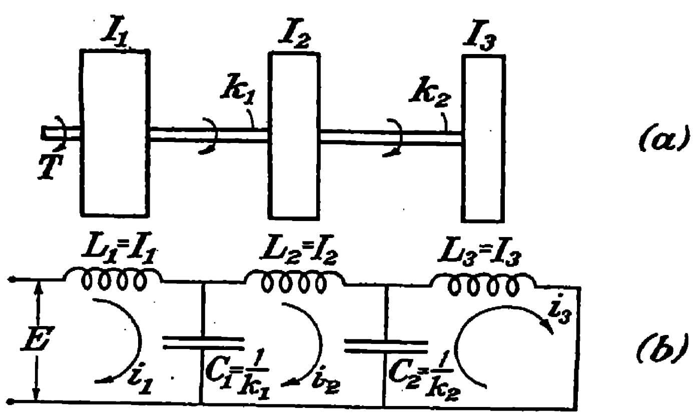

FIG. 4.4.—Mechanical system consisting of three elastically connected disks and the equivalent electric network.

The frequencies of the free oscillations of the shaft are given by the roots of $Z(p) = 0$, as was mentioned in the last section. The reader will verify that $p = 0$ is a root of $Z(p) = 0$. The physical meaning of this root is that the shaft is free to rotate with uniform angular velocity (cf. Chapter V, section 9).

5. Superposition of Periodic Motions.—So far we have considered the response to a force which is a harmonic function of the time. If the force is given by a series of harmonic functions, the response can be obtained by *superposition* of the elementary solutions. The *principle of superposition* can be applied to all cases in which the effect of simultaneous superposed actions is the sum of the effects of each individual action. It can be applied, therefore, to linear systems, *i.e.*, to physical systems governed by linear differential or integral equations. Hence, the principle is valid for the small oscillations of elastic systems and

for electric circuits if the impedances of the elements can be assumed to be constant.

Let us assume that the force is given by a series of the form:

$$F(t) = \sum_{n=1}^{m} F_n e^{in\omega t} \tag{5.1}$$

and the impedance of the system is $Z(i\omega)$; then the displacement $x(t)$ is given by

$$x(t) = \sum_{n=1}^{m} F_n \frac{e^{in\omega t}}{Z(in\omega)} \tag{5.2}$$

Let us apply the principle of superposition, as expressed by Eq. (5.2), to one example of a mechanical system and one example of an electrical system.

Example 1.—Consider the steady-state motion of a mass with elastic restraint and small damping under the action of the periodic force

$$F(t) = F_0\left(\sin \omega t + \frac{\sin 3\omega t}{3}\right) \tag{5.3}$$

We assume that, owing to the damping, the motion after a sufficient time is independent of the initial conditions and that the effect of the damping on the steady state can be neglected. Then the steady-state motion is given with fair approximation by the equation

$$m\frac{d^2x}{dt^2} + kx = F_0\left(\sin \omega t + \frac{1}{3}\sin 3\omega t\right) \tag{5.4}$$

The steady-state motion corresponding to the force $F_1 = F_0 \sin \omega t$ is given by Eq. (6.4) (Chapter IV)

$$x_1 = \frac{F_0}{k}\frac{\sin \omega t}{1 - (\omega/\omega_0)^2}$$

where $\omega_0^2 = k/m$. Correspondingly, the steady-state motion produced by the force $F_2 = \dfrac{F_0}{3}\sin 3\omega t$ is given by

$$x_2 = \frac{F_0}{3k}\frac{\sin 3\omega t}{1 - (3\omega/\omega_0)^2}$$

Since the sum $x = x_1 + x_2$ is a solution of Eq. (5.4), we obtain the resulting displacement by superposition of the displacements x_1 and x_2:

$$x = \frac{F_0}{k}\left[\frac{\sin \omega t}{1 - (\omega/\omega_0)^2} + \frac{1}{3}\frac{\sin 3\omega t}{1 - (3\omega/\omega_0)^2}\right] \tag{5.5}$$

If, for instance, $\omega = \omega_0/2$,

$$x = \frac{F_0}{k}\left(\frac{4}{3}\sin \omega t - \frac{4}{15}\sin 3\omega t\right) \tag{5.6}$$

The force $F_0(\sin \omega t + \frac{1}{3}\sin 3\omega t)$ is plotted in Fig. 5.1, the displacement x in Fig. 5.2. In the vectorial representation both x and F are given by

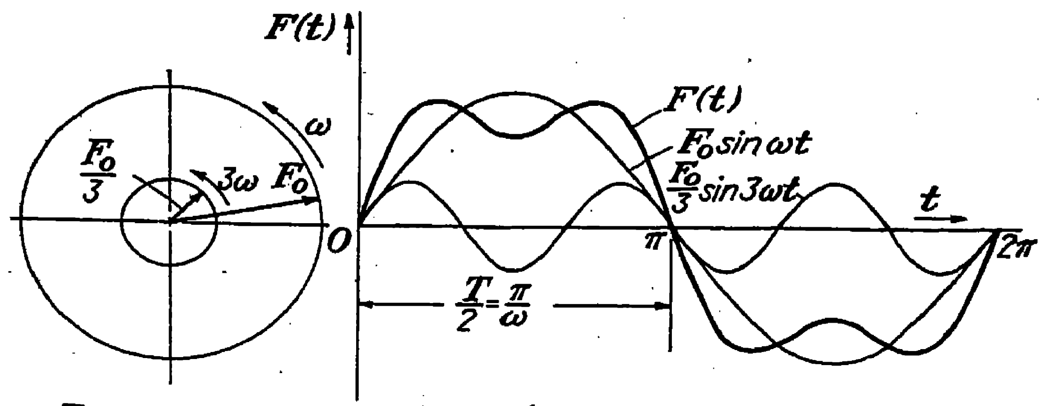

FIG. 5.1.—Vectorial representation of the periodic force $F(t) = F_0 (\sin \omega t + \frac{1}{3}\sin 3\omega t)$.

the vertical components of the resultants of two vectors, one rotating with the angular velocity ω, the other with the angular velocity 3ω. Note

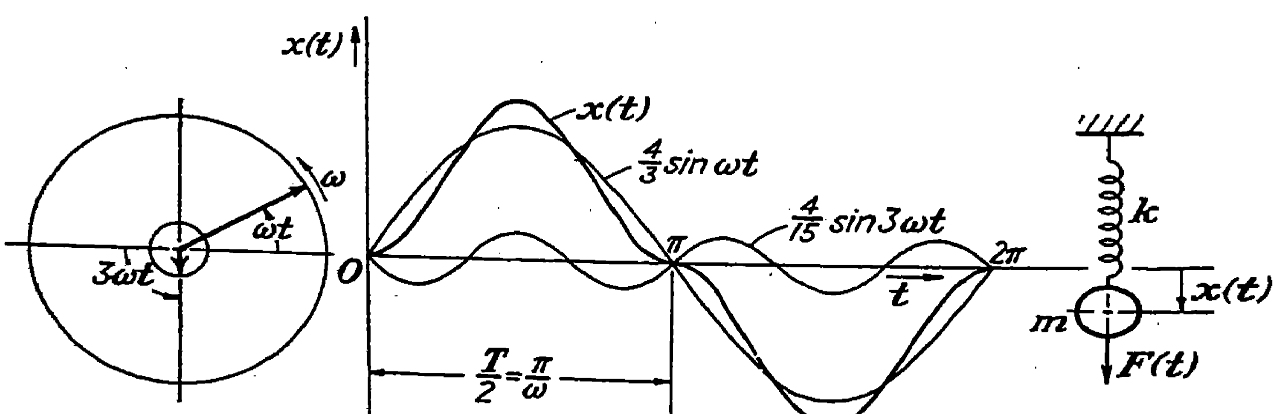

FIG. 5.2.—Forced oscillation of a mass with elastic restraint under the action of the force represented in Fig. 5.1.

that the response to the force whose frequency is 3ω has a phase difference of 180°. We note that the two periodic functions representing F and x may have very different appearances.

Example 2.—In order to illustrate the application of the principle of superposition to electric circuits, we consider a circuit composed of a resistance R and an inductance L in series (Fig. 5.3). An alternating current generator G supplies to the circuit a periodic voltage given by

$$E(t) = E_0(\sin \omega t + \tfrac{1}{3} \sin 3\omega t) \tag{5.7}$$

The expression (5.7) is the imaginary part of $E_0(e^{i\omega t} + \frac{1}{3}e^{3i\omega t})$. The current corresponding to the voltage $E_0e^{i\omega t}$ is equal to [cf. Eqs. (2.6) and (2.7)]

$$\frac{E}{Z(p)} = \frac{E_0e^{i\omega t}}{Z(i\omega)} = \frac{E_0e^{i\omega t}}{Li\omega + R} \tag{5.8}$$

The current corresponding to $\frac{1}{3}E_0e^{3i\omega t}$ is given by

$$\frac{1}{3}\,\frac{E_0e^{3i\omega t}}{3Li\omega + R} \tag{5.9}$$

The current corresponding to the voltage $E_0(e^{i\omega t} + \frac{1}{3}e^{3i\omega t})$ can be obtained by superposition of (5.8) and (5.9) thus:

$$\frac{E_0e^{i\omega t}}{Li\omega + R} + \frac{E_0e^{3i\omega t}}{3(3Li\omega + R)} \tag{5.10}$$

The imaginary part of (5.10) gives the current corresponding to the voltage E given by Eq. (5.7). In order to separate the real and imaginary parts

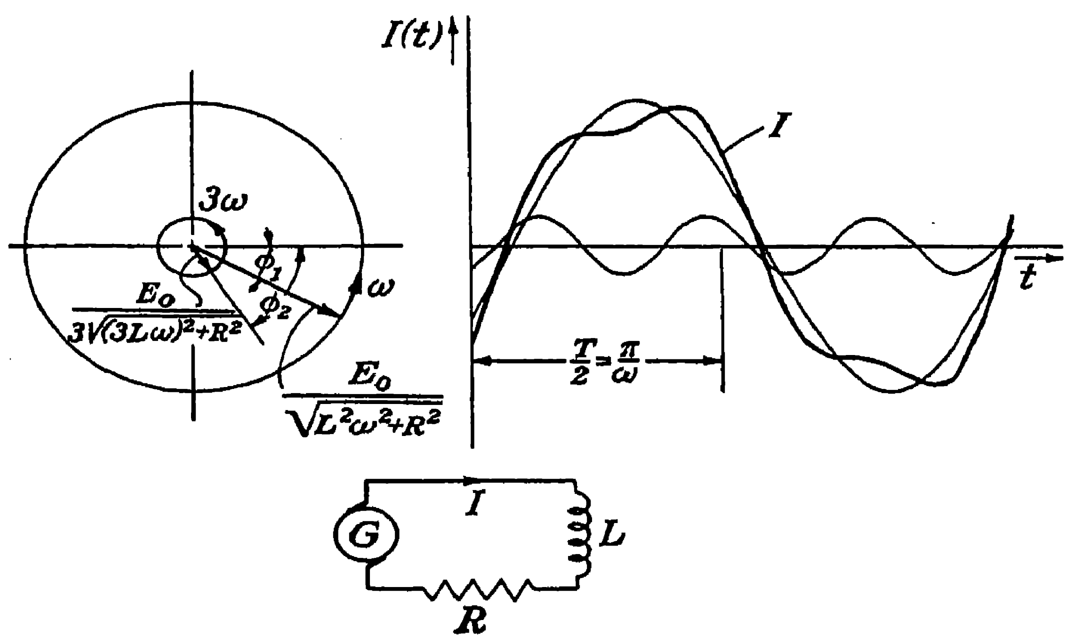

FIG. 5.3.—Vectorial representation of the current produced by a periodic voltage impressed on a circuit composed of a resistance and an inductance.

of (5.10), we introduce the phase differences $\varphi_1 = \tan^{-1}\frac{R}{L\omega}$, $\varphi_2 = \tan^{-1}\frac{R}{3L\omega}$. Then we obtain for the expression (5.10)

$$E_0\left(\frac{e^{i(\omega t-\varphi_1)}}{\sqrt{L^2\omega^2 + R^2}} + \frac{e^{i(3\omega t-\varphi_2)}}{3\sqrt{9L^2\omega^2 + R^2}}\right) \tag{5.11}$$

Therefore, the actual current is equal to

$$I(t) = E_0\left[\frac{\sin(\omega t - \varphi_1)}{\sqrt{L^2\omega^2 + R^2}} + \frac{1}{3}\frac{\sin(3\omega t - \varphi_2)}{\sqrt{9L^2\omega^2 + R^2}}\right] \tag{5.12}$$

We see that the current I is equal to the sum of the vertical components of two vectors. One vector has the modulus $E_0/\sqrt{L^2\omega^2 + R^2}$ and rotates with the same angular velocity as the voltage vector $E_0e^{i\omega t}$ but trails the voltage by the constant phase difference φ_1. The other vector has the modulus $\frac{1}{3}E_0/\sqrt{9L^2\omega^2 + R^2}$ and rotates with the angular velocity 3ω with a phase lag φ_2 behind the voltage vector $\frac{1}{3}E_0e^{3i\omega t}$. The two vectors and the resulting current I are shown in Fig. 5.3.

6. Response to an Arbitrary Periodic Force. The Complex Form of Fourier Series.—So far we have considered forces given by a finite trigonometric series. Let us assume now that the function $F(t)$ which represents the force acting on a mechanical system or the voltage imposed on an electrical system is an arbitrary periodic function of t, *i.e.*, $F(t + T) = F(t)$, where T is the period. If the function $F(t)$ satisfies the conditions given in Chapter VIII for the convergence of the Fourier expansion, we can write $F(t)$ in the form:

$$F(t) = \sum_{n=1}^{\infty} a_n \sin n\omega t + \sum_{n=0}^{\infty} b_n \cos n\omega t \tag{6.1}$$

where $\omega = 2\pi/T$.

If the function $F(t)$ is given analytically, numerically, or graphically, we can determine the coefficients of Eq. (6.1) by means of the relations developed in Chapter VIII, Eqs. (2.5), (2.6), and (2.7). We obtain

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T F(t)\sin n\omega t\,dt \\ b_0 &= \frac{1}{T}\int_0^T F(t)\,dt \\ b_n &= \frac{2}{T}\int_0^T F(t)\cos n\omega t\,dt, \qquad n = 1, 2, \ldots \end{aligned} \tag{6.2}$$

We notice that since $F(t)$ is a periodic function of t with the period T, we can take any values t_0 and $t_0 + T$ as limits of the integrals in Eq. (6.2). The coefficient b_0 is equal to the mean value of the function taken over the period T.

The results obtained for the Fourier series representing a function defined in a finite interval apply also to the series representing a periodic function. For example, the finite sum

$$S_m = \sum_{n=1}^{m} a_n \sin n\omega t + \sum_{n=0}^{m} b_n \cos n\omega t \tag{6.3}$$

whose coefficients are the coefficients of the infinite series (6.1), approximates the function $F(t)$ with less error than any other

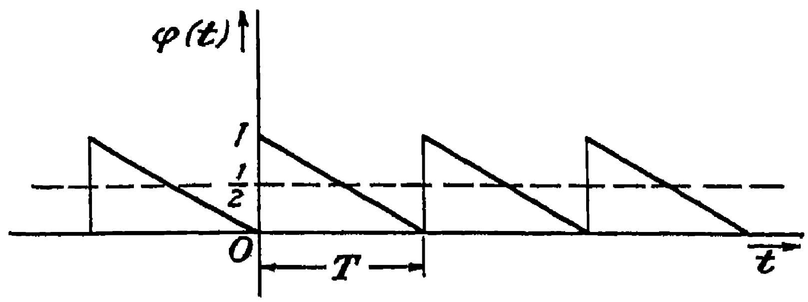

FIG. 6.1.—The *saw-toothed* function.

sum of the same form. It follows that if we want to approximate $F(t)$ by a sum of trigonometric functions of the form:

$$S_{m+1} = \sum_{n=1}^{m+1} a_n \sin n\omega t + \sum_{n=0}^{m+1} b_n \cos n\omega t \tag{6.4}$$

the terms of (6.4) remain unaltered, and the additional two terms

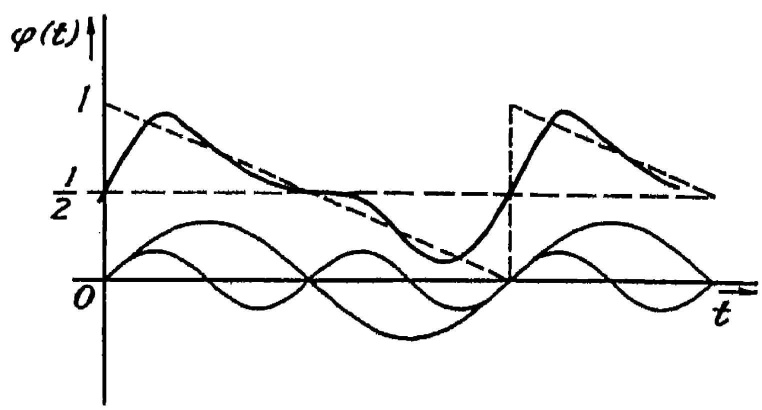

FIG. 6.2—Approximation of the saw-toothed function by the sum of the first two terms of a Fourier series.

merely improve the approximation. The mean square of the error of the approximation S_m is given by

$$\frac{1}{T}\int_t^{t+T} F(t)^2\, dt - \frac{1}{2}\sum_{n=1}^{m} (a_n^2 + b_n^2) - b_0^2$$

Using Eqs. (6.2), we find, for example, the following expansion for the so-called *saw-toothed* function shown in Fig. 6.1:

$$\varphi(t) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1,2}^{\infty} \frac{(-1)^n}{n} \sin \frac{2\pi nt}{T} \tag{6.5}$$

Figures 6.2 and 6.3 show the approximation obtained by taking the first two terms and the first five terms, respectively, under the summation sign.

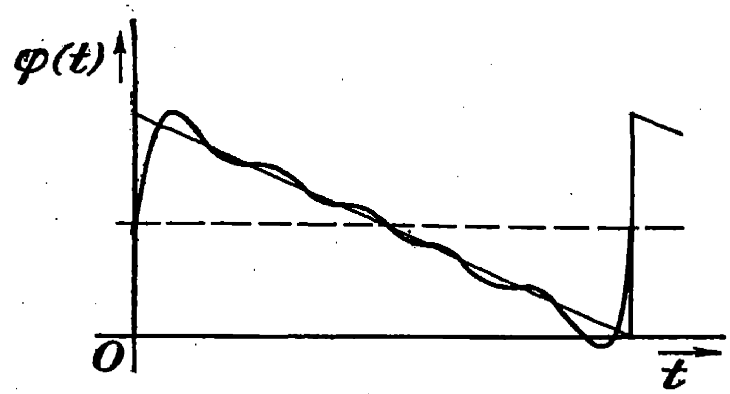

FIG. 6.3.—Approximation of the saw-toothed function by the sum of the first five terms of a Fourier series.

For many applications it is convenient to write the Fourier series (6.1) in complex form. We remember that

$$\sin n\omega t = \frac{1}{2i}(e^{in\omega t} - e^{-in\omega t})$$
$$\cos n\omega t = \frac{1}{2}(e^{in\omega t} + e^{-in\omega t})$$

Substituting these expressions in (6.1), we obtain

$$F(t) = \frac{1}{2i}\sum_{n=1}^{\infty}(a_n + ib_n)e^{in\omega t} - \frac{1}{2i}\sum_{n=0}^{\infty}(a_n - ib_n)e^{-in\omega t} \tag{6.6}$$

We shall write Eq. (6.6) in the form:

$$F(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega t} \tag{6.7}$$

where the coefficients C_n are defined by

$$
\begin{aligned}
C_n &= \frac{a_n + ib_n}{2i} = \frac{b_n - ia_n}{2} \\
C_0 &= b_0 \\
C_{-n} &= -\frac{a_n - ib_n}{2i} = \frac{b_n + ia_n}{2}
\end{aligned}
\tag{6.8}
$$

Substituting a_n and b_n in Eq. (6.2), we obtain

$$
C_n = \frac{1}{T}\int_0^T F(t)(\cos n\omega t - i \sin n\omega t)\, dt = \frac{1}{T}\int^T F(t)e^{-in\omega t}\, dt \tag{6.9}
$$

and it is easily seen that Eq. (6.9) holds also for $n = 0$ and $n < 0$.

We call Eq. (6.7) the *complex Fourier series* for the real function $F(t)$, and Eq. (6.9) *the complex Fourier coefficient formula.*

If a mechanical or electrical force is given in the form (6.7), we find the response by the formula

$$
x(t) = \sum_{n=-\infty}^{\infty} \frac{C_n}{Z(in\omega)} e^{in\omega t} \tag{6.10}
$$

As an example, let us assume that the alternating voltage impressed on a circuit is given by a complex Fourier series

$$
E = \sum_{n=-\infty}^{\infty} E_n e^{in\omega t}
$$

where the E_n's are complex coefficients. Then the current corresponding to the voltage E is given by

$$
I = \sum_{n=-\infty}^{\infty} \frac{E_n}{Z(ni\omega)} e^{in\omega t} \tag{6.11}
$$

If the circuit consists of an inductance L and a resistance R, then (6.11) becomes

$$
I = \sum_{n=-\infty}^{\infty} \frac{E_n}{inL\omega + R} e^{in\omega t} \tag{6.12}
$$

Problems

1. A periodic electromotive force $E = E_0(\sin \omega t - \frac{1}{3} \sin 3\omega t)$ is applied to an inductance L. Plot the value of the periodic current and compare with

the curve representing E. Plot the current when E is applied to a capacitance C.

2. The equivalent electric circuit of an induction motor is shown in Fig. P.2. The numbers on the figure indicate impedances corresponding to a 60-cycle current. The real numbers are resistances in ohms; the imaginary numbers are equal to $2\pi 60Li$, where L is the corresponding inductance in henrys. One of the resistances is given as function of the slip γ, which is proportional to the velocity difference between the rotating magnetic field and the rotor ($\gamma \cong 0$ when the motor is unloaded, $\gamma = 1$ when the load brakes the rotor to a stop). The voltage applied is 577 volts a.c. Draw a vector diagram in which the voltage is represented by a fixed vector along the imaginary axis and the current vector varies with γ. Show that the end point of the current vector describes a circle and that this is a general property for induction motors independent of the particular numerical value of the circuit parameters. Draw the circle by calculating three points for $\gamma = 0$, $\gamma = 0.1$, and $\gamma = \infty$.

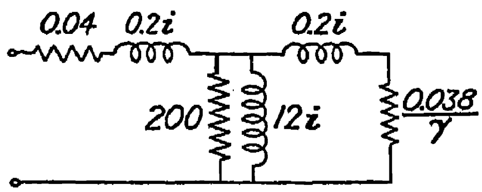

FIG. P.2.

3. An elastic shaft carries three equidistant disks of equal moment of inertia I. The spring constant between disks is c. A periodic torque $M = M_0(\sin \omega t + \frac{1}{4} \sin 3\omega t)$ is applied at one of the end disks. Calculate the motion of the disk at the other end. Indicate the frequencies at which the amplitude is infinite.

4. An elastic shaft carries four equidistant identical disks. Find the dynamic spring constant for a torque applied to one of the end disks. Establish the frequency equation by putting the dynamic spring constant equal to zero. Derive the same frequency equation starting from the equations of motion of the disks.

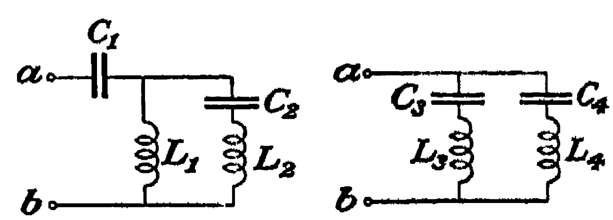

FIG. P.5.

5. Show that the two networks in Fig. P.5 have the same impedance between the terminals a, b if

$$\frac{1}{L_1} + \frac{1}{L_2} = \frac{1}{L_3} + \frac{1}{L_4}$$

$$C_1 = C_3 + C_4$$

$$\frac{1}{C_1}\left(\frac{1}{L_1} + \frac{1}{L_2}\right) + \frac{1}{L_2C_2} = \frac{1}{L_3C_3} + \frac{1}{L_4C_4}$$

$$L_1L_2C_1C_2 = L_3L_4C_3C_4$$

6. The equivalent network of a transformer is shown in Fig. P.6. All impedances are given for 60 cycles/sec. in ohm units (cf. Prob. 2). (*a*) Calculate the imput current when a voltage of 130 volts a.c. is applied at the terminals A,B and a resistance of 3 ohms is connected across the terminals C,D. (*b*) Calculate the current for a frequency of 500 cycles/sec. when the terminals C,D are short-circuited.

Prove that the network in Fig. P.7 is equivalent to a pure resistance R een the terminals A,B if $R = \sqrt{L/C}$.

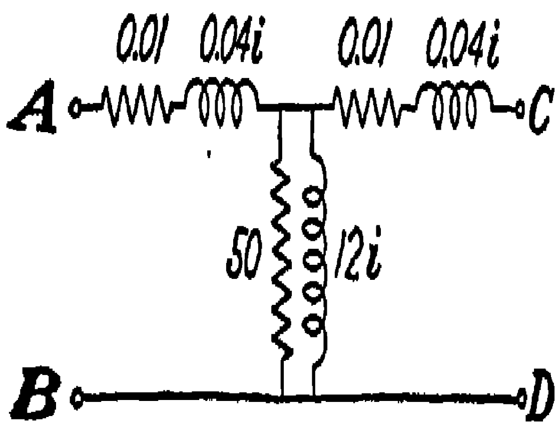

FIG. P.6.

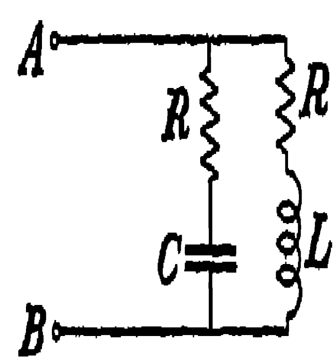

FIG. P.7.

References

eferences 1 to 3, Chapter VIII (for Fourier series).

UILLEMIN, E. A.: "Communication Networks," vol. 1, John Wiley & ons, Inc., New York, 1931.

AHL, O. G. C.: "Electric Circuits—Theory and Applications," vol. 1, cGraw-Hill Book Company, Inc., New York, 1928.

CHAPTER X

TRANSIENT PHENOMENA. OPERATIONAL CALCULUS

"Shall I refuse my dinner because I do not fully understand the process of digestion?"

—O. HEAVISIDE.

Introduction.—In the preceding chapter we were concerned with the *steady-state oscillations* of mechanical and electrical systems. In this chapter we consider the *transient state, i.e.*, the state of motion that is essentially determined by the initial conditions. Problems with given initial conditions have been treated in Chapter IV by calculating the constants of integration in such a way that the initial conditions are complied with. This procedure becomes very cumbersome for complicated systems. The methods presented in this chapter avoid such lengthy calculations. They are especially suitable for a quick and tearless determination of the particular solution needed for special practical problems. In addition, methods using the *principle of superposition* are given for determining the response of mechanical and electrical systems to arbitrary mechanical and electromotive forces imposed on them. The investigations of this chapter are also restricted to linear systems.

1. Application of the Fourier Integral to Nonperiodic Phenomena.—To establish the transition from periodic to nonperiodic phenomena, we consider the motion of a damped system in the time interval between $t = -T/2$ and $t = T/2$ under the assumption that the force $F(t)$ is zero for $-T/2 < t < 0$ and is only different from zero for $0 < t < T/2$. For simplicity's sake we consider a system which is fully determined by one coordinate x. However, the results hold for any number of degrees of freedom. We assume that at $t = -T/2$ the system is at rest, *i.e.*, the displacement x and the initial velocity dx/dt are equal to zero. Let us determine the motion of the system corresponding to these initial conditions.

We expand the function $F(t)$ in the interval $-T/2 \leqq t \leqq T/2$ in a complex Fourier series [cf. Chapter IX, Eq. (6.7)]:

$$F(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t} \tag{1.1}$$

where $\omega_0 = 2\pi/T$. The Fourier coefficient C_n is given by the formula [cf. Chapter IX, Eq. (6.9)]:

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} F(\tau)e^{-in\omega_0\tau}\,d\tau \tag{1.2}$$

We denote the impedance of the system by $Z(p)$, the two characteristic exponents of the system by

$$p_1 = \alpha_1 + i\beta_1, \qquad \text{and} \qquad p_2 = \alpha_2 + i\beta_2,$$

where (since the system is damped) α_1 and α_2 are negative. Then the solution of the problem can be written in the form:

$$x(t) = A_1 e^{(\alpha_1+i\beta_1)\left(t+\frac{T}{2}\right)} + A_2 e^{(\alpha_2+i\beta_2)\left(t+\frac{T}{2}\right)} + \sum_{n=-\infty}^{\infty} \frac{C_n e^{in\omega_0 t}}{Z(in\omega_0)} \tag{1.3}$$

and the constants A_1 and A_2 can be determined so that the initial conditions for $t = -T/2$ are satisfied.

Let us now increase the period T and investigate what happens when we proceed to the limit $T \to \infty$, $\omega_0 \to 0$. Since α_1 and α_2 are negative, the first two terms become vanishingly small for all positive values of t and also for $t < 0$, provided we keep t finite in the limiting process. Hence, only the last term contributes to the limit, and we have

$$x(t) = \lim_{\omega_0\to 0} \sum_{n=-\infty}^{\infty} \frac{C_n e^{in\omega_0 t}}{Z(in\omega_0)} \tag{1.4}$$

The expression for the force $F(t)$, Eq. (1.1), becomes, similarly,

$$F(t) = \lim_{\omega_0\to 0} \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t} \tag{1.5}$$

We carry out the limiting process first in Eq. (1.5). We denote $n\omega_0$ by ω and consider C_n as a function of ω or of $i\omega$. To

be sure, C_n is defined only for values of ω corresponding to integral values of n. Denoting the difference between $n\omega_0$ and $(n+1)\omega_0$ by $\Delta\omega$ (*i.e.*, $\Delta\omega = \omega_0 = 2\pi/T$), and putting $C_n = \omega_0 G(i\omega)$, Eq. (1.5) becomes

$$F(t) = \lim_{\Delta\omega \to 0} \sum_{n=-\infty}^{\infty} \Delta\omega G(i\omega)e^{i\omega t} \tag{1.6}$$

or

$$F(t) = \int_{\omega=-\infty}^{\infty} G(i\omega)e^{i\omega t}\, d\omega \tag{1.7}$$

Since according to the definition of $G(i\omega)$, $G(i\omega) = \dfrac{C_n}{\omega_0} = C_n\dfrac{T}{2\pi}$, from Eq. (1.2) we obtain for $T \to \infty$

$$G(i\omega) = \frac{1}{2\pi}\int_{\tau=-\infty}^{\infty} F(\tau)e^{-i\omega\tau}\, d\tau \tag{1.8}$$

Carrying out the same limiting process as above in the expression (1.4), we have

$$x(t) = \int_{\omega=-\infty}^{\infty} \frac{G(i\omega)}{Z(i\omega)}e^{i\omega t}\, d\omega \tag{1.9}$$

Equation (1.8) is analogous to Eq. (1.2) which gives the coefficients of the complex Fourier series. Substituting (1.8) into Eq. (1.7), we obtain

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} F(\tau)e^{i\omega(t-\tau)}\, d\tau \tag{1.10}$$

Equation (1.10) is the complex form of the *Fourier integral theorem* given in Chapter VIII. Separating the real and imaginary parts of (1.10), we see that the latter part vanishes, since $\sin \omega(t - \tau)$ is an odd function of ω, and the integration is extended from $\omega = -\infty$ to $\omega = +\infty$. The real part of the integrand when integrated between $-\infty$ and 0 and between 0 and ∞ gives identical values. Therefore,

$$F(t) = \frac{1}{\pi}\int_{0}^{\infty} d\omega \int_{-\infty}^{\infty} F(\tau) \cos \omega(t - \tau)\, d\tau \tag{1.11}$$

which is in agreement with Eq. (6.4) of Chapter VIII, if the time coordinates are replaced by length coordinates and the variable ω is replaced by λ.

We notice a certain *duality* in the form of the expressions (1.7) and (1.8). Equation (1.8) may be considered as solution of Eq. (1.7), for the unknown function $G(i\omega)$ if $F(t)$ is given, whereas Eq. (1.7) gives $F(t)$ for a given $G(i\omega)$. We say that Eq. (1.8) solves Eq. (1.7) by *inversion.*

Let us now consider the physical significance of these results. According to (1.6) or (1.7) the force $F(t)$ is built up by superposition of forces which are harmonic functions of the time. Then the quantity $G(i\omega)\,d\omega$ is the contribution of those *harmonic components* of the force whose frequencies are included in the frequency interval between ω and $\omega + d\omega$.

Equation (1.9) gives the response of the system to the force $F(t)$, provided the system is at rest from $t = -\infty$ to $t = 0$. According to Eq. (1.9) the response is also composed of the superposition of pure harmonic motions, whose frequencies are distributed over the total frequency interval $-\infty < \omega < \infty$. The contribution of the harmonic components, whose frequencies are between ω and $\omega + d\omega$, is equal to $\dfrac{G(i\omega)}{Z(i\omega)}\,d\omega$.

We obtain, therefore, the following rule for the determination of the response of a system which is initially at rest and subjected to the action of a force starting at a certain instant: We resolve the force into harmonic components by means of the Fourier integral (1.8) and divide the amplitude of every component by the impedance. This gives us the amplitude distribution of the harmonic components of the response. If these harmonic components are summed by means of the integral (1.9), we obtain the final solution of the problem. The integral is zero for $t < 0$.

This method holds for any number of degrees of freedom and for electrical or other physical systems, provided they are governed by linear equations with constant coefficients and are damped.

To apply the method the integral in (1.8) must be convergent. If we separate the real and imaginary parts of (1.8), we obtain

$$G(i\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\tau)\cos\omega\tau\,d\tau + \frac{i}{2\pi}\int_{-\infty}^{\infty} F(\tau)\sin\omega\tau\,d\tau \qquad (1.12)$$

Since $F(\tau) = 0$ for $\tau < 0$, we can replace the lower limit by zero. It appears that the method is restricted to the case where the

integrals $\int_0^\infty F(\tau)\cos\omega\tau\,d\tau$ and $\int_0^\infty F(\tau)\sin\omega\tau\,d\tau$ converge. It is seen that, if $F(\tau)$ does not vanish for $\tau \to \infty$, the integrals are divergent. In some examples treated later we shall show how in certain cases this difficulty can be avoided.

If we are concerned with oscillations of a system composed of a finite number of harmonic components, we say that the discrete frequencies of the system constitute the *spectrum* of the system. In an analogous way we assign to the force $F(t)$ and the response $x(t)$ a *continuous spectrum* of harmonic components. The interval $d\omega$ is called a *spectrum interval.* In the case of a discrete spectrum we generally plot the amplitude as ordinate and use the discrete values of the frequency as abscissa. In the case of the continuous spectrum we plot the variable ω as abscissa and the real and imaginary parts of the amplitude distribution functions $G(i\omega)$ and $G(i\omega)/Z(i\omega)$, respectively, as ordinates.

Sound vibration in general contains so many discrete frequencies that they can be replaced by a continuous frequency spectrum. The quality of sound transmission depends on the difference between the amplitude and phase distributions of the functions $G(i\omega)$ and $G(i\omega)/Z(i\omega)$ in the audible range of the frequency ω.

We assumed for the deduction of (1.9) that the system is damped, *i.e.*, the roots of $Z(p) = 0$ are real negative quantities or complex quantities with negative real parts. If $Z(p) = 0$ has pure imaginary roots, *i.e.*, if the system is capable of carrying out undamped harmonic oscillations, the integral in Eq. (1.9) in general becomes an *improper integral.* In this case we can proceed in the following way: We introduce a damping term in the equations of the system, calculate the response of the damped system, and carry out the limiting process to zero damping in the resulting Eq. (1.9). This can be done by replacing the impedance $Z(p)$ by the function $Z(p + \beta)$, where β is a positive constant. It is seen that if the root p_k of $Z(p) = 0$ is pure imaginary, the corresponding root of $Z(p + \beta) = 0$ will be equal to $-\beta + p_k$, and will have a negative real part. Then we obtain the response to the force $F(t)$ from the equation

$$x(t) = \lim_{\beta \to 0} \int_{-\infty}^{\infty} \frac{G(i\omega)e^{i\omega t}}{Z(i\omega + \beta)}\,d\omega \tag{1.13}$$

This procedure will be clarified by examples in the next sections.

Example.—Let us assume that $F(t) = 0$ for $t < 0$ and $F(t) = e^{-\beta t}$ for $t > 0$ (Fig. 1.1*a*).

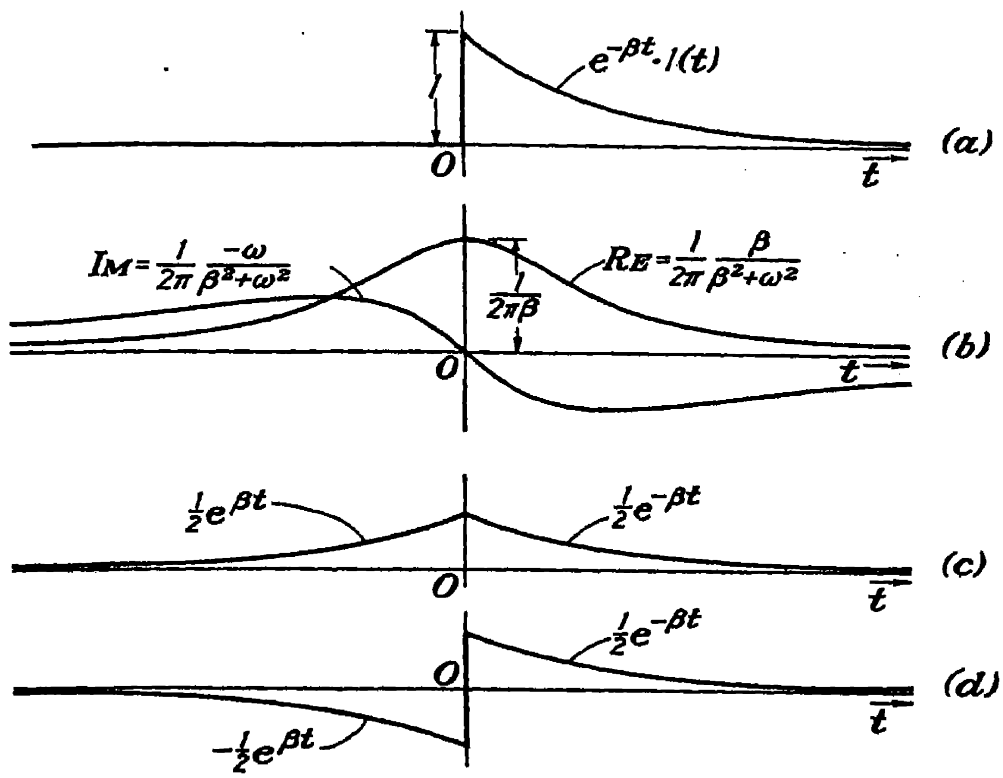

FIG. 1.1.—*a*. The function $F(t) = 0$ for $t < 0$, $F(t) = e^{-\beta t}$ for $t > 0$. (The notation used in the figure is explained in the next section.) *b*. The real and imaginary parts of the amplitude distribution $G(i\omega)$. *c* and *d*. The even and odd parts of $F(t)$.

The formula for the amplitude distribution $G(i\omega)$ gives [cf. Eq. (1.8)]

$$G(i\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{-i\omega t}\,dt = \frac{1}{2\pi}\int_{0}^{\infty} e^{-\beta t}e^{-i\omega t}\,dt$$

or (1.14)

$$G(i\omega) = \frac{1}{2\pi}\frac{1}{\beta + i\omega} = \frac{1}{2\pi}\frac{\beta - i\omega}{\beta^2 + \omega^2}$$

Figure 1.1*b* shows the real part of $G(i\omega)$, $\frac{\beta}{2\pi(\beta^2 + \omega^2)}$, and the imaginary part $\frac{-\omega}{2\pi(\beta^2 + \omega^2)}$ as functions of ω. Substituting $G(i\omega)$ in the Fourier integral (1.7), we obtain

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\beta - i\omega}{\beta^2 + \omega^2}(\cos \omega t + i \sin \omega t)\,d\omega \tag{1.15}$$

or, since the imaginary part vanishes,

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\beta}{\beta^2 + \omega^2} \cos \omega t \, d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega}{\beta^2 + \omega^2} \sin \omega t \, d\omega \qquad (1.16)$$

The first integral is an even function of the time. We can easily verify that it is equal to $\frac{1}{2}e^{\beta t}$ for $t < 0$ and $\frac{1}{2}e^{-\beta t}$ for $t > 0$ (Fig. 1.1c). The second integral is an odd function of t and is equal to $-\frac{1}{2}e^{\beta t}$ for $t < 0$ and $\frac{1}{2}e^{-\beta t}$ for $t > 0$ (Fig. 1.1d). Hence, the sum is equal to zero for $t < 0$ and equal to $e^{-\beta t}$ for $t > 0$.

2. The Unit-step and the Unit-impulse Functions.—The method given in the last section requires the evaluation of the function $G(i\omega)$ or of the *spectrum of the external action.* We carry out the calculation for two especially important cases.

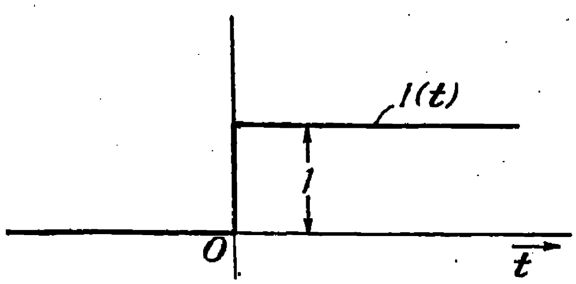

FIG. 2.1—The unit-step function.

Example 1.—Let us assume that

$$\begin{aligned} F(t) &= 0 \quad \text{for} \quad t < 0 \\ F(t) &= 1 \quad \text{for} \quad t \geqq 0 \end{aligned} \qquad (2.1)$$

This function is represented in Fig. 2.1. We shall denote such a function by the symbol $1(t)$ and call it *the unit-step function.* If we apply expression (1.8) to this function, we find that

$$G(i\omega) = \frac{1}{2\pi}\int_{0}^{\infty} e^{-i\omega\tau} \, d\tau \qquad (2.2)$$

is not a convergent integral. It is possible, however, to find a Fourier integral representation for the unit-step function by the following artifice already mentioned in the last section. We can apply the formula (1.8) without difficulty to the function $e^{-\beta t} 1(t)$ where β is real and positive. This function is obviously identical with the function $F(t)$ shown in Fig. 1.1a. We have

$$G(i\omega) = \frac{1}{2\pi}\int_{0}^{\infty} e^{-(\beta+i\omega)\tau} \, d\tau = \frac{1}{2\pi}\frac{1}{\beta + i\omega} \qquad (2.3)$$

Then, according to Eq. (1.7) we have the Fourier integral representation

$$e^{-\beta t} 1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\beta + i\omega} \, d\omega \qquad (2.4)$$

If we go to the limit $\beta \rightarrow 0$, we obtain

$$1(t) = \frac{1}{2\pi} \lim_{\beta \to 0} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\beta + i\omega}\, d\omega \tag{2.5}$$

If we put $\beta = 0$ in the integral (2.4), we obtain an improper integral. However, we may avoid this difficulty by separating the real and imaginary parts of Eq. (2.5) as we did in section 1 [cf. Eq. (1.16)]:

$$e^{-\beta t}\, 1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\beta \cos \omega t + \omega \sin \omega t}{\beta^2 + \omega^2}\, d\omega \tag{2.6}$$

By means of the substitution $\omega/\beta = \zeta$ we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{\beta \cos \omega t}{\beta^2 + \omega^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\zeta \frac{\cos \beta\zeta t}{1 + \zeta^2}$$

We can now substitute $\beta = 0$, and the value of the integral becomes

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\zeta}{1 + \zeta^2} = \frac{1}{2\pi} \Big[\tan^{-1} \zeta \Big]_{-\infty}^{\infty} = \frac{1}{2}$$

In the integral $\int_{-\infty}^{\infty} d\omega \frac{\omega \sin \omega t}{\beta^2 + \omega^2}$ we can substitute $\beta = 0$ without difficulty. Thus, we obtain

$$1(t) = \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin \omega t}{\omega}\, d\omega \tag{2.7}$$

The integral in Eq. (2.7) is known as *Dirichlet's integral.*

Example 2.—As a second example, we take a function $F(t)$ which is zero for $t < -\epsilon$ and $t > \epsilon$ and which in the interval between $-\epsilon$ and ϵ is equal to F (Fig. 2.2*a*). This is the case, for instance, if a constant force F acts during a certain time 2ϵ. We notice that $2F\epsilon$ is the *total impulse* transmitted to the system.

The Fourier representation of $F(t)$ is given by Eqs. (1.7) and (1.8). In the present example $F(t)$ is equal to zero if $t < -\epsilon$, whereas in section 1 it was assumed that $F(t) = 0$ for $t < 0$. However, it is obvious that a shift of the zero instant does not alter Eq. (1.8). Hence, we write

$$G(i\omega) = \frac{F}{2\pi} \int_{-\epsilon}^{\epsilon} e^{-i\omega\tau}\, d\tau = \frac{F}{2\pi} \left[-\frac{e^{-i\omega\tau}}{i\omega} \right]_{\tau=-\epsilon}^{\tau=\epsilon}$$

or

$$G(i\omega) = \frac{F}{\pi} \frac{\sin \omega\epsilon}{\omega} \tag{2.8}$$

Equation (2.8) gives the *amplitude distribution* of the force $F(t)$ and is plotted in Fig. 2.2*b*.

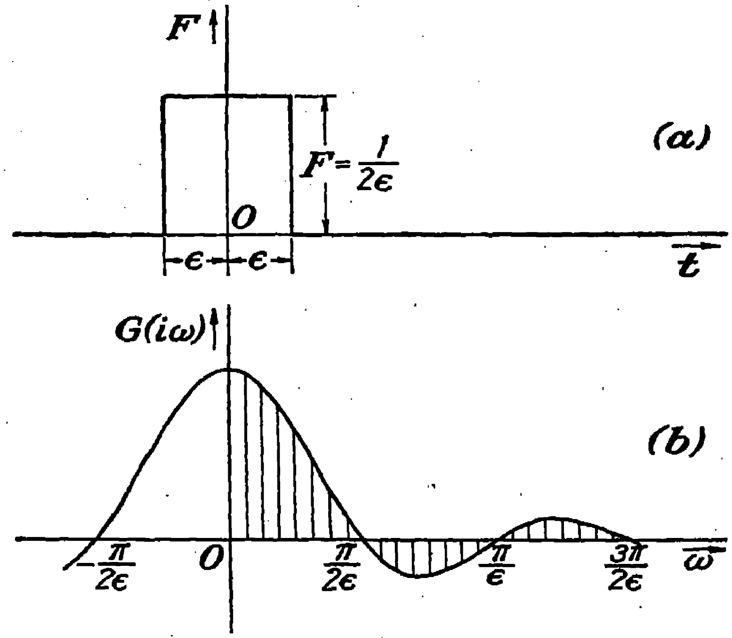

FIG. 2.2.—A function representing a constant force acting over a short time interval and the corresponding amplitude distribution.

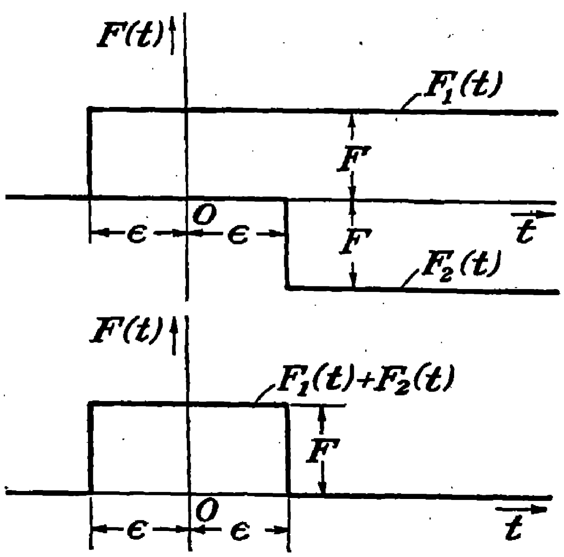

FIG. 2.3.—Superposition of a positive and a negative step F.

The function $F(t)$ is given by

$$F(t) = \frac{F}{\pi}\int_{-\infty}^{\infty} \frac{\sin \omega\epsilon}{\omega} e^{i\omega t}\, d\omega \qquad (2.9)$$

The same result can be obtained by the use of the principle of superposition. We superpose two functions $F_1(t)$ and $F_2(t)$ defined by

$F_1(t) = 0$ for $t < -\epsilon$, $F_1(t) = 1$ for $t > -\epsilon$, $F_2(t) = 0$ for $t < \epsilon$ and $F_2(t) = -1$ for $t > \epsilon$, so that $F_1 + F_2$ has the values specified above (Fig. 2.3). Obviously,

$$F_1 + F_2 = 1(t + \epsilon)F - 1(t - \epsilon)F \qquad (2.10)$$

Then, applying Eq. (2.7),

$$F_1(t) = \frac{F}{2} + \frac{F}{2\pi}\int_{-\infty}^{\infty} \frac{\sin \omega(t + \epsilon)}{\omega}\, d\omega$$

$$F_2(t) = -\frac{F}{2} - \frac{F}{2\pi}\int_{-\infty}^{\infty} \frac{\sin \omega(t - \epsilon)}{\omega}\, d\omega$$

$$F(t) = F_1(t) + F_2(t) = \frac{F}{\pi}\int_{-\infty}^{\infty} \frac{\cos \omega t \sin \omega\epsilon}{\omega}\, d\omega \tag{2.11}$$

which is equivalent to Eq. (2.9).

We remember that $F(t)$ is zero outside of the interval between $-\epsilon$ and ϵ. Hence, if we put $F = 1/2\epsilon$ in this interval and make ϵ tend to zero, we obtain a function which is zero everywhere, except for $t = 0$, where it has an infinite value such that the integral $\int_{-\epsilon}^{\epsilon} F(t)\, dt = 1$.

The function obtained in the limit $\epsilon = 0$ is called the *unit-impulse function* and is denoted by $S(t)$. It is seen from Eq. (2.10), upon substituting $F = \frac{1}{2}\epsilon$, that

$$S(t) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon}\,[1(t + \epsilon) - 1(t - \epsilon)] \tag{2.12}$$

We can express $S(t)$ also by the limit of a Fourier integral. From Eq. (2.9), it follows that

$$S(t) = \lim_{\epsilon \to 0} \frac{1}{2\pi\epsilon}\int_{-\infty}^{\infty} \frac{\sin \omega\epsilon\, e^{i\omega t}}{\omega}\, d\omega \tag{2.13}$$

3. Indicial Admittance and Response to Unit Impulse.—The theory of the transient state motion to be presented in the following sections applies the notion of the unit-step function and the unit impulse. The response of the system to arbitrary forces or impulses will be built up by superposition of responses to sudden increases of the force by unit steps or responses to unit impulses. The main problem is, therefore, the determination of these responses when the parameters of the system are given. There are two methods for this purpose: One method uses the expansions in Fourier integrals given in the last section; the other employs special so-called *operational* rules.

Before discussing these methods we give in this section a few examples to show how the responses to a unit step and a unit impulse can be determined in an elementary way by integration of the differential equations of the systems.

a. Single Mass with Elastic Restraint.—Let us assume that a mass m is attached to a fixed base by a spring whose spring constant is k (Fig. 3.1). Up to the instant $t = 0$, the mass is at rest; at the instant $t = 0$ a constant unit force begins to act

and persists for $t > 0$. Hence, the force is given by $1(t)$. Th equation of motion for $t > 0$ is

$$m \frac{d^2x}{dt^2} + kx = 1 \tag{3.1}$$

The initial conditions are $x = dx/dt = 0$ for $t = 0$. Th general solution of (3.1) is

$$x = \frac{1}{k} + C_1 \sin \sqrt{\frac{k}{m}}t + C_2 \cos \sqrt{\frac{k}{m}}t \tag{3.2}$$

The constants of integration determined by the initial condition are $C_1 = 0$, $C_2 = -1/k$. Then Eq. (3.2) becomes

$$x = \frac{1}{k}\left(1 - \cos \sqrt{\frac{k}{m}}t\right)$$

We call the function $x(t)$ the response to the unit-step functio $1(t)$ or the *indicial admittance* of our system. To indicate tha $x = 0$ for $t < 0$, we write

$$x = A(t) = \frac{1}{k}\left(1 - \cos \sqrt{\frac{k}{m}}t\right) 1(t) \tag{3.3}$$

The term *indicial admittance* was originated in the theory c electric networks to designate the current due to the sudde

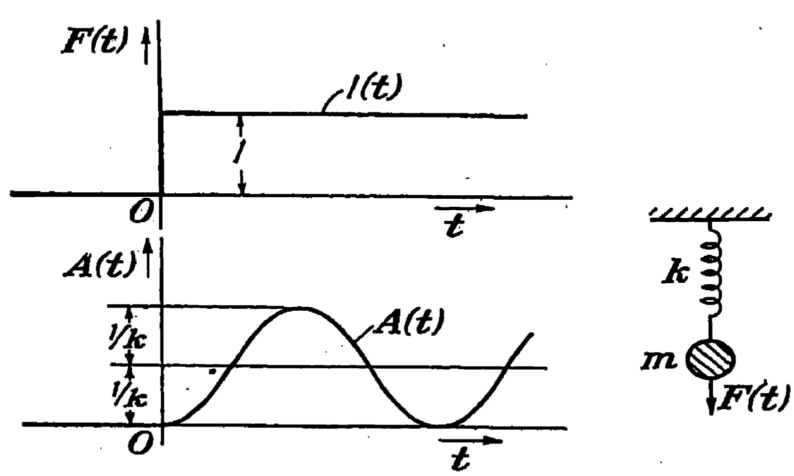

FIG. 3.1.—Indicial admittance. The response $x = A(t)$ of an elastical restrained mass to a unit force applied for $t > 0$.

application of a unit voltage. It is denoted by $A(t)$. In tl following discussion we shall use the same expression and notatic to designate also the displacement due to the application of sudden unit force on a mechanical system. The function $A($ for our case, as given by Eq. (3.3), is shown in Fig. 3.1.

Let us now calculate the response to a unit impulse. In this case the initial conditions are $x = 0$ and $m\dfrac{dx}{dt} = 1$ for $t = 0$. The quantity $m\dfrac{dx}{dt}$ is the impulse transmitted by a force acting over a small interval Δt. In fact, if $m\dfrac{d^2x}{dt^2} = F$, $\left[m\dfrac{dx}{dt}\right]_0^{\Delta t} = \displaystyle\int_0^{\Delta t} F\,dt$. Hence, if we go to the limit $\Delta t \to 0$, $F \to \infty$, the integral $\int_0^{\Delta t} F\,dt$ remains finite, and we obtain as the initial condition $m\dfrac{dx}{dt} = 1$ for $t = 0 + \epsilon$. The contribution of the force $-kx$ to the impulse is negligible during the time interval Δt.

With the initial conditions $x = 0$ and $dx/dt = 1/m$, we obtain

$$x = \frac{1}{\sqrt{mk}} \sin \sqrt{\frac{k}{m}}t \qquad (3.4)$$

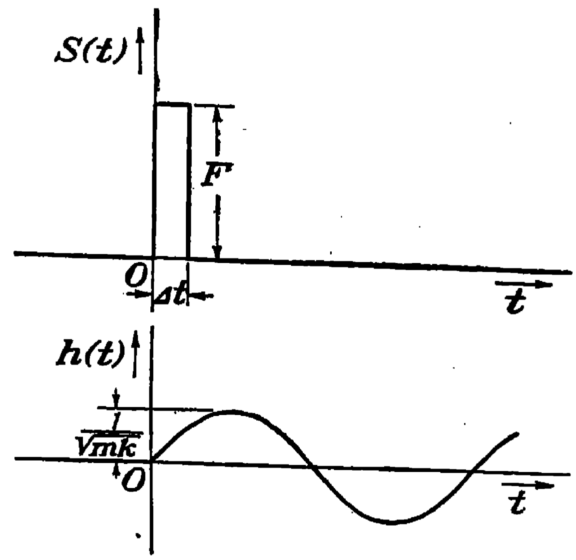

FIG. 3.2.—The response $x = h(t)$ of an elastically restrained mass to a unit impulse applied at $t = 0$.

We call $x(t)$ in this case *the response to a unit impulse.* In general, for the response to the unit impulse $S(t)$ we shall use in the following considerations the symbol $h(t)$ (cf. Fig. 3.2). Hence, in our case,

$$h(t) = \frac{1}{\sqrt{mk}} \sin \sqrt{\frac{k}{m}}t\, 1(t) \qquad (3.5)$$

b. Electric Circuit.—An electric circuit (Fig. 3.3) consists of a coil of inductance L, a resistance R, and a capacitor C. By closing the switch S a constant electromotive force can be suddenly impressed on the circuit. By quickly closing and opening the switch, a voltage impulse, so-called short rectangular pulse, can be sent through the circuit. If we consider as the unknown function the charge Q accumulated in the capacitor, the differential equation for Q is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E \qquad (3.6)$$

Let us now assume that $E = 0$ for $t < 0$ and $E = 1$ for $t >$
The initial conditions for $t = 0$ are $Q = 0$ and $dQ/dt = I =$
The general solution of (3.6) when $E = 1$ is then

$$Q = C + B_1 e^{-\alpha t} \sin \beta t + B_2 e^{-\alpha t} \cos \beta t \qquad (3$$

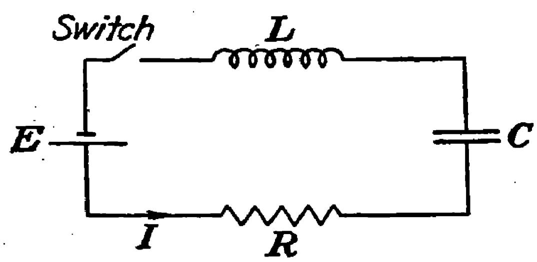

FIG. 3.3.—Electric circuit on which a constant electromotive force is suddenly impressed by closing a switch.

where $\alpha = R/2L$ and

$$\beta = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}.$$

The initial conditions lead to

$$Q = C\left(1 - e^{-\alpha t} \cos \beta t - \frac{\alpha}{\beta} e^{-\alpha t} \sin \beta t \qquad (3$$

We are especially interested in the current, which is equal to first derivative of Q, *viz.*,

$$I = \frac{1}{L\beta} e^{-\alpha t} \sin \beta t \qquad (3$$

If there is no capacitor, *i.e.*, if $C = \infty$, $\beta = \dfrac{iR}{2L}$ is imagin and $\sin \beta t$ becomes $i \sinh \dfrac{R}{2L} t$. Hence,

$$I = \frac{1}{R}(1 - e^{-\frac{R}{L}t}) \qquad (3$$

Equations (3.9) and (3.10) give the indicial admittance resulting from a sudden application of constant unit voltage.

To indicate that $A(t) = 0$ for $t < 0$, we write Eq. (3.9), example, in the form

$$A(t) = \frac{1}{L\beta} e^{-\alpha t} \sin \beta t \, 1(t) \qquad (3$$

Let us now consider the response to a unit voltage imp defined by $\int_0^{\Delta t} E\,dt = 1$. Since $E = 0$ for $t > 0$, the gen solution of Eq. (3.6) is in this case,

$$Q = A_1 e^{-\alpha t} \sin \beta t + A_2 e^{-\alpha t} \cos \beta t$$

The initial conditions are $Q = 0$, and from (3.6)

$$\int_0^{\Delta t} E\,dt = \left[L\frac{dQ}{dt}\right]_0^{\Delta t} = \left[L\frac{dQ}{dt}\right]_{0+\epsilon} = 1$$

If we determine A_1 and A_2 so that they comply with these conditions, we obtain

$$Q = \frac{1}{L\beta}e^{-\alpha t}\sin\beta t$$

and by differentiation

$$I = -\frac{\alpha}{L\beta}e^{-\alpha t}\sin\beta t + \frac{1}{L}e^{-\alpha t}\cos\beta t$$

We call this function $h(t)$ and write

$$h(t) = \left(-\frac{\alpha}{L\beta}e^{-\alpha t}\sin\beta t + \frac{1}{L}e^{-\alpha t}\cos\beta t\right) 1(t) \tag{3.12}$$

For the special case $C = \infty$,

$$h(t) = \frac{1}{L}e^{-\frac{Rt}{L}}\, 1(t) \tag{3.13}$$

Relation between $A(t)$ and $h(t)$.—We shall now show that, in general, $h(t)$ is the derivative of $A(t)$ with respect to the time. It was shown in section 2 of this chapter, Eq. (2.12), that

$$S(t) = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}[1(t+\epsilon) - 1(t-\epsilon)]$$

Using for 2ϵ the symbol Δt,

$$S(t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[1\left(t+\frac{\Delta t}{2}\right) - 1\left(t-\frac{\Delta t}{2}\right)\right] \tag{3.14}$$

By the principle of superposition, the same relation must prevail between the respective response functions. Hence,

$$h(t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[A\left(t+\frac{\Delta t}{2}\right) - A\left(t-\frac{\Delta t}{2}\right)\right] \tag{3.15}$$

or

$$h(t) = \frac{dA}{dt} \tag{3.16}$$

Figure 3.4 illustrates the derivation of the function $h(t)$ from the function $A(t)$. It is easily seen, for example, that the expression (3.5) can be obtained from (3.3), and (3.12) from (3.11), by differentiation.

The deduction leading to Eq. (3.16) is not valid for $t = 0$. The behavior of $A(t)$ and $h(t)$ in the neighborhood of $t = 0$ requires special consideration. For $t < 0$ both $A(t)$ and $h(t)$ are equal to zero. If $A(t)$ is discontinuous at $t = 0$, *i.e.*,

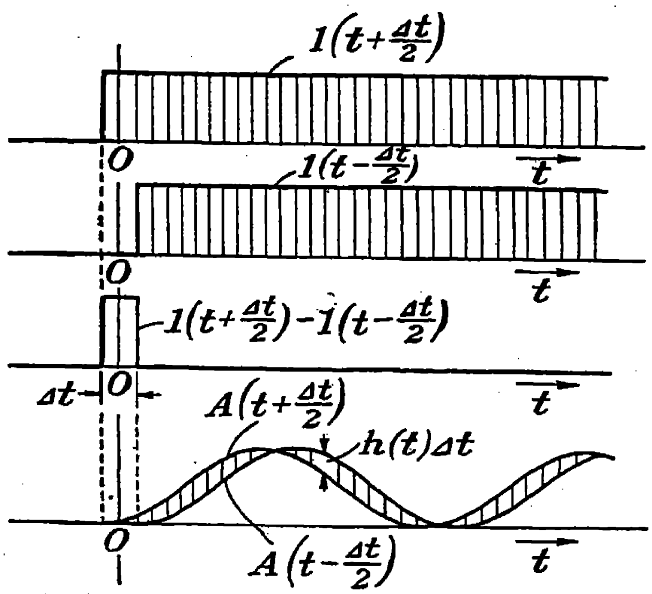

FIG. 3.4.—Derivation of the response to a unit impulse by superposition of two successive indicial admittances.

$$A(0) = \lim_{\epsilon \to 0} A(0 + \epsilon) \neq 0$$

the function $h(t)$ behaves for $t = 0$ similar to $S(t)$; its value jumps to infinity and then decreases to zero or a finite value, whereas $\lim\limits_{\epsilon \to 0} \int_{0-\epsilon}^{0+\epsilon} h(t)\, dt$ remains finite and is equal to $A(0)$.

Substituting $t = 0$ in (3.15) and taking Eq. (3.14) into account, we obtain

$$h(0) = A(0)S(0) \tag{3.17}$$

We can formally combine Eqs. (3.16) and (3.17) in one equation:

$$h(t) = A(0)S(t) + \frac{dA}{dt} \tag{3.18}$$

For $t \neq 0$, $S(t) = 0$ and Eq. (3.18) is reduced to Eq. (3.16); for $t = 0$, the first term is infinite; therefore, the second term is irrelevant and we obtain (3.17).

Consider, for example, a circuit consisting of a resistance R and a condenser of capacity C, and having zero inductance. The response to a suddenly applied unit voltage is

$$A(t) = \frac{1}{R} e^{-\frac{t}{CR}}$$

Hence $A(0) = 1/R$, *i.e.*, the unit voltage produces instantaneously a current of finite magnitude. The response to a unit impulse is, according to Eq. (3.18),

$$h(t) = \frac{1}{R}S(t) - \frac{1}{CR^2}e^{-\frac{t}{CR}}$$

The unit impulse, *i.e.*, an infinitesimally short infinite pulse, produces a flash of infinite current, which charges the condenser. This is expressed by the first term; the second term represents the gradual discharge of the condenser.

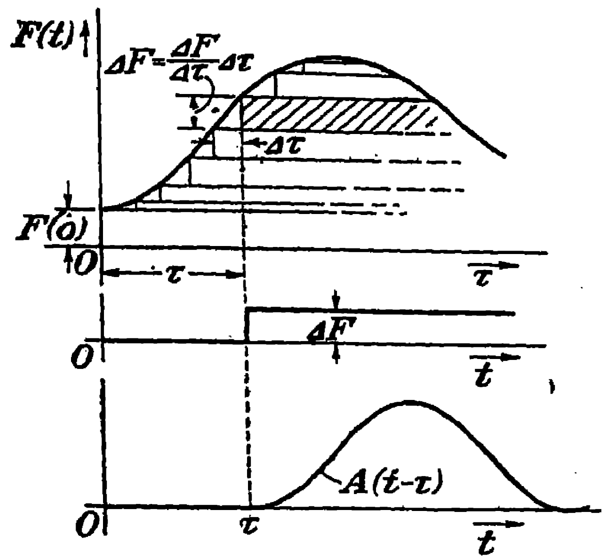

FIG. 4.1.—Illustration for Duhamel's integral.—Application of the indicial admittance.

4. Duhamel's Integral.—The response to a unit impulse and the indicial admittance can be used to obtain the response to a force that is an arbitrary function of time by application of the principle of superposition. We pointed out that the principle of superposition can be applied to any linear system, *e.g.*, to a system described by linear differential equations. However, the application of the following method is subjected to another restricting condition. Let us assume that a unit impulse is applied at the time τ, and that we observe the response at the instant t; then we also assume that this response is only a function of the elapsed time $t - \tau$ and does not depend either on t or on τ separately. This will be the case if the coefficients of the differential equations of the system are constants; it will not be the case, in general, if the coefficients are functions of the time.

Let us now consider the force applied to the system at the time τ to be given by a function $F(\tau)$. We advance along the time axis by steps $\Delta\tau$. Then, we approximate the action of the force F by application of steps ΔF at equidistant instants $0, \Delta\tau, 2\Delta\tau, 3\Delta\tau, \ldots$ (Fig. 4.1). If $F = 0$ for $\tau < 0$, the step ΔF corresponding to $\tau = 0$ is equal to $F(0)$. Then, if the indicial admittance is $A(t)$, we obtain.

$$x(t) = F(0)A(t) + \sum_{\tau = \Delta t}^{\tau = t} \Delta F A(t - \tau)$$

If we write $\Delta F = \dfrac{\Delta F}{\Delta\tau}\Delta\tau$, we have

$$x(t) = F(0)A(t) + \sum_{\tau=\Delta\tau}^{\tau=t} \frac{\Delta F}{\Delta\tau} A(t-\tau)\,\Delta\tau \tag{4.1}$$

or in the limit $\Delta\tau \to 0$,

$$x(t) = F(0)A(t) + \int_0^t \frac{dF}{d\tau} A(t-\tau)\,d\tau \tag{4.2}$$

The expression on the right side of Eq. (4.2) is known under the name of the French mathematician Duhamel. An equivalent form of *Duhamel's integral* is obtained by integrating by parts in Eq. (4.2). We obtain

$$x(t) = F(t)A(0) + \int_0^t F(\tau)A'(t-\tau)\,d\tau \tag{4.3}$$

If $A(0) = 0$, the first term in Eq. (4.3) drops out, and [cf. Eq. (3.18)] $A'(t-\tau) = h(t-\tau)$, where $h(t)$ is the response to a unit impulse. Then the second term can be interpreted in the following way: We approximate the action of the force F by discrete impulses equal to $F\,\Delta t$ (Fig. 4.2). The response at the instant t will be equal to the sum of the responses to all impulses which have taken place from $\tau = 0$ to $\tau = t$. Hence, if we denote the response by $x(t)$, then

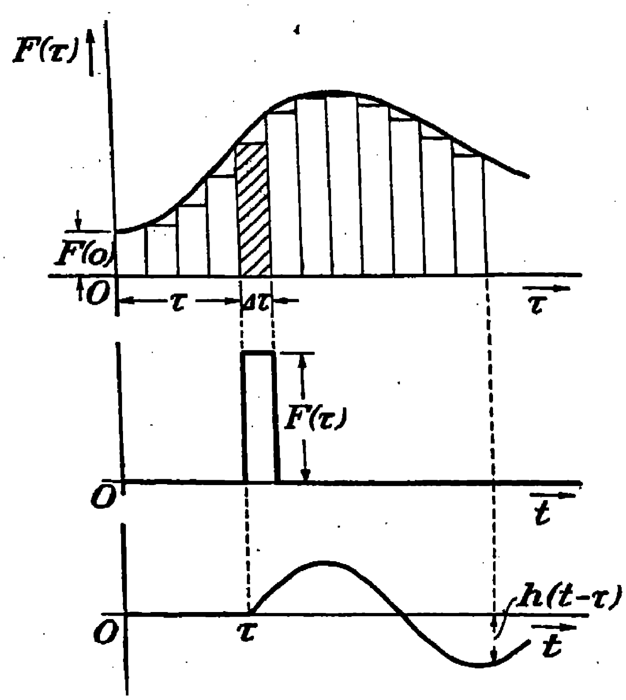

FIG. 4.2.—Illustration for Duhamel's integral.—Application of the response to a unit impulse.

$$x(t) = \sum_{\tau=0}^{\tau=t} F(\tau)\,\Delta\tau\, h(t-\tau) \tag{4.4}$$

or if $\Delta\tau \to 0$,

$$x(t) = \int_0^t F(\tau)\,h(t-\tau)\,d\tau \tag{4.5}$$

Let us apply Duhamel's integral to the mechanical system considered in the last section. Let us assume that the action of a periodic force whose frequency is equal to the frequency of the

free oscillation $\omega_0 = \sqrt{k/m}$ starts at the instant $\tau = 0$. We write

$$F(\tau) = F_0 \sin \omega_0\tau \, 1(\tau)$$

Then the response at the instant t is given, using Eq. (4.5), by

$$x = F_0 \int_0^t \sin \omega_0\tau \, h(t - \tau) \, d\tau$$

and substituting $h(t - \tau)$ from Eq. (3.5)

$$x = \frac{F_0}{\sqrt{mk}} \int_0^t \sin \omega_0\tau \sin \omega_0(t - \tau) \, d\tau$$

Carrying out the integration,

$$x = \frac{F_0}{2m\omega_0} \left(\frac{\sin \omega_0 t}{\omega_0} - t \cos \omega_0 t \right)$$

This result can be obtained from Eq. (6.10), Chapter IV, by direct computation of the constants of integration if we put $x = 0$ and $dx/dt = 0$ for $t = 0$.

5. Application of Fourier Integrals to the Determination of the Response to Unit Step. Bromwich's Integral.—Let us return now to the problem of the determination of the indicial admittance $A(t)$. The elementary method used in section 3 becomes very cumbersome for complicated systems. However, the methods explained in section 1 show a direct way for the calculation of the response to any force or other external action that can be expanded in a Fourier integral. The expansion for the unit-step function $1(t)$ was given in section 2. We shall now apply the equations of these two sections.

We found for the unit-step function the expression [Eq. (2.5)]:

$$1(t) = \frac{1}{2\pi} \lim_{\beta \to 0} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\beta + i\omega} \, d\omega \tag{5.1}$$

If we denote the impedance of the system by $Z(i\omega)$, the response $A(t)$ to the force $1(t)$ is given by [cf. Eq. (1.9)]

$$A(t) = \frac{1}{2\pi} \lim_{\beta \to 0} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\beta + i\omega)Z(i\omega)} \, d\omega \tag{5.2}$$

Consider, for example, the response of an electric circuit of resistance R and inductance L to a constant voltage of unit magnitude applied at $t = 0$ and kept constant for $t > 0$. The impedance of the system is $Z(p) = Lp + R$, and, therefore, the response is given by

$$A(t) = \lim_{\beta \to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\beta + i\omega)(Li\omega + R)}\, d\omega \tag{5.3}$$

By expansion in partial fractions we obtain

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\beta + i\omega)(Li\omega + R)}\, d\omega = \frac{1}{R - L\beta} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\beta + i\omega}\, d\omega - \frac{L}{R - L\beta} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{Li\omega + R}\, d\omega \tag{5.4}$$

According to Eq. (2.4), we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\beta + i\omega}\, d\omega = e^{-\beta t}\, 1(t)$$

On the other hand,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{R + Li\omega}\, d\omega = \frac{1}{2\pi} \frac{1}{L} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\frac{R}{L} + i\omega}\, d\omega = \frac{1}{L} e^{-\frac{R}{L}t}\, 1(t)$$

Hence, substituting these expressions into Eq. (5.4), we obtain

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\beta + i\omega)(Li\omega + R)}\, d\omega = \frac{e^{-\beta t}}{R - L\beta}\, 1(t) - \frac{e^{-\frac{R}{L}t}}{R - L\beta}\, 1(t)$$

Proceeding to the limit $\beta \to 0$, we find the response

$$A(t) = \frac{1}{R}\left(1 - e^{-\frac{R}{L}t}\right) 1(t) \tag{5.5}$$

It is seen that this result is in accordance with Eq. (3.10).

If the system is not damped, we have to apply Eq. (1.13), and Eq. (5.2) becomes

$$A(t) = \frac{1}{2\pi} \lim_{\beta \to 0} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\beta + i\omega) Z(\beta + i\omega)}\, d\omega \tag{5.6}$$

In the literature on the subject $i\omega$ is replaced by the symbol p, and Eqs. (5.2) and (5.6) are written in the same form, *viz.*,

$$A(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{pt}}{pZ(p)}\,dp \tag{5.7}$$

This relation is known as *Bromwich's integral formula.*

The notation used in Eq. (5.7) is based on more advanced methods for the evaluation of improper integrals by means of integration of analytic functions of a complex variable.

Equations (5.2) and (5.6) yield explicit values for $A(t)$ if we know the impedance. However, the evaluation of the integrals by elementary methods is generally rather tedious. Their efficient treatment requires the use of the more advanced theory of complex variables and integration in the complex plane. In the next section we shall find another relation between $Z(p)$ and $A(t)$. This relation does not involve complex integration and will help us to establish simple *operational rules* which in many cases lead directly to the determination of the response to a unit step.

6. Carson's Integral Equation.—In Chapter IX we considered the response of linear systems to a force F that is proportional to $e^{i\omega t}$. We found that the following procedure leads to particular solutions of the equations which govern the system: In the expression for the force we replaced $e^{i\omega t}$ by e^{pt}, the differential operator d/dt by p, the integral operator $\int^t dt$ by $1/p$. By this substitution we obtained algebraic linear equations for the unknown quantity sought. The solution of the linear equations could be written in the form $x = F/Z(p)$, where $Z(p)$ is the impedance. Substituting $p = i\omega$, we obtained a periodic solution for x, proportional to $e^{i\omega t}$.

If we assume that the force F is equal to e^{pt} where p is a positive real quantity, then the formal procedure remains the same as described above, and we obtain a particular solution of the form $x = e^{pt}/Z(p)$, which satisfies the equations governing the system. Let us now assume that $F = 1(t)e^{pt}$, *i.e.*, F is equal to zero for $t < 0$, and $F = e^{pt}$ for $t > 0$. The response to this force for $t > 0$ will have the form:

$$x(t) = G(t) + \frac{e^{pt}}{Z(p)} \tag{6.1}$$

where $G(t)$ is a solution of the homogeneous differential or integral equation associated with the problem, *i.e.*, $G(t)$ represents free oscillations of the system. On the other hand, if we start to apply the force e^{pt} at the instant $t = 0$ and calculate the response by Duhamel's integral, we find

$$G(t) + \frac{e^{pt}}{Z(p)} = A(t) + \int_0^t pe^{p\tau}A(t - \tau)\, d\tau \tag{6.2}$$

We introduce $t - \tau$ as a new variable ζ. Then $d\tau = -d\zeta$, and

$$G(t) + \frac{e^{pt}}{Z(p)} = A(t) + p\int_0^t e^{p(t-\zeta)}A(\zeta)\, d\zeta$$

The integral contains the factor e^{pt}. Dividing both sides of the equation by this factor, we find

$$\frac{G(t)}{e^{pt}} + \frac{1}{Z(p)} = \frac{A(t)}{e^{pt}} + p\int_0^t e^{-p\zeta}A(\zeta)\, d\zeta \tag{6.3}$$

Now $G(t)$ and $A(t)$ remain finite when $t \to \infty$, provided the system is not unstable, *i.e.*, its free oscillations are not of the type that have an increasing amplitude. Hence, proceeding to $t \to \infty$ in Eq. (6.3), $\frac{G(t)}{e^{pt}} \to 0$, $\frac{A(t)}{e^{pt}} \to 0$, and we obtain the relation

$$\frac{1}{Z(p)} = p\int_0^\infty e^{-p\zeta}A(\zeta)\, d\zeta \tag{6.4}$$

Equation (6.4) holds also if p is a complex quantity, provided its real part is positive. If, for example, $p = \alpha + i\beta$, the force e^{pt} will have the form $e^{\alpha t}(\cos \beta t + i \sin \beta t)$ and all conclusions we used above hold as long as $\alpha > 0$. We also note that for a damped system $G(t)$ and $A(t)$ vanish for $t \to \infty$, and, therefore, the relation

$$\frac{e^{pt}}{Z(p)} = \int_0^t pe^{p\tau}A(t - \tau)\, d\tau$$

holds also for $t \to \infty$ if e^{pt} remains finite, *i.e.*, if p is a pure imaginary quantity, for example, $p = i\omega$. Hence, in the case of a damped system, Eq. (6.4) holds as long as p has no negative real part.

We shall replace the variable ζ in the definite integral (6.4) by τ, and write

$$\frac{1}{Z(p)} = p\int_0^\infty e^{-p\tau}A(\tau)\, d\tau \tag{6.5}$$

Equation (6.5) is a so-called *linear integral equation of the first kind* for the unknown function $A(\tau)$, provided the left side is a given function of p. It is known in the literature as *Carson's integral equation.*

The function $\Lambda(p)$ defined by the definite integral

$$\Lambda(p) = \int_0^\infty e^{-p\tau} f(\tau)\, d\tau \tag{6.6}$$

is sometimes called the *Laplace transform* of the function $f(\tau)$. Hence, to solve Eq. (6.5) for $A(\tau)$ means to find the function $f(\tau)$ whose Laplace transform is $1/pZ(p)$.

Bromwich's integral formula actually solves this mathematical problem. However, as we mentioned before, the evaluation of integrals of the type (5.2) requires rather lengthy calculations or the use of complex variables. Therefore, for engineering purposes it is often preferable to apply methods that do not involve this type of analytical work. The rules of the so-called *operational calculus,* first invented by Heaviside, permit the unknown function in Eq. (6.5) to be calculated directly in many cases. The heuristical discovery of such rules was a great and useful achievement of the famous British electrical engineer. He was not able to give mathematically rigorous proofs for all his formulas, but mathematicians later found them to be correct.

Example.—Let us verify that the response of an electric circuit of resistance R and inductance L to a unit-step voltage satisfies Carson's equation. The response $A(t)$ was found to be [Eq. (3.10)]

$$A(t) = \frac{1}{R}\left(1 - e^{-\frac{Rt}{L}}\right) 1(t)$$

Substituting this expression into Eq. (6.5), we have

$$\frac{p}{R}\int_0^\infty e^{-p\tau}\left(1 - e^{-\frac{R}{L}\tau}\right) d\tau = \frac{1}{Lp + R} \tag{6.7}$$

The right side of Eq. (6.7) is equal to the reciprocal value of the impedance $Z(p) = Lp + R$ of the circuit, as is required by Eq. (6.5).

7. The Notion of Operators.—We have shown in the last section that the determination of the response $A(t)$ to a unit step requires the solution of an integral equation of the form:

$$\frac{1}{Z(p)} = p\int_0^\infty e^{-pt} A(t)\, dt \tag{7.1}$$

where p is either a positive real quantity or a complex quantity with positive real part. The determination of $A(t)$ is merely a mathematical problem, independent of the physical significance and derivation of Eq. (7.1). However, the physically inspired mind of Heaviside was not satisfied with the treatment of physical problems from a purely mathematical point of view. He tried to visualize the meaning of the mathematical operations. To follow the general lines of his reasoning, we start with the steady-state motion. We remember that the response to a periodic force is obtained by dividing the imposed force by the impedance $Z(p)$. Let us denote the reciprocal value $1/Z(p)$ by $\Omega(p)$; then we obtain the response to a force by multiplying it by $\Omega(p)$. The multiplication by $\Omega(p)$ transforms the force into the response. Heaviside suggested that to obtain the response we perform an operation or impose the operator $\Omega(p)$ on the force. Let us take, for instance, the simplest electric circuit, whose impedance is a constant resistance R. Then $\Omega(p) = 1/R$ and the operation that transforms the electromotive force E into the current I consists of a mere multiplication by $1/R$. Then let us take the next simplest case of a coil whose impedance is equal to Lp. Let us assume that $L = 1$, then the impedance is $Z(p) = p$ and $\Omega(p) = 1/p$. Therefore, the current $I = E/p = \Omega(p)E$. Let us see now what the operation $\Omega(p) = 1/p$ means physically. We know that $L\,dI/dt = E$, or with $L = 1$, $dI/dt = E$. Therefore, the division by p is equivalent to an integration with respect to time; if $E = 0$ at $t = 0$, then $I = E/p = \int_0^t E\,dt$.

Let us transfer these ideas to the theory of the transient state. We remember that $A(t)$ is defined as the response to the unit step $1(t)$. Consequently, if the step is imposed on a system whose impedance is $Z(p)$, we say that *the operator* $\Omega(p) = 1/Z(p)$ *transforms* $1(t)$ *into* $A(t)$. We express this idea by the symbolic equation

$$\Omega(p)1(t) = A(t) \tag{7.2}$$

The problem now is to learn the meaning of the operation $\Omega(p)$ when it is no longer applied to a *periodic force* but is applied to a *suddenly applied unit force*. We derive the meaning of $\Omega(p)$ from the simple operators p and $1/p$ by the *operational rules* which we shall present now. The rules are verified by Carson's

integral relation, since the symbolic equation (7.2) is equivalent to Carson's integral relation (7.1); therefore, a solution $A(t)$ is correct if it satisfies Carson's relation.

8. Operational Rules.

I. *Addition of Operators.*—If

$$\begin{aligned} \Omega_1(p)1(t) &= A_1(t) \\ \Omega_2(p)1(t) &= A_2(t) \end{aligned} \tag{8.1}$$

and we write $\Omega(p) = \Omega_1(p) + \Omega_2(p)$; then,

$$\Omega(p)1(t) = A_1(t) + A_2(t) \tag{8.2}$$

We prove (8.2) by using Carson's equation (7.1). The integral relations equivalent to (8.1) are

$$\Omega_1(p) = p\int_0^\infty e^{-pt}A_1(t)\,dt$$
$$\Omega_2(p) = p\int_0^\infty e^{-pt}A_2(t)\,dt$$

Therefore, $\Omega_1(p) + \Omega_2(p) = p\int_0^\infty e^{-pt}[A_1(t) + A_2(t)]\,dt$, which is equivalent to the symbolic equation (8.2).

II. *Multiplication by a Constant.*—If $\Omega(p)1(t) = A(t)$ then $a\Omega(p)1(t) = aA(t)$, where a is a constant. This follows from the linearity of Eq. (7.1).

III. *Multiplication by p.*—If

$$\Omega(p)1(t) = A(t) \tag{8.3}$$

and $A(0) = 0$, then,

$$p\Omega(p)1(t) = \frac{dA(t)}{dt} \tag{8.4}$$

This follows from (7.1) in the following way: Assume that

$$\Omega(p) = p\int_0^\infty e^{-pt}A(t)\,dt$$

then,

$$p\Omega(p) = p\int_0^\infty pe^{-pt}A(t)\,dt$$

Integrating by parts, we obtain

$$p\Omega(p) = p\Big[-e^{-pt}A(t)\Big]_0^\infty + p\int_0^\infty e^{-pt}\frac{dA}{dt}\,dt$$

The expression in the brackets vanishes since $e^{-pt} \to 0$ for $t \to \infty$, and it was assumed that $A(t)$ vanishes for $t = 0$. The rest is equivalent to the symbolic equation (8.4).

If $A(0) \neq 0$, we obtain

$$p\Omega(p) = pA(0) + p\int_0^\infty e^{-pt}\frac{dA}{dt}\,dt$$

This equation is equivalent to

$$[\Omega(p) - A(0)]p1(t) = \frac{dA}{dt} \tag{8.5}$$

The meaning of the operator $A(0)p$ will be discussed in the next section.

IV. *Division by p.*—If

$$\Omega(p)1(t) = A(t) \tag{8.6}$$

then,

$$\frac{1}{p}\Omega(p)1(t) = \int_0^t A(t)\,dt \tag{8.7}$$

This follows again from

$$\Omega(p) = p\int_0^\infty e^{-pt}A(t)\,dt$$

If we divide both sides by p, we have

$$\frac{\Omega(p)}{p} = \int_0^\infty e^{-pt}A(t)\,dt = \left[-e^{-pt}\int^t A(t)\,dt\right]_0^\infty + p\int_0^\infty dt\,e^{-pt}\int_0^t A(t)\,dt$$

The expression in the brackets vanishes, since $e^{-pt} \to 0$ for $t \to \infty$ and $A(t)$ is finite. The rest is equivalent to Eq. (8.7).

V. *Multiplication of p by a Constant.*—If

$$\Omega(p)1(t) = A(t) \tag{8.8}$$

then,

$$\Omega(ap)1(t) = A\left(\frac{t}{a}\right) \tag{8.9}$$

where a is a positive real constant.

We start with Eq. (7.1)

$$\Omega(p) = p\int_0^\infty e^{-pt}A(t)\,dt$$

Then, substituting a new variable p' defined by $p = ap'$,

$$\Omega(ap') = ap'\int_0^\infty e^{-ap't}A(t)\,dt$$

Now we substitute $at = t'$ and obtain

$$\Omega(ap') = p'\int_0^\infty e^{-p't'}A\left(\frac{t'}{a}\right)dt'$$

If we change the notations from p' to p and from t' to t,

$$\Omega(ap) = p\int_0^\infty e^{-pt}A\left(\frac{t}{a}\right)dt$$

in accordance with (8.9).

This rule means that if we change the time scale of all frequencies, the time scale of the response is reduced correspondingly.

VI. *Addition of a Constant to p.*—If

$$\Omega(p)1(t) = A(t) \tag{8.10}$$

then,

$$\frac{p}{p+a}\Omega(p+a)\,1(t) = e^{-at}A(t) \tag{8.11}$$

where a is an arbitrary constant whose real part is positive or zero. This restriction must be imposed since Carson's integral equation was proved under the assumption that the real part of the factor multiplying t in the exponent is not negative.

According to Carson's integral (7.1),

$$\frac{1}{Z(p)} = p\int_0^\infty e^{-pt}A(t)\,dt$$

Hence, after replacing p by $p + a$,

$$\frac{1}{(p+a)Z(p+a)} = \int_0^\infty e^{-pt-at}A(t)\,dt$$

or

$$\frac{p}{p+a}\,\frac{1}{Z(p+a)} = p\int_0^\infty e^{-pt}[e^{-at}A(t)]\,dt$$

This is expressed symbolically by (8.11).

9. Some Fundamental Operators. *a. The Operator* $1/p^n$.—If we apply the operator 1 to $1(t)$ we obtain $1(t)$ since $p\int_0^\infty e^{-pt}1(t)\,dt = 1$. Then applying rule IV repeatedly (Fig. 9.1), we obtain

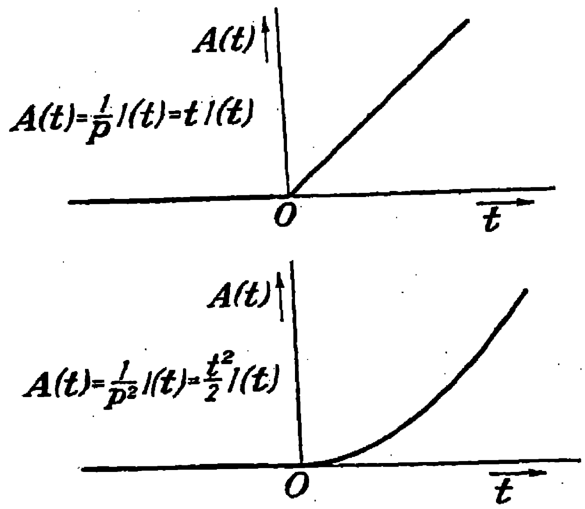

FIG. 9.1.—The functions determined by the operational expressions $\frac{1}{p}1(t)$ and $\frac{1}{p^2}1(t)$.

$$\frac{1}{p}\,1(t) = \int_0^t 1(t)\,dt = t\,1(t)$$

$$\frac{1}{p^2}\,1(t) = \int_0^t t\,1(t)\,dt = \frac{t^2}{1\cdot 2}\,1(t)$$

and, in general,

$$\frac{1}{p^n}\,1(t) = \frac{t^n}{n!}\,1(t) \tag{9.1}$$

b. The Operator $\dfrac{1}{p+a}$.—Applying rule VI to the equation

$$1\,1(t) = 1(t) \tag{9.2}$$

we obtain (Fig. 9.2*a*)

$$\frac{p}{p+a}\,1(t) = e^{-at}\,1(t) \tag{9.3}$$

Let us assume that a is purely imaginary, say $a = i\omega_0$, then we have

$$\frac{1}{p+a} = \frac{p - i\omega_0}{p^2+\omega_0^2}; \qquad e^{-at} = \cos\omega_0 t - i\sin\omega_0 t$$

and separating the real and imaginary parts on both sides of Eq. (9.3), we obtain two operational formulas, which are used very frequently (Fig. 9.2*b* and *c*),

$$\begin{aligned} \frac{p^2}{p^2+\omega_0^2}\,1(t) &= \cos\omega_0 t\,1(t) \\ \frac{\omega_0 p}{p^2+\omega_0^2}\,1(t) &= \sin\omega_0 t\,1(t) \end{aligned} \tag{9.4}$$

Using rule IV, we obtain from Eq. (9.3)

$$\frac{1}{p+a}\,1(t) = \int_0^t e^{-at}\,dt = \frac{1}{a}(1 - e^{-at})\,1(t) \tag{9.5}$$

If $a = i\omega_0$, we obtain

$$\begin{aligned} \frac{p}{p^2 + \omega_0^2}\, 1(t) &= \frac{1}{\omega_0} \sin \omega_0 t\, 1(t) \\ \frac{\omega_0}{p^2 + \omega_0^2}\, 1(t) &= \frac{1 - \cos \omega_0 t}{\omega_0}\, 1(t) \end{aligned} \tag{9.6}$$

The reader will easily verify the results of section 3, especially Eqs. (3.3) and (3.5) by this method. It is seen that the opera-

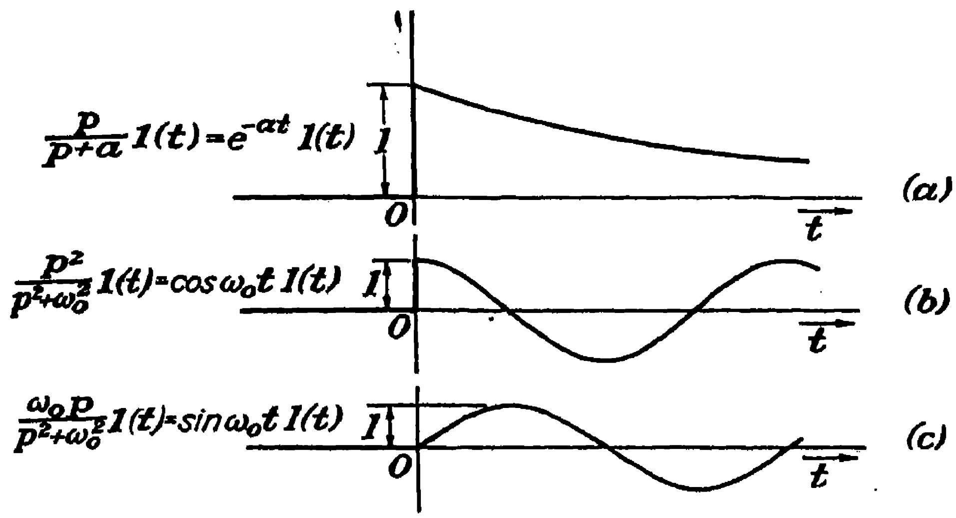

FIG. 9.2.—The functions determined by the operational expressions $\frac{p}{p + a} 1(t)$, $\frac{p^2}{p^2 + \omega_0^2} 1(t)$, $\frac{\omega_0 p}{p^2 + \omega_0^2} 1(t)$.

tional method gives directly the particular solutions that fit the initial conditions.

c. The Operator p.—The solution of the symbolic equation

$$p 1(t) = A(t) \tag{9.7}$$

has to satisfy the relation

$$p = p \int_0^\infty e^{-pt} A(t)\, dt \tag{9.8}$$

It is seen that the unit impulse function $S(t)$ satisfies this equation. For if we write

$$\int_0^\infty e^{-pt} A(t)\, dt = \int_{0-\epsilon}^{0+\epsilon} S(t) e^{-pt}\, dt + \int_{0+\epsilon}^\infty S(t) e^{-pt}\, dt$$

the first integral is equal to $\int_{0-\epsilon}^{0+\epsilon} S(t)\, dt = 1$ and the second integral vanishes, since $S(t) = 0$ for $t > 0 + \epsilon$. Hence we have

$$p 1(t) = S(t) \tag{9.9}$$

We are now able to interpret rule III as given in Eq. (8.5). Substituting for the operator p according to Eq. (9.9), this rule becomes

$$p\Omega(p)1(t) = A(0)S(t) + \frac{dA}{dt} \tag{9.10}$$

or, according to Eq. (3.18),

$$p\Omega(p)1(t) = h(t) \tag{9.11}$$

where $h(t)$ is the response to a unit impulse. Equation (9.11) can be used for calculating $h(t)$ directly from $Z(p)$.

10. Expansion Methods of the Operational Calculus.—The rules I to VI furnish two efficient methods for handling more complicated operators.

a. We expand $\Omega(p)$ in a series of ascending powers of $1/p$,

$$\Omega(p) = a_0 + \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots \tag{10.1}$$

Then applying rules I and II we have

$$\Omega(p)1(t) = \sum_{k=0}^{\infty} \frac{a_k}{p^k} 1(t) \tag{10.2}$$

Then applying Eq. (9.1), we obtain

$$A(t) = \left(a_0 + \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots\right) 1(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} 1(t) \tag{10.3}$$

Let us apply this equation to the response of a circuit whose resistance is R and whose inductance is L, assuming that a unit constant voltage is applied starting from $t = 0$. The impedance of the circuit is $Z(p) = Lp + R$. Expanding $\Omega(p) = \dfrac{1}{Lp + R}$ in a series of ascending powers of $1/p$, we have

$$\Omega(p) = \frac{1}{R} \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{R}{L}\right)^k \frac{1}{p^k} \tag{10.4}$$

Applying Eq. (10.3), we obtain

$$A(t) = \frac{1}{R}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\left(\frac{Rt}{L}\right)^k 1(t) \tag{10.5}$$

Using the known expansion for the exponential function,

$$e^{-\frac{Rt}{L}} = 1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\left(\frac{Rt}{L}\right)^k$$

Eq. (10.5) becomes

$$A(t) = \frac{1}{R}\left(1 - e^{-\frac{Rt}{L}}\right) 1(t) \tag{10.6}$$

This result is in accordance with the Eq. (3.10) obtained in section 3 by means of integration of the differential equation and with Eq. (9.5) of the last section.

b. Let us now assume that $\Omega(p) = 1/Z(p)$ can be expressed as the quotient of two polynomials:

$$\Omega(p) = \frac{1}{Z(p)} = \frac{M(p)}{N(p)} \tag{10.7}$$

where the degree of $M(p)$ is lower than the degree of $N(p)$. If $N(p)$ has n roots, all distinct, $p_1, p_2, p_3, \ldots, p_n$, it is known that $M(p)/N(p)$ can be expanded in the form:

$$\frac{M(p)}{N(p)} = \sum_{k=1}^{n}\frac{B_k}{p - p_k} \tag{10.8}$$

where

$$B_k = \frac{M(p_k)}{N'(p_k)} \tag{10.9}$$

and $N'(p)$ is the first derivative of $N(p)$.

We call the sum in Eq. (10.8) an *expansion in partial fractions*.

Equations (10.8) and (10.9) can be verified in the following way: If the coefficient of p^n in $N(p)$ is denoted by N_0, the polynomial $N(p)$ can be written in the form:

$$N(p) = N_0(p - p_1)(p - p_2) \cdots (p - p_n) \tag{10.10}$$

Multiplying both sides of Eq. (10.8) by $N(p)$, we obtain

$$\begin{aligned} M(p) = N_0[B_1(p - p_2)(p - p_3) \cdots (p - p_n) \\ + B_2(p - p_1)(p - p_3) \cdots (p - p_n) + \cdots \\ + B_n(p - p_1)(p - p_2) \cdots (p - p_{n-1})] \end{aligned} \tag{10.11}$$

Now it is seen by differentiation of (10.10) that the derivative of $N(p)$ is equal to

$$\begin{aligned} N'(p) = N_0[(p - p_2)(p - p_3) \cdots (p - p_n) \\ + (p - p_1)(p - p_3) \cdots (p - p_n) + \cdots \\ + (p - p_1)(p - p_2) \cdots (p - p_{n-1})] \end{aligned} \tag{10.12}$$

Substituting $p = p_1$ into Eq. (10.11), all products vanish except the first, in which the factor $p - p_1$ does not appear. The same holds for Eq. (10.12). Therefore,

$$M(p_1) = N_0B_1(p_1 - p_2)(p_1 - p_3) \cdots (p_1 - p_n)$$

and

$$N'(p_1) = N_0(p_1 - p_2)(p_1 - p_3) \cdots (p_1 - p_n)$$

We obtain by division

$$\frac{M(p_1)}{N'(p_1)} = B_1$$

In general, substituting p_k into (10.11) and (10.12), we have

$$B_k = \frac{M(p_k)}{N'(p_k)}$$

Using the expansion (10.8), it follows from rule I and from the operational formula (9.5) that

$$\Omega(p)\, 1(t) = \sum_{k=1}^{n} \frac{B_k}{p_k}(e^{p_kt} - 1)\, 1(t) \tag{10.13}$$

From Eq. (10.8) it follows that

$$\Omega(0) = \frac{1}{Z(0)} = \frac{M(0)}{N(0)} = -\sum_{k=1}^{n} \frac{B_k}{p_k}$$

Hence, Eq. (10.13) can be written in the form:

$$\Omega(p)\, 1(t) = \frac{1}{Z(0)}\, 1(t) + \sum_{k=1}^{n} \frac{B_k}{p_k} e^{p_kt} \tag{10.14}$$

This relation is known as *Heaviside's expansion theorem.*

Note.—If $M(p)$ and $N(p)$ are of the same degree,

$$\frac{M(p)}{N(p)} = \frac{M_0}{N_0} + \sum_{k=1}^{n} \frac{B_k}{p - p_k} \tag{10.15}$$

where M_0 and N_0 are the coefficients of the highest degree term in $M(p)$ and $N(p)$, respectively.

Then an additive term $\frac{M_0}{N_0} 1(t)$ appears in (10.13). However, in this case $\frac{1}{Z(0)} = \frac{M_0}{N_0} + \sum_{k=1}^{n} \frac{B_k}{p_k}$ and, therefore, Eq. (10.14) remains valid. It is seen that whenever M and N are of the same degree, $A(0) \neq 0$.

The rules used in this deduction are based on the assumption that the real part of $-p_k$ is not negative. This is satisfied if the system is damped or capable of carrying out oscillations with constant amplitude. Let us first consider a damped system. In this case the real parts of the roots p_k are negative, and for $t \rightarrow \infty$ the sum in Eq. (10.14) vanishes. Hence, the limiting value of the indicial admittance for $t \rightarrow \infty$ is given by

$$A(\infty) = \frac{1}{Z(0)} \tag{10.16}$$

This expression may be considered as the *static response* of the system under action of a unit force; for example, in the case of an elastic system it is equal to the static deflection produced by unit load. If some of the p_k's are pure imaginary, the corresponding terms in (10.14) represent harmonic oscillations about the static equilibrium position.

The case of $N(p)$ with multiple roots can be treated in a similar way.

For example, if p_1 is a double root, the expansion of $\Omega(p)$ in partial fractions contains a term of the form $\frac{B_1 + B_2 p}{(p - p_1)^2}$. Replacing p by $p + p_1$ (where the real part of p_1 is not positive), and applying rule VI, we obtain

$$\frac{p}{p+p_1}\frac{B_1 + B_2(p+p_1)}{p^2}\,1(t) = e^{-p_1 t}A(t)$$

or

$$A(t) = e^{p_1 t}\left[\frac{B_1}{p_1}\left(\frac{1}{p} - \frac{1}{p+p_1}\right) + \frac{B_2}{p}\right]1(t)$$

Using Eqs. (9.1) and (9.5) we find

$$A(t) = \frac{1}{p_1^2}[(B_1 + B_2 p_1)p_1 t e^{p_1 t} + B_1(1 - e^{-p_1 t})] \quad (10.17)$$

11. Response of an Electric Network to the Sudden Application of a Constant Voltage.—We consider the network shown in Fig. 11.1 and assume that the switch S is closed at $t = 0$. The battery B furnishes a constant electromotive force E. We investigate the transient state. The impedance of the total circuit is

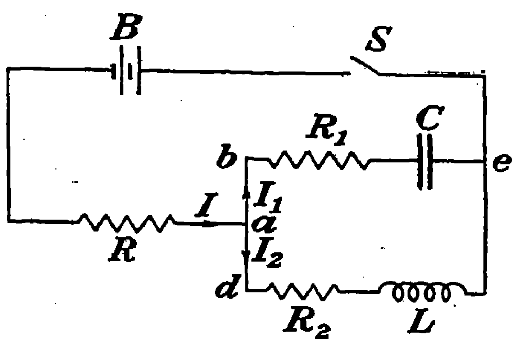

FIG. 11.1.—Example of a network on which a constant electromotive force is suddenly impressed by closing of a switch.

$$Z(p) = R + Z_{ae} \quad (11.1)$$

where Z_{ae} is the resulting impedance of the two parallel branches abe and ade. Hence,

$$\frac{1}{Z_{ae}} = \frac{1}{R_1 + 1/Cp} + \frac{1}{Lp + R_2} \quad (11.2)$$

Substituting expression (11.2) in Eq. (11.1), we obtain

$$Z(p) = R + \frac{1}{Cp/(R_1Cp+1) + 1/(Lp+R_2)} \quad (11.3)$$

or

$$Z(p) = R + \frac{(R_1Cp+1)(Lp+R_2)}{CLp^2 + C(R_1+R_2)p + 1}$$

The reciprocal value of $Z(p)$ is

$$\Omega(p) = \frac{1}{Z(p)} = \frac{M(p)}{N(p)} \quad (11.4)$$

with

$$M(p) = LCp^2 + (R_1+R_2)Cp + 1$$
$$N(p) = LC(R+R_1)p^2 + [L + C(R_1R_2 + RR_1 + RR_2)]\,p + R + R_2$$

With the abbreviations

$$\begin{aligned} P &= LC(R + R_1) \\ Q &= L + C(R_1R_2 + RR_1 + RR_2) \\ T &= R + R_2 \end{aligned}$$

we have

$$N(p) = Pp^2 + Qp + T \tag{11.5}$$

The roots of $N(p)$ are

$$\begin{aligned} p_1 &= -\frac{Q}{2P} + \sqrt{\left(\frac{Q}{2P}\right)^2 - \frac{T}{P}} \\ p_2 &= -\frac{Q}{2P} - \sqrt{\left(\frac{Q}{2P}\right)^2 - \frac{T}{P}} \end{aligned} \tag{11.6}$$

Using Eq. (10.14) we could write the final result directly, but for the sake of physical interpretation we prefer to repeat the intermediate steps of the solution. We note here that the two polynomials $M(p)$ and $N(p)$ are of the same degree; for the expansion of $M(p)/N(p)$ in partial fractions we apply, therefore, the formula (10.15) and write

$$\frac{M(p)}{N(p)} = \frac{1}{R + R_1} + \frac{B_1}{p - p_1} + \frac{B_2}{p - p_2} \tag{11.7}$$

with

$$B_1 = \frac{M(p_1)}{N'(p_1)}, \qquad B_2 = \frac{M(p_2)}{N'(p_2)}$$

The values of the coefficients are

$$\begin{aligned} B_1 &= \frac{LCp_1^2 + (R_1 + R_2)Cp_1 + 1}{2Pp_1 + Q} \\ B_2 &= \frac{LCp_2^2 + (R_1 + R_2)Cp_2 + 1}{2Pp_2 + Q} \end{aligned} \tag{11.8}$$

The symbolic equation for the indicial admittance is

$$\left(\frac{1}{R + R_1} + \frac{B_1}{p - p_1} + \frac{B_2}{p - p_2}\right) 1(t) = A(t) \tag{11.9}$$

Applying Eq. (9.5), we obtain

$$A(t) = \left[\frac{1}{R + R_1} + \frac{B_1}{p_1}(e^{p_1t} - 1) + \frac{B_2}{p_2}(e^{p_2t} - 1)\right] 1(t) \tag{11.10}$$

The function $A(t)$ gives the current through the battery when the switch S is closed at the instant $t = 0$ and a unit electromotive force is imposed by the battery. The value of the current at the instant $t = 0$ is equal to $\frac{1}{R + R_1}$. The limiting value for $t \to \infty$ is equal to $\frac{1}{R + R_1} - \frac{B_1}{p_1} - \frac{B_2}{p_2}$. According to Eq. (11.7), this expression is equal to the value of $M(p)/N(p)$ for $p = 0$, *i.e.*, equal to $\frac{1}{R + R_2}$. Hence, the transient current starts with the value $\frac{1}{R + R_1}$ and approaches asymptotically the value $\frac{1}{R + R_2}$. This can be seen physically in the following way. At the instant $t = 0$ the whole current goes through the branch *abe* because of the presence of the inductance in the branch *ade*. The condenser C is initially without charge, and, therefore, its counterelectromotive force is zero. The only potential drop occurs in the resistances R and R_1. In the steady state due to the presence of the condenser in the branch *abe*, the whole current passes through the branch *ade*, and the total potential drop occurs in the resistances R and R_2.

Equation (11.10) gives the total current passing through the two branches *abe* and *ade*. We shall now calculate the current through the capacitor C. We denote the current through the branch *abe* by I_1, the impedance of this branch by Z_1, the current through *ade* by I_2, and the impedance of this branch by Z_2. The total current is $I = I_1 + I_2$. On the other hand, $Z_1 I_1 = Z_2 I_2$ and, therefore,

$$I_1 = \frac{Z_2}{Z_1 + Z_2} I \tag{11.11}$$

With $I = E/Z$, we have

$$I_1 = \frac{Z_2}{Z_1 + Z_2} \frac{1}{Z} E \tag{11.12}$$

Hence, the *transfer impedance* for the current in the branch *abe* produced by an electromotive force applied to B is given by

$$Z_{B1} = Z \frac{Z_1 + Z_2}{Z_2} \tag{11.13}$$

With $Z_1 = R_1 + \dfrac{1}{Cp}$, $Z_2 = R_2 + Lp$, and $Z = \dfrac{N(p)}{M(p)}$, we obtain the symbolic equation for the current $I_1 = A_1(t)$:

$$\frac{R_2 + Lp}{R_1 + R_2 + Lp + 1/Cp}\,\frac{M(p)}{N(p)}\,1(t) = A_1(t) \qquad (11.14)$$

or

$$\frac{Cp(R_2 + Lp)}{N(p)}\,1(t) = A_1(t) \qquad (11.15)$$

We expand the operator in partial fractions:

$$\frac{Cp(R_2 + Lp)}{N(p)} = \frac{1}{R_1 + R} + \frac{B_1'}{p - p_1} + \frac{B_2'}{p - p_2} \qquad (11.16)$$

where B_1' and B_2' are constants containing the parameters of the system.

Then the solution of (11.15) becomes

$$A_1(t) = \frac{1}{R + R_1} + \frac{B_1'}{p_1}(e^{p_1t} - 1) + \frac{B_2'}{p_2}(e^{p_2t} - 1) \qquad (11.17)$$

If $t \to \infty$,

$$A_1(t) = \frac{1}{R + R_1} - \frac{B_1'}{p_1} - \frac{B_2'}{p_2} \qquad (11.18)$$

Substituting $p = 0$ in Eq. (11.16), we see that the right side of (11.18) vanishes.

This example shows an important feature of the operational method. If we are interested in a particular result only, *e.g.*, in the current passing through a certain branch of the network, we can calculate this unknown quantity without solving the whole problem.

If we are especially interested in the beginning phase of a transient process, it is convenient to use the expansion theorem (10.3) which generally leads with little labor to a satisfactory answer.

12. Response of Undamped Mechanical Systems to Suddenly Applied Forces.—The response of undamped mechanical systems to sudden applications of external forces can be calculated by means of the methods given in Chapter V, section 7. We have seen that if we know the principal oscillations of the system it is not difficult to describe its motion by means of the displace-

ments ξ_r of the normal modes and the normal components Ξ_r of the external forces. The displacements ξ_r satisfy equations of the form

$$M(\ddot{\xi}_r + \omega_r^2 \xi_r) = \Xi_r \tag{12.1}$$

If the forces are suddenly applied at $t = 0$, Ξ_r is of the form

$$\Xi_r = C_r\, 1(t)$$

In this case the solution of Eq. (12.1) represents the motion of a mass M with elastic restraint under a suddenly applied force. Applying the result already found in section 3 for this case we have

$$\xi_r = \frac{C_r}{M\omega_r^2}(1 - \cos \omega_r t) \tag{12.2}$$

This gives the amplitude of each normal mode as function of time, due to a sudden application of external forces.

Using arbitrary coordinates, each coordinate q_i is given by an equation of the form [cf. Chapter V, Eq. (6.17)]:

$$q_i = \sum_{r=1}^{n} \frac{\varphi_i^{(r)} C_r}{M\omega_r^2}(1 - \cos \omega_r t) \tag{12.3}$$

If the system has no restraints for one degree of freedom—for example, it can rotate freely around an axis—its lowest frequency is $\omega_1 = 0$. The amplitude ξ_1 describing the free rotation satisfies the equation

$$M\ddot{\xi}_1 = C_1\, 1(t) \tag{12.4}$$

The solution satisfying the initial conditions $\xi_1 = \dot{\xi}_1 = 0$ is

$$\xi_1 = \frac{\frac{1}{2}C_1 t^2}{M} \tag{12.5}$$

which is a uniformly accelerated motion. The complete solution consists of the forms (12.2) and (12.5).

The same problem may be considered from the point of view of the operational calculus. We shall see that the method of expansion of $1/Z(p)$ in partial fractions leads to the same representation of the motion as we obtained by superposition of principal oscillations. Both methods require the solution of the

frequency equation $Z(p) = 0$. Their respective advantages will depend on each individual problem. The operational method is illustrated by the two following examples.

Example 1.—Consider two disks connected by an elastic shaft, and assume that a sudden unit torque is applied to the disk D_1 (Fig. 12.1).* We denote the moment of inertia of the disk D_1 by I_1, that of the disk D_2 by I_2, and the torsional spring constant of the shaft by k. We define the impedance $Z_1(p)$ as the quotient of the alternating torque T applied to D_1 and the angular displacement of the same disk.

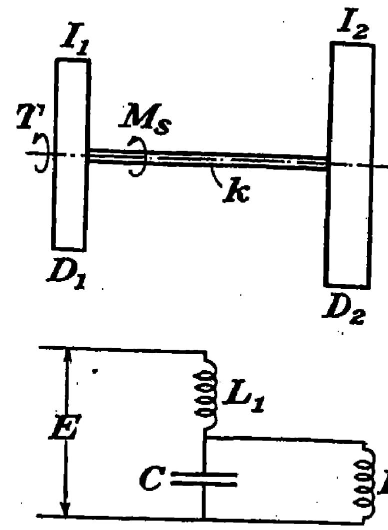

FIG. 12.1.—Mechanical system consisting of two elastically connected disks and the equivalent electrical network.

To find this impedance we write the equations of motion for the angular displacements θ_1 and θ_2 of the disks D_1 and D_2, respectively. With $p = d/dt$, we obtain

$$\begin{aligned} I_1p^2\theta_1 &= -k(\theta_1 - \theta_2) + T \\ I_2p^2\theta_2 &= -k(\theta_2 - \theta_1) \end{aligned} \tag{12.6}$$

Eliminating θ_2 from these equations we find,

$$\frac{I_1I_2p^4 + kp^2(I_1 + I_2)}{I_2p^2 + k}\theta_1 = T$$

and therefore

$$Z(p) = p^2\frac{[I_1I_2p^2 + k(I_1 + I_2)]}{I_2p^2 + k} \tag{12.7}$$

The same result can be obtained by the methods of Chapter IX, section 3. From Eq. (4.9) of that chapter we have

$$Z_1(p) = I_1p^2 + \frac{1}{(1/k) + (1/I_2p^2)} \tag{12.8}$$

which expression is identical with (12.7).

Eliminating θ_1 from Eqs. (12.6), we obtain

$$(p^2/k)[I_1I_2p^2 + k(I_1 + I_2)]\,\theta_2 = T$$

* Figure 12.1 also shows the electrical network corresponding to the mechanical system. Thus, the calculation presented in the text can be applied to compute the effect of a voltage suddenly switched on the network.

Hence, we have

$$Z_{12}(p) = (p^2/k)[I_1I_2p^2 + k(I_1 + I_2)] \tag{12.9}$$

where Z_{12} is the ratio of the torque T applied to D_1 to the angular displacement of the disk D_2.

The roots of $Z_1(p) = 0$ and $Z_{12}(p) = 0$ are $p = 0$, corresponding to a free rotation, and $p = \pm i\omega_0 = \pm i\sqrt{k\left(\frac{1}{I_1} + \frac{1}{I_2}\right)}$, where ω_0 is the frequency of the relative oscillation of the two disks, which is in this case the only principal oscillation.

The symbolic equation for the response $\theta_1 = A_1(t)$ of disk D_1 to a unit step is

$$\frac{1}{Z_1(p)}\,1(t) = \frac{1}{I_1I_2}\,\frac{I_2p^2 + k}{p^2(p^2 + \omega_0^2)}\,1(t) = A_1(t) \tag{12.10}$$

or, expanding the operator $1/Z_1(p)$ in partial fractions,

$$\frac{1}{I_1I_2}\left(\frac{k}{\omega_0^2p^2} - \frac{k - I_2\omega_0^2}{\omega_0^2(p^2 + \omega_0^2)}\right)1(t) = A_1(t) \tag{12.11}$$

According to Eqs. (9.1) and (9.6),

$$\frac{1}{p^2}\,1(t) = \frac{t^2}{2}\,1(t)$$

$$\frac{1}{p^2 + \omega_0^2}\,1(t) = \frac{1 - \cos \omega_0 t}{\omega_0^2}\,1(t)$$

Therefore, Eq. (12.11) becomes

$$A_1(t) = \left(\frac{k}{I_1I_2\omega_0^2}\,\frac{t^2}{2} - \frac{k - I_2\omega_0^2}{I_1I_2\omega_0^2}\,\frac{1 - \cos \omega_0 t}{\omega_0^2}\right)1(t)$$

or substituting $k = \frac{I_1I_2}{I_1 + I_2}\omega_0^2$,

$$A_1(t) = \frac{1}{I_1 + I_2}\left(\frac{t^2}{2} + I_2\frac{1 - \cos \omega_0 t}{I_1\omega_0^2}\right)1(t) \tag{12.12}$$

The first term represents a uniformly accelerated rotation with the angular acceleration $\frac{1}{I_1 + I_2}$. However, the acceleration immediately after the beginning of the motion is equal to $1/I_1$;

this shows that at the first instant the presence of the second disk has no effect.

It is interesting to calculate the angular displacement of the disk D_2. The response $\theta_2 = A_2(t)$ of this disk is given by the symbolic equation

$$\frac{1}{Z_{12}(p)}\, 1(t) = \frac{k}{I_1 I_2}\,\frac{1}{p^2(p^2+\omega_0^2)}\, 1(t) = A_2(t) \qquad (12.13)$$

By the method used above, we obtain

$$A_2(t) = \frac{1}{I_1 + I_2}\left(\frac{t^2}{2} - \frac{1-\cos\omega_0 t}{\omega_0^2}\right) 1(t) \qquad (12.14)$$

Since there is no damping, the sum of the moments of momentum of the two disks must be equal to the total impulse imposed on the system. The moments of momentum of the disks are $I_1 \frac{dA_1}{dt}$ and $I_2 \frac{dA_2}{dt}$. From Eqs. (12.12) and (12.14), we obtain

$$I_1 \frac{dA_1}{dt} + I_2 \frac{dA_2}{dt} = t \qquad (12.15)$$

The expressions (12.12) and (12.14) represent combinations of Eqs. (12.2) and (12.5). The difference $\theta_1 - \theta_2$ is given by

$$\theta_1 - \theta_2 = A_1(t) - A_2(t) = \frac{1}{I_1\omega_0^2}(1 - \cos\omega_0 t)\, 1(t) \qquad (12.16)$$

The torque M_s that occurs in the shaft is equal to $k(\theta_1 - \theta_2)$ or, substituting the value of ω_0^2;

$$M_s = \frac{I_2}{I_1 + I_2}(1 - \cos\omega_0 t)\, 1(t) \qquad (12.17)$$

Therefore, the ratio of the maximum torque occurring in the shaft to the external torque T is equal to

$$\frac{M_s}{T} = \frac{2I_2}{I_1 + I_2} \qquad (12.18)$$

This quotient approaches zero if $I_2/I_1 \to 0$ and approaches the value 2 for $I_2/I_1 \to \infty$.

Example 2.—The same method of treatment can be applied to more complicated mechanical devices. Figure 12.2 shows a

motor driving, by means of a gear G and two pinions P_1 and P_2, two elastic shafts S_1 and S_2 carrying the disks D_3 and D_4 at their free ends. The moment of inertia of the masses rigidly connected to the motor shaft is denoted by I, the moments of inertia of the pinions by I_1 and I_2, those of the disks by I_3 and I_4.

We define as the impedance of the system the ratio of the alternating torque T acting on the motor shaft to the angular

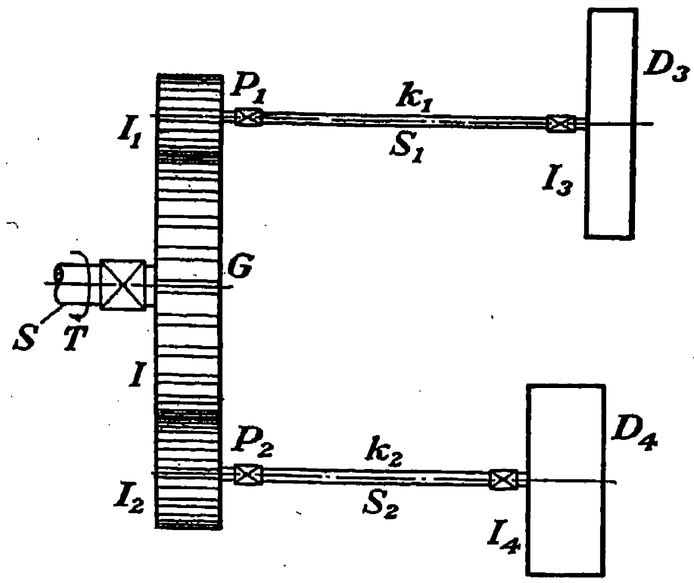

FIG. 12.2.—Mechanical system consisting of two disks mounted at the free ends of elastic shafts and driven from a central shaft by means of a gear system.

deflection θ of the same shaft. We show that the torque T is given by

$$T = Z\theta = (Ip^2 + g_1^2 Z_1 + g_2^2 Z_2)\theta \tag{12.19}$$

where Z_1 and Z_2 are the impedances of the two systems consisting of pinion, shaft, and disk, and g_1, g_2 are the gear ratios. Let us denote, for example, the angular deflection of the pinion P_1 by θ_1. Then $\theta = \theta_1/g_1$, whereas the torques acting about the shaft of the gear G and that of the pinion P_1 are in the ratio $g_1:1$. Hence, the torque transferred to the gear is equal to $g_1 Z_1 \theta_1 = g_1^2 Z_1 \theta$. Therefore, the total torque is equal to the expression (12.19).

The impedances Z_1 and Z_2 are obtained by using the method given in Chapter IX, section 4.

We find

$$Z_1 = I_1 p^2 + \frac{1}{(1/k_1) + (1/I_3 p^2)}$$

$$Z_2 = I_2 p^2 + \frac{1}{(1/k_2) + (1/I_4 p^2)}$$

and, therefore,

$$Z = (I + I_1 g_1^2 + I_2 g_2^2)p^2 + g_1^2 \frac{k_1 I_3 p^2}{k_1 + I_3 p^2} + g_2^2 \frac{k_2 I_4 p^2}{k_2 + I_4 p^2} \qquad (12.20)$$

We put $\omega_1^2 = k_1/I_3$, $\omega_2^2 = k_2/I_4$, $\gamma_1 = g_1^2 k_1$, $\gamma_2 = g_2^2 k_2$ and

$$I_r = I + g_1^2 I_1 + g_2^2 I_2$$

Then,

$$\frac{1}{Z(p)} = \frac{1}{I_r p^2} \frac{(p^2 + \omega_1^2)(p^2 + \omega_2^2)}{(p^2 + \omega_1^2)(p^2 + \omega_2^2) + \frac{\gamma_1}{I_r}(p^2 + \omega_2^2) + \frac{\gamma_2}{I_r}(p^2 + \omega_1^2)}$$

The quantities ω_1, ω_2 represent the frequencies of oscillation of the disks when the pinions are rigidly fixed. The roots of the denominator are given by $p = 0$, $p^2 = -\mu_1^2$, and $p^2 = -\mu_2^2$ where μ_1 and μ_2 are the two natural frequencies of oscillation of the system. Hence,

$$\frac{1}{Z(p)} = \frac{1}{I_r} \frac{(p^2 + \omega_1^2)(p^2 + \omega_2^2)}{p^2(p^2 + \mu_1^2)(p^2 + \mu_2^2)} \qquad (12.21)$$

Let us now calculate the effect of a unit torque suddenly applied by the motor. We have to evaluate $\frac{1}{Z(p)} 1(t) = A(t)$. We expand the operator $1/Z(p)$ in partial fractions, as follows:

$$\frac{1}{Z(p)} 1(t) = \frac{1}{I_r}\left[\left(\frac{\omega_1\omega_2}{\mu_1\mu_2}\right)^2 \frac{1}{p^2} + \frac{C_1}{p^2 + \mu_1^2} + \frac{C_2}{p^2 + \mu_2^2}\right] 1(t) \qquad (12.22)$$

The coefficients C_1 and C_2 may be obtained very simply by using the formula (10.9), considering p^2 as the variable instead of p. We find

$$C_1 = \frac{(\omega_1^2 - \mu_1^2)(\omega_2^2 - \mu_1^2)}{\mu_1^2(\mu_2^2 - \mu_1^2)}, \qquad C_2 = \frac{(\omega_1^2 - \mu_2^2)(\omega_2^2 - \mu_2^2)}{\mu_2^2(\mu_1^2 - \mu_2^2)}$$

Using the operational formulas (9.1) and (9.6), we have

$$A(t) = \frac{1}{Z(p)} 1(t) = \frac{1}{I_r}\left[\left(\frac{\omega_1\omega_2}{\mu_1\mu_2}\right)^2 \frac{t^2}{2} + \frac{C_1}{\mu_1^2}(1 - \cos \mu_1 t) + \frac{C_2}{\mu_2^2}(1 - \cos \mu_2 t)\right] \qquad (12.23)$$

The motion of the motor due to the sudden application of a constant torque consists of a uniformly accelerated motion and two superposed harmonic oscillations. If the frequencies ω_1 and ω_2 are equal, we find that one of the roots, for example μ_1, is equal to ω_1, and, therefore, $C_1 = 0$. The angular displacement of the motor shaft is then given by

$$A(t) = \frac{1}{I_r}\left[\left(\frac{\omega_2}{\mu_2}\right)^2 \frac{t^2}{2} + \frac{C_2}{\mu_2^2}(1 - \cos \mu_2 t)\right] 1(t) \qquad (12.24)$$

In this case only one oscillation is excited by the application of the torque.

13. Multiplication of Operators. Borel's Theorem.—Let us assume that $\Omega_1(p)$ and $\Omega_2(p)$ are two operators, such that

$$\begin{aligned} \Omega_1(p)1(t) &= A_1(t) \\ \Omega_2(p)1(t) &= A_2(t) \end{aligned} \qquad (13.1)$$

Then we prove that the operator $\Omega_1(p)\Omega_2(p)/p$ transforms the unit step into the function

$$A_{12}(t) = \int_0^t A_1(\tau)A_2(t-\tau)\,d\tau \qquad (13.2)$$

or

$$A_{21}(t) = \int_0^t A_2(\tau)A_1(t-\tau)\,d\tau \qquad (13.3)$$

In operational form,

$$(1/p)\Omega_1(p)\Omega_2(p)1(t) = A_{12}(t) = A_{21}(t) \qquad (13.4)$$

This relation is known as *Borel's theorem.* First, we prove that $A_{12} = A_{21}$. If we replace $t - \tau$ by ζ and $-d\tau$ by $d\zeta$, Eq. (13.2) becomes

$$A_{12} = -\int_t^0 A_2(\zeta)A_1(t-\zeta)\,d\zeta = \int_0^t A_2(\zeta)A_1(t-\zeta)\,d\zeta$$

The last expression is identical with A_{21} [Eq. (13.3)], if we replace the variable of integration by τ.

For the proof of the multiplication theorem, we make use of Carson's integral equation (6.5). This equation states that the operator that transforms $1(t)$ into the function $A(t)$ is p times the Laplace transform of that function. Hence, we must show that $\Omega_1(p)\Omega_2(p)/p^2$ is the Laplace transform of $A_{12}(t)$ or $A_{21}(t)$.

The Laplace transform of $A_{12}(t)$ is given by

$$\Lambda(p) = \int_0^\infty e^{-pt}A_{12}(t)\,dt = \int_0^\infty e^{-pt}\,dt\int_0^t A_1(\tau)A_2(t-\tau)\,d\tau \qquad (13.5)$$

We write Eq. (13.5) in the form:

$$\Lambda(p) = \int_0^\infty dt\int_0^t e^{-p(t-\tau)}A_2(t-\tau)e^{-p\tau}A_1(\tau)\,d\tau \qquad (13.6)$$

The right side of Eq. (13.6) is a double integral whose domain of integration AOB is shown in Fig. 13.1. It consists of the infinitely extended area between the axis $\tau = 0$ and the 45°

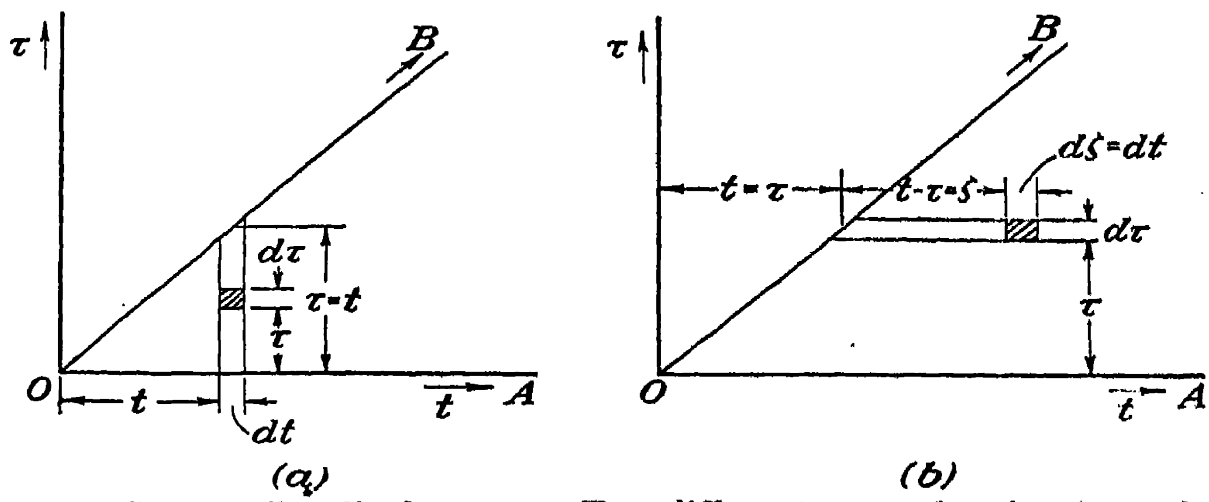

FIG. 13.1.—Proof of Borel's theorem. Two different ways of evaluating a double integral extended over the infinite triangular area AOB.

line OB represented by the equation $\tau = t$. In Eq. (13.6) the integration is performed first along vertical strips extending from $\tau = 0$ to $\tau = t$ (Fig. 13.1*a*). However, we can perform the integration also along horizontal strips (Fig. 13.1*b*), starting from the 45° line OB, and add their contributions. Thus Eq. (13.6) becomes

$$\Lambda(p) = \int_0^\infty d\tau\int_{t=\tau}^{t=\infty} e^{-p(t-\tau)}A_2(t-\tau)A_1(\tau)e^{-p\tau}\,dt \qquad (13.7)$$

Using $\zeta = t - \tau$ and τ as variables of integration, we put $dt = d\zeta$ and obtain

$$\Lambda(p) = \int_0^\infty d\tau\int_0^\infty e^{-p\zeta}A_2(\zeta)A_1(\tau)e^{-p\tau}\,d\zeta$$

This double integral can be broken up into a product of two single integrals:

$$\Lambda(p) = \int_0^\infty e^{-p\tau}A_1(\tau)\,d\tau\int_0^\infty e^{-p\zeta}A_2(\zeta)\,d\zeta \qquad (13.8)$$

Now $\Omega_1(p)/p$ is the Laplace transform of $A_1(\tau)$, and $\Omega_2(p)/p$ is that of $A_2(\tau)$, *i.e.*,

$$\int_0^\infty e^{-p\tau}A_1(\tau)\,d\tau = \Omega_1(p)/p$$

$$\int_0^\infty e^{-p\zeta}A_2(\zeta)\,d\zeta = \Omega_2(p)/p$$

Therefore,

$$\Lambda(p) = \Omega_1(p)\Omega_2(p)/p^2 \qquad \text{Q.E.D.} \tag{13.9}$$

It is easy to see that we can apply the multiplication theorem to the calculation of the response $x(t)$ of a system whose impedance is $Z(p)$ to a force $F(t)$, which is an arbitrary function of t, provided that $F(t)$ is expressed by a symbolic equation of the form:

$$\Omega_2(p)1(t) = F(t) \tag{13.10}$$

If $A(t)$ is the response of the system to a unit step, we have the symbolic equation

$$\frac{1}{Z(p)}1(t) = A(t) \tag{13.11}$$

We now denote $1/Z(p)$ by $\Omega_1(p)$ and apply Borel's theorem, substituting $A(t)$ for $A_1(t)$, and $F(t)$ for $A_2(t)$. According to Eqs. (13.2) and (13.4),

$$\frac{1}{p}\Omega_1(p)\Omega_2(p)1(t) = \int_0^t F(t-\tau)A(\tau)d\tau \tag{13.12}$$

Multiplying the operator on the left side by p, we obtain, according to rule III,

$$\Omega_1(p)\Omega_2(p)1(t) = \frac{d}{dt}\int_0^t F(t-\tau)A(\tau)d\tau$$

or

$$\Omega_1(p)\Omega_2(p)1(t) = F(0)A(t) + \int_0^t F'(t-\tau)A(\tau)d\tau \tag{13.13}$$

Interchanging the variables τ and $t-\tau$ in the integral, we obtain

$$\Omega_1(p)\Omega_2(p)1(t) = F(0)A(t) + \int_0^t F'(\tau)A(t-\tau)d\tau \tag{13.14}$$

The expression on the right side of Eq. (13.14) is Duhamel's formula [cf. Eq. (4.2)] for the response $x(t)$ to the force $F(t)$.

Equation (13.14) is useful in many cases. It shows that when an operational representation for $F(t)$ is known, we can evaluate the response by direct operational methods without actually carrying out the integration in Duhamel's formula. This is illustrated by the example in the following section.

14. Response of a Network to a Suddenly Applied Alternating Voltage.—We consider the same network (Fig. 11.1) as in section 11 but assume that the battery is replaced by an alternating generator G. The alternating voltage is switched on the network at $t = 0$. We assume that the voltage applied is represented by

$$F(t) = \sin \omega t \, 1(t) \tag{14.1}$$

so that the network is switched in when the voltage is zero. The function $\sin \omega t \, 1(t)$ is the solution of the symbolic equation [cf. Eq. (9.4)]:

$$\frac{\omega p}{p^2 + \omega^2} 1(t) = \sin \omega t \, 1(t) \tag{14.2}$$

Hence, we know that the operator $\Omega_2(p) = \dfrac{\omega p}{p^2 + \omega^2}$ transforms $1(t)$ into the external action $F(t)$, and we can apply Borel's theorem. The impedance of the system is, according to Eq. (11.4),

$$Z(p) = \frac{N(p)}{M(p)}$$

Substituting this value for $Z(p)$ and $\dfrac{\omega p}{p^2 + \omega^2}$ for $\Omega_2(p)$ in Eq. (13.14), we find that the response, *i.e.*, the current $I(t)$ through the generator, becomes

$$I(t) = \frac{M(p)}{N(p)} \frac{\omega p}{p^2 + \omega^2} 1(t) \tag{14.3}$$

This symbolic equation is solved by expanding the operator in partial fractions. The roots of the denominator are p_1, p_2, $i\omega$, $-i\omega$, where p_1 and p_2 have the values given in section 11.

We note that the derivative of the denominator is

$$(p^2 + \omega^2)N'(p) + 2pN(p) = (p^2 + \omega^2)(2Pp + Q) + 2pN(p)$$

Therefore, we obtain the expansion

$$\frac{M(p)}{N(p)} \frac{\omega p}{p^2 + \omega^2} = \frac{B_1 p}{p - p_1} + \frac{B_2 p}{p - p_2} + \frac{B_3 p}{p - i\omega} + \frac{B_4 p}{p + i\omega} \tag{14.4}$$

with

$$B_1 = \omega \frac{M(p_1)}{(p_1^2 + \omega^2)(2Pp_1 + Q)}$$
$$B_2 = \omega \frac{M(p_2)}{(p_2^2 + \omega^2)(2Pp_2 + Q)}$$
$$B_3 = \frac{M(i\omega)}{2iN(i\omega)}$$
$$B_4 = -\frac{M(-i\omega)}{2iN(-i\omega)}$$

The current is

$$I(t) = \frac{B_1 p}{p - p_1} 1(t) + \frac{B_2 p}{p - p_2} 1(t) + \frac{B_3 p}{p - i\omega} 1(t) + \frac{B_4 p}{p + i\omega} 1(t) \tag{14.5}$$

Applying Eq. (9.3),

$$I(t) = (B_1 e^{p_1 t} + B_2 e^{p_2 t} + B_3 e^{i\omega t} + B_4 e^{-i\omega t})\, 1(t) \tag{14.6}$$

The terms $B_1 e^{p_1 t} + B_2 e^{p_2 t}$ represent the transient part of the current; the last two terms correspond to the steady state.

Problems

1. A sinusoidal voltage $\sin \omega t\, 1(t)$ is switched at $t = 0$ on a circuit of inductance L and resistance R. Find the current by using the general solution of the differential equation.

2. In Prob. 1 calculate the currents by applying Borel's theorem.

3. An elastic shaft carries a disk of moment of inertia I at one end and a clutch of moment of inertia I_1 at the other. The system is driven by an electric motor. The shaft is initially at rest and started in such a way that the torque applied to the shaft increases proportionally with time until it reaches a certain value M_0, after which it stays constant (Fig. P.3). Determine the maximum torque in the shaft. Compare it to the case where the driving torque is applied suddenly at $t = 0$.

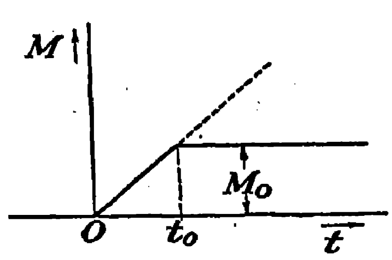

FIG. P.3.

Hint: The torque may be represented by $\frac{M_0}{t_0}[t\, 1(t) - (t - t_0)\, 1(t - t_0)]$

4. Find the current flowing through a capacity and a resistance where the voltage applied increases proportionally to the time and then decreases linearly with the same constant rate until it reaches zero again.

Hint: We note that the voltage may be represented by the superposition of three functions: $(E_0/t_0)[t\, 1(t) - 2(t - t_0)\, 1(t - t_0) + (t - 2t_0)\, 1(t - 2t_0)]$ (Fig. P.4). We therefore calculate the current $I(t)$ due to a voltage

$(E_0/t_0)t\, 1(t)$ and apply the principle of superposition. The total current will be

$$I(t)\, 1(t) - 2I(t - t_0)\, 1(t - t_0) + I(t - 2t_0)\, 1(t - 2t_0)$$

To calculate $I(t)$ we can make use of the relation $\frac{1}{p}\, 1(t) = t\, 1(t)$ and use Borel's theorem.

5. The triangular voltage curve of Prob. 4 lasts $2t_0$ sec. It is repeated periodically with the period $2t_0$ (Fig. P.5) and applied to a circuit with capacitance and resistance. Compute the steady-state current and the transient current by using a representation for the voltage similar to that suggested for Prob. 4. Compare the transient with the steady-state current.

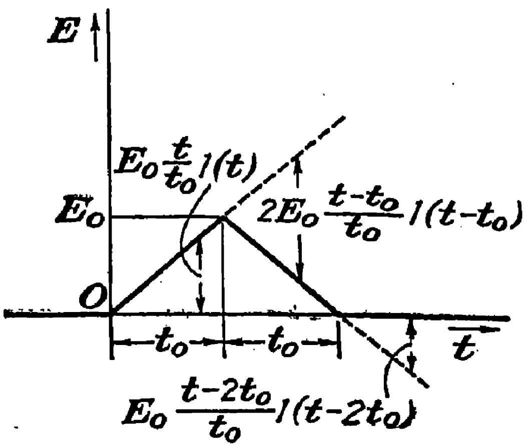

FIG. P.4.

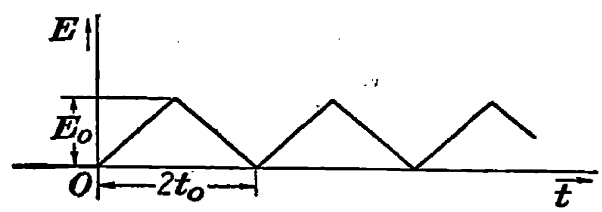

FIG. P.5.

6. Replace the capacitance in Prob. 5 by an inductance.

7. Calculate the current due to a voltage $\cos \omega t\, 1(t)$ applied to the network considered in section 11 of this chapter. Apply Borel's theorem.

8. Replace the voltage applied in Prob. 7 by $\sin (\omega t + \varphi)\, 1(t)$. Plot the current for various values of φ.

Hint: Write $\sin (\omega t + \varphi)\, 1(t) = \sin \omega t \cos \varphi\, 1(t) + \cos \omega t \sin \varphi\, 1(t)$ and apply the principle of superposition.

9. A valve-operating mechanism can be approximated by a mass m attached on one side to a fixed point by a spring k, while on the other side it receives the push of a cam through another spring (Fig. P.9). The cam profile is such that the vertical displacement of the roller is

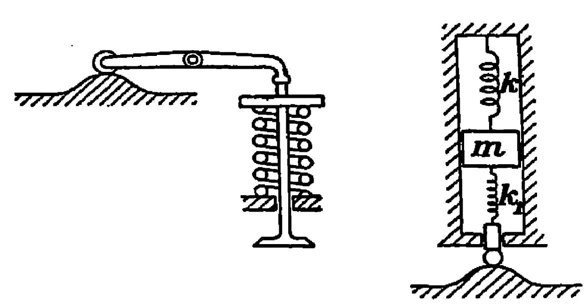

FIG. P.9.

$$x_1 = a\left(1 - \cos \frac{2\pi t}{T}\right)$$

between $t = 0$ and $t = T$ and zero for $t > T$. The springs are initially strained under a compression P. Find the motion of the mass m (value) and the pressure of the roller on the cam. Show that $\frac{2\pi}{T} < \sqrt{\frac{k}{m} + \frac{k_1}{m} \frac{P}{P + 2ak_1}}$ is a sufficient condition for the roller not to leave the cam.

10. Calculate the deceleration of the last car of the electric train considered in Chapter VI, Prob. 4, when a braking force of 7,000 lb. is suddenly applied to the locomotive. Consider the cases where $\beta = 2\sqrt{km}$, and $\beta = \frac{3}{8}\sqrt{km}$.

11. What is the current produced by the sudden application of a constant pressure to the diaphragm of the condenser microphone in Chapter VI, Prob. 3? Assume $m = 0$ and $\frac{1}{LC_0} < \frac{E^2}{kLa^2} + \left(\frac{R}{2L}\right)^2$ where $C_0 = A/a$.

References

1. HEAVISIDE, O.: "Electromagnetic Theory," The Electrician, London, 2d ed., 1922.
2. BUSH, V.: "Operational Circuit Analysis" John F. Wiley & Sons, Inc., New York, 1929.
3. CAMPBELL, G. A., and R. M. FOSTER: "Fourier Integrals for Practical Applications," *Bell Telephone System, Technical Publications*, September, 1931.
4. CARSON, J. R.: "Electric Circuit Theory and the Operational Calculus," McGraw-Hill Book Company, Inc., New York, 1926.
5. BERG, E. J.: "Heaviside's Operational Calculus as Applied to Engineering and Physics," McGraw-Hill Book Company, Inc., New York, 2d ed., 1936.
6. JEFFREYS, H.: "Operational Methods in Mathematical Physics," Cambridge University Press, London, 1927.
7. PIPES, L. A.: "The Operational Calculus," *Journal of Applied Physics*, Vol. 10, 1939.
8. MCLACHLAN, N. W.: "Complex Variable and Operational Calculus," Cambridge University Press, London, 1939.

CHAPTER XI

EQUATIONS OF FINITE DIFFERENCES APPLIED TO ENGINEERING

"Brooks Taylor, in his book, 'The Method of Increments' (1715–1717), was the first to consider equations of finite differences. As a matter of fact, the relations between the terms of arithmetic and geometric progressions, which were known for a long time, represent the simplest examples of difference equations. But they were not considered from the point of view of their relation to a general theory. This viewpoint was a veritable discovery."

—LE MARQUIS DE LAPLACE,
"Théorie Analytique des Probabilités" (1820).

Introduction.—This last chapter is devoted to the calculus of finite differences, especially to linear difference equations. Such equations are being applied more and more to engineering problems. Sometimes they are used as approximations to differential equations. However, their most important application is made in connection with systems—mechanical or electrical—that consist of many identical components. Because of the large number of degrees of freedom of such systems, the general methods usually resorted to involve lengthy and complicated calculations, whereas the special methods developed in this chapter greatly reduce the labor required and simplify the solution.

1. The Calculus of Finite Differences.—Let us assume that a function $f(x)$ of the variable x is defined for equidistant values of x. If x is one of the values for which $f(x)$ is defined, $f(x)$ is also defined for the values $x + k\,\Delta x$, where Δx is the interval between two successive values of x, and k is an integer. We denote the value of the function $y = f(x)$ for $x + k\,\Delta x$, for the sake of simplicity, by writing y with a subscript:

$$f(x + k\,\Delta x) = y_{x+k\,\Delta x} \tag{1.1}$$

We now define the *first difference* or the *difference of first order* Δy_x of y at the point x as the increment of the value of y in going from x to $x + \Delta x$:

$$\Delta y_x = y_{x+\Delta x} - y_x \tag{1.2}$$

It is seen that we have chosen arbitrarily the step in the direction of increasing x; we could also define Δy_x by the difference $y_x - y_{x-\Delta x}$.

Continuing this process, we call the increment of the first difference in going from x to $x + \Delta x$ *the difference of second order* of y at x, *i.e.*,

$$\Delta^2 y_x = \Delta y_{x+\Delta x} - \Delta y_x \tag{1.3}$$

or

$$\Delta^2 y_x = y_{x+2\Delta x} - 2y_{x+\Delta x} + y_x$$

In general, we define the *difference of the order* n by

$$\Delta^n y_x = \Delta^{n-1} y_{x+\Delta x} - \Delta^{n-1} y_x \tag{1.4}$$

It is seen that if we build up the difference of nth order $\Delta^n y_x$, starting from the first difference Δy_x, then $\Delta^n y_x$ appears as a linear expression in $y_x, y_{x+\Delta x}, \ldots, y_{x+n\Delta x}$.

In most applications it is convenient to choose the interval Δx equal to unity. Then we write $y_{x+n\Delta x} = y_{x+n}$. Using this notation, the sequence of differences becomes

$$\begin{aligned}
\Delta y_x &= y_{x+1} - y_x \\
\Delta^2 y_x &= y_{x+2} - 2y_{x+1} + y_x \\
\Delta^3 y_x &= y_{x+3} - 3y_{x+2} + 3y_{x+1} - y_x \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\Delta^n y_x &= \sum_{r=0}^{n} (-1)^r \frac{n!}{r!(n-r)!} y_{x+n-r}
\end{aligned} \tag{1.5}$$

It is seen that the coefficients of $\Delta^n y_x$ are the same coefficients which appear in the expansion of $(a - b)^n$, *i.e.*, they are equal to the *binomial coefficients* multiplied by $(-1)^r$.

In many physical problems only differences of even order occur. In such cases it is more convenient to define the differences $\Delta^{2m} y_x$ in a somewhat different way. We shall write

$$\Delta^2 y_x = y_{x-1} - 2y_x + y_{x+1} \tag{1.6}$$

i.e., $\Delta^2 y_x$ is the increment of the first difference taken on the right and left sides of the point x. Similarly, we write

$$\Delta^4(y_x) = \Delta^2(\Delta^2 y_x)$$

or in general,

$$\Delta^{2m}(y_x) = \Delta^2(\Delta^{2m-2}y_x) \tag{1.7}$$

In this case a difference of the order $2m$ represents a linear expression in $y_{x-m}, y_{x-m+1}, \ldots, y_x, \ldots, y_{x+m-1}, y_{x+m}$.

2. Linear Difference Equations with Constant Coefficients.—A *linear difference equation of the order* n is a linear relation between an unknown function y_x and its first n differences. Its general form is

$$A_n\,\Delta^n y_x + A_{n-1}\,\Delta^{n-1}y_x + \cdots + A_1\,\Delta y_x + A_0 y_x = \phi(x) \tag{2.1}$$

The simplest example of such an equation is

$$\Delta y_x = a$$

where a is a constant, or, substituting the value of Δy_x from Eq. (1.5),

$$y_{x+1} - y_x = a$$

The solution is obviously

$$y_x = ax + \text{const.}$$

In other words the values y_x constitute an *arithmetic progression* with the difference a.

If we write

$$\Delta y_x + by_x = 0$$

we obtain

$$y_{x+1} - y_x + by_x = 0$$

or

$$\frac{y_{x+1}}{y_x} = 1 - b$$

It is seen that in this case the values y_x constitute a *geometric progression*. The coefficients $A_n, \ldots, A_0$ of Eq. (2.1) are, in general, functions of the variable x, defined for the equidistant values $x = x_0 + k$ where k is an integer. However, we shall consider in detail only the case of the equation with constant coefficients. The function $\phi(x)$ is given for the same equidistant values of x. If $\phi(x) = 0$, the difference equation is called *homogeneous*.

As Eq. (2.1) represents a linear relation with constant coefficients between $n + 1$ successive values of y_x, it can be also written in the form:

$$a_n y_{x+n} + a_{n-1} y_{x+n-1} + \cdots + a_1 y_{x+1} + a_0 y_x = \phi(x) \tag{2.2}$$

If we use the differences in the form (1.7), a linear difference equation of the order $2m$ can be written in the following way:

$$b_{-m} y_{x-m} + b_{-m+1} y_{x-m+1} + \cdots + b_0 y_x + \cdots + b_m y_{x+m} = \phi(x) \tag{2.3}$$

As a matter of fact, in most practical problems the physical setup leads to this form of the difference equations.

Let us start from Eq. (2.2) containing the unknowns $y_x, \ldots, y_{x+n}$. If the values of $y_x, y_{x+1}, \ldots, y_{x+n-1}$ are arbitrarily chosen, then y_{x+n} is determined by the equation. We now proceed to the next equation, increasing the subscripts by 1. The values of y occurring in this equation are $y_{x+1}, y_{x+2}, \ldots, y_{x+n+1}$. Since $y_{x+1}, \ldots, y_{x+n}$ are known, we can determine y_{x+n+1}. If we continue this process, we see that the solution is determined by the arbitrary choice of the n initial values $y_x, \ldots, y_{x+n-1}$. Therefore, the solution depends on n arbitrarily chosen initial values. Assume that $f_1(x), f_2(x), \ldots, f_n(x)$ are n particular solutions of the homogeneous equation, each corresponding to a particular choice of the initial values. Then, owing to the linear character of the equations, the $f_i(x)$'s multiplied by arbitrary constants, $C_1f_1(x), C_2f_2(x), \ldots, C_nf_n(x)$, are also solutions of the system, and we can write the general solution in the form:

$$y_x = C_1f_1(x) + C_2f_2(x) + \cdots + C_nf_n(x) \tag{2.4}$$

As in the case of linear differential equations, (2.4) gives the general solution of the difference equation (2.2), if none of the $f_i(x)$'s can be expressed as a linear combination of the others. In this case we say the particular solutions $f_i(x)$ are independent.

We obtain the general solution of the nonhomogeneous equation by adding to (2.4) a particular solution of the nonhomogeneous equation. In many cases such a particular solution can easily be found.

The general solution of a homogeneous difference equation can be obtained in a manner similar to that employed in the

theory of linear differential equations with constant coefficients. We substitute a solution of the form:

$$y_x = Ce^{\lambda x} \tag{2.5}$$

where C is an arbitrary constant and λ is an undetermined factor. Putting $e^\lambda = \beta$, we may also write

$$y_x = C\beta^x \tag{2.6}$$

and substitute this expression into the difference equation. In some cases it is more convenient to use the form (2.5), in other cases (2.6). The difference equation furnishes an algebraic equation for λ or β, respectively.

Let us consider as an example an equation of the second order, *e.g.*,

$$a_2y_{x+2} + a_1y_{x+1} + a_0y_x = 0 \tag{2.7}$$

Substituting $y_x = C\beta^x$, we obtain

$$C\beta^x(a_2\beta^2 + a_1\beta + a_0) = 0$$

Hence, excluding the cases $C = 0$ and $\beta = 0$, which would mean that $y_x = 0$ for all values of x, we have

$$a_2\beta^2 + a_1\beta + a_0 = 0 \tag{2.8}$$

If the quadratic equation (2.8) has two distinct roots, β_1 and β_2, the general solution of (2.7) is

$$y_x = C_1\beta_1^x + C_2\beta_2^x \tag{2.9}$$

If $\beta_1 = \beta_2$, Eq. (2.8) furnishes only one solution. In this case we follow a method suggested also by the theory of differential equations. We first write Eq. (2.8) in the form:

$$\beta^2 + \frac{a_1}{a_2}\beta + \frac{a_0}{a_2} = 0 \tag{2.10}$$

and as β_1 is a double root, the left side of Eq. (2.10) must be identically equal to $(\beta - \beta_1)^2$. Therefore,

$$\beta^2 + \frac{a_1}{a_2}\beta + \frac{a_0}{a_2} = \beta^2 - 2\beta\beta_1 + \beta_1^2$$

Hence, $a_1/a_2 = -2\beta_1$, $a_0/a_2 = \beta_1^2$, and the difference equation (2.7) appears in the form:

$$y_{x+2} - 2\beta_1y_{x+1} + \beta_1^2y_x = 0 \tag{2.11}$$

We verify by elementary calculation, that

$$y_x = x\beta_1^x \tag{2.12}$$

solves Eq. (2.11); in fact, we obtain, by substituting (2.12),

$$\beta_1^{x+2}[x + 2 - 2(x + 1) + x] = 0 \tag{2.13}$$

Therefore, in the case of a double root, the general solution of (2.7) is given by

$$y_x = C_1\beta_1^x + C_2x\beta_1^x \tag{2.14}$$

In the case of the difference equation (2.2) of order n, the general solution of the homogeneous equation is given by

$$y_x = \sum_{i=1}^{n} C_i\beta_i^x \tag{2.15}$$

if the n roots $\beta_1, \beta_2, \ldots, \beta_n$ are distinct. If one root, say β_1, is an m-fold root, the first m terms in (2.15) are to be replaced by $(C_1 + C_2x + \cdots + C_mx^{m-1})\beta_1^x$.

3. Application to Continuous Beams.—The analysis of continuous beams requires the evaluation of the statically indeterminate moments acting at the supports. These moments satisfy the so-called *three-moment equation,* which is a relation between the moments acting at three successive supports. If, for instance, the supports are equidistant and there is no load acting between them, the moments at successive supports are related by the equation*

$$M_{x-1} + 4M_x + M_{x+1} = 0 \tag{3.1}$$

The supports are numbered $0, 1, 2, \ldots, x$. Let us assume that we have a beam extending to infinity to the right (Fig. 3.1) and supported at equidistant points $0, 1, 2, \ldots, x$, etc. At a distance a to the left of the first support, a load P is acting on the beam. Now, if we want to calculate the moments directly by using Eq. (3.1), we can start from the known value of the moment at the first support $M_0 = -Pa$; then we assume an arbitrary value for M_1 and calculate M_2. The consecutive equations (3.1) give us $M_3, M_4, M_5, \ldots$. However, the values of $M_3, M_4, M_5 \ldots$ are dependent on the arbitrary value of M_1. This arbitrary value is determined by the boundary condition that $M_x \to 0$ as $x \to \infty$, which follows from the physical con-

*See S. Timoshenko, "Strength of Materials," vol. 1, p. 226.

sideration that the effect of a finite load cannot extend to infinity. Consequently, the value of M_1 has to be determined so that the sequence $M_0, M_1, M_2, \ldots, M_x$ tends to zero as x increases to infinity. This requires the application of a trial-and-error method.

However, if we consider (3.1) as a difference equation of second order, the method given in the last section leads to a

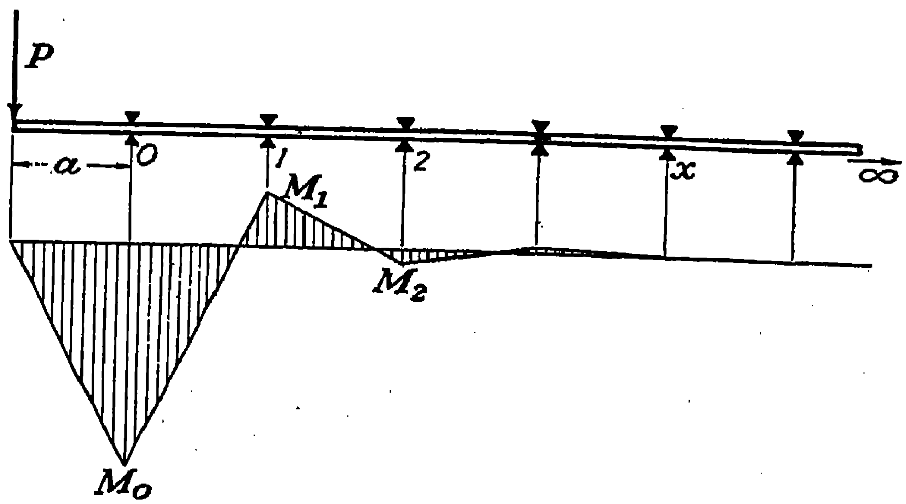

FIG. 3.1.—Bending moment diagram for a continuous beam supported at an infinite number of equidistant points and loaded at its free end by a concentrated load.

simple solution of the problem. Putting $M_x = \beta^x$, we obtain the equation

$$\beta^2 + 4\beta + 1 = 0$$

The roots of the equation are

$$\beta_1 = -2 + \sqrt{3} = -0.26795$$
$$\beta_2 = -2 - \sqrt{3} = -3.73205$$

The general solution of Eq. (3.1) is, therefore,

$$M_x = C_1(-2 + \sqrt{3})^x + C_2(-2 - \sqrt{3})^x \tag{3.2}$$

The boundary conditions are $M_0 = -Pa$ and $M_\infty = 0$. Since $|\beta_1| < 1$ and $|\beta_2| > 1$, $|\beta_1^x| \to 0$ and $|\beta_2^x| \to \infty$ as $x \to \infty$. Therefore, we put $C_2 = 0$. Then $M_0 = -Pa$ leads to

$$M_x = -Pa(-1)^x(0.26795)^x \tag{3.3}$$

Equation (3.3) gives the values of the moments at the supports. The distribution of the bending moment is shown in Fig. 3.1.

As a second example, we consider the case of a continuous beam with n equal spans and a triangular load distribution, as shown in Fig. (3.2). The three-moment equation for this case is

$$M_{x-1} + 4M_x + M_{x+1} = -\frac{Pl}{n^2}x \tag{3.4}$$

where P is the total load on the beam. Equation (3.4) is a nonhomogeneous difference equation of second order given in a form corresponding to Eq. (2.3). As a particular solution, we

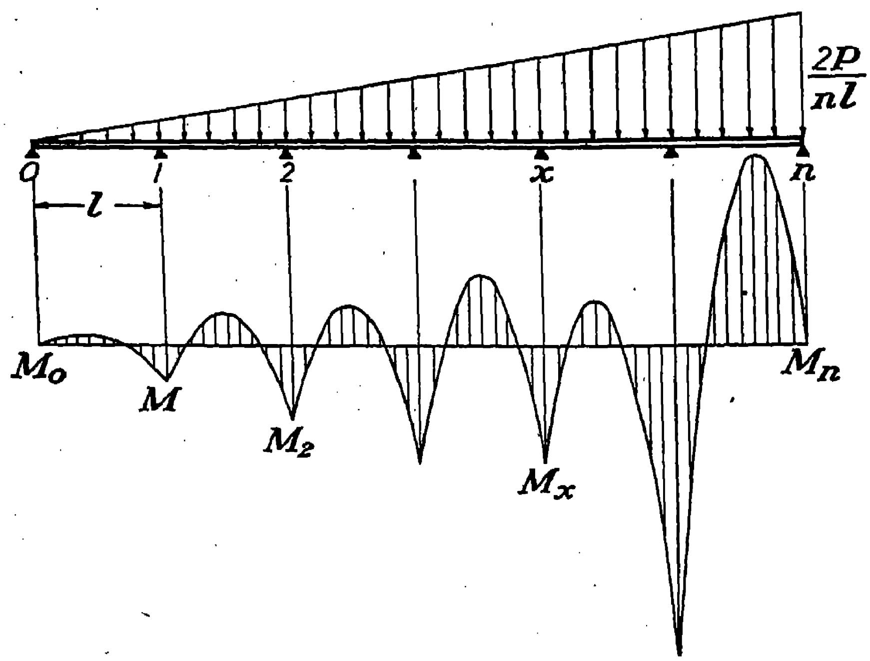

FIG. 3.2.—Bending moment diagram for a continuous beam supported at $n + 1$ points and loaded by a linearly distributed load.

try a linear function of the form $M_x = Cx$. Substituting M_x in Eq. (3.4), we obtain

$$C(x - 1) + 4Cx + C(x + 1) = -\frac{Pl}{n^2}x$$

satisfied by

$$C = -\frac{Pl}{6n^2} \tag{3.5}$$

The general solution of Eq. (3.4) will be the sum of the general solution of the homogeneous equation (3.1) and the particular solution (3.5) found above:

$$M_x = C_1(-2 + \sqrt{3})^x + C_2(-2 - \sqrt{3})^x - \frac{Pl}{6n^2}x$$

Since

$$\beta_1\beta_2 = (2 - \sqrt{3})(2 + \sqrt{3}) = 1$$

we can write

$$M_x = C_1\beta_1^x + C_2\beta_1^{-x} - \frac{Pl}{6n^2}x$$

The boundary conditions require that the moments at the supports $x = 0$ and $x = n$ be zero. From $M_0 = M_n = 0$, it follows that

$$C_1 = -C_2 = -\frac{Pl}{6n}\frac{\beta_1^n}{1 - \beta_1^{2n}}$$

and the solution becomes

$$M_x = -\frac{Pl}{6n}\left(\frac{x}{n} - \frac{\beta_1^{n-x} - \beta_1^{n+x}}{1 - \beta_1^{2n}}\right)$$

Putting $e^\gamma = -\beta_1 = 0.268$, we can introduce hyperbolic functions and obtain

$$M_x = -\frac{Pl}{6n}\left[\frac{x}{n} - (-1)^{x+n}\frac{\sinh \gamma x}{\sinh \gamma n}\right] \tag{3.6}$$

The bending moment diagram is shown in Fig. 3.2. It must be kept in mind, of course, that formula (3.6) does not represent the bending moment along the span, but only at the supports, *i.e.*, it applies only to integral values of x. To find the complete moment diagram we first connect the end points of the ordinates M_1, M_2, . . . by straight lines and then add the moments corresponding to the load under the assumption that every section of the continuous beam may be considered as simply supported.

4. Buckling of a Rectangular Lattice Truss.—A rectangular truss consisting of two vertical columns and a number of crossbeams forming n identical frames is clamped at both ends. An axial load P is applied to each column. We assume that the variation of the length of the columns can be neglected and that both columns buckle in the same way. In other words, the truss is assumed to buckle only by shear (Fig. 4.1). The crossbeams are numbered from 1 to $n - 1$.

Consider first the crossbeams in the buckled shape. They are bent into a shape with an inflection point at the center.

Hence, the bending moment at the center of each crossbean equal to zero, and if we cut the crossbeams into two halves, action of the other half can be replaced by a force acting at free end of the half crossbeam. This force is equal to the shear force occurring at the inflection point and is (Fig. 4.2) of sı magnitude, that the deflection of the crossbeam at that pc is zero. Hence, we consider half of each crossbeam as a be clamped at the joint at a right angle to the column. We den

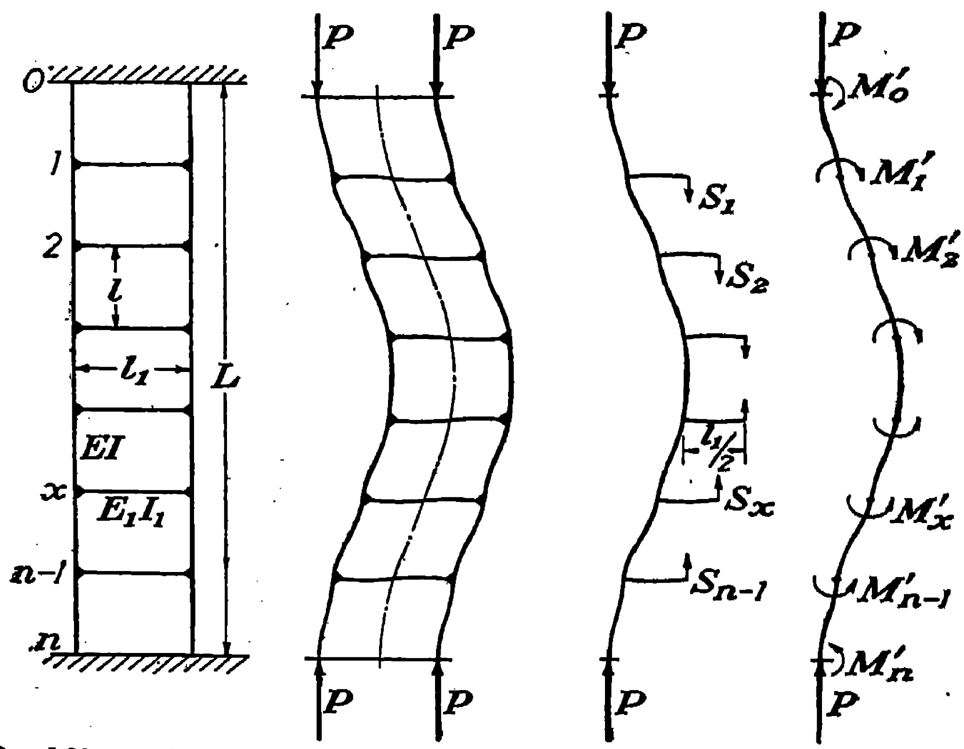

FIG. 4.1.—Buckling of a truss under axial load. Replacement of the effect the crossbeams by restraining moments.

the slope of the column at the joint of the crossbeam meml x by v_x (Fig. 4.2); the shearing force acting at the center S_x (Fig. 4.1); the length of the crossbeam by l_1; the mom of inertia of its cross section by I_1; and the elastic modulus the material by E_1. Then the deflection of the center is eq to $-v_x\frac{l_1}{2} + \frac{S_x(l_1/2)^3}{3I_1E_1} = 0$ and, therefore, $S_x = \frac{12E_1I_1}{l_1^2}v_x$. T moment at the joint is equal to $\frac{S_xl_1}{2} = \frac{6E_1I_1}{l_1}v_x$. Hence t effect of the crossbeams can be replaced by the moments acti at the joints, if we neglect the effects of second and higher orc in v_x. Therefore, the problem of the buckling of the truss reduced to the problem of the buckling of a single column wi

elastic restoring moments acting at the joints, the restoring moment being of the magnitude

$$M'_x = \frac{6E_1I_1}{l_1}v_x \tag{4.1}$$

In order to solve this problem we consider the slopes v_x of the columns at the joints as unknowns and express the bending moments acting at the ends of the vertical elements of the frames by the v_x's. Then the equilibrium conditions of the joints furnish the necessary number of equations for the determination of the v_x's and the buckling load P.

Let us consider the vertical element of the frame that extends between the joints x and $x + 1$ (Fig. 4.2). The deflection of

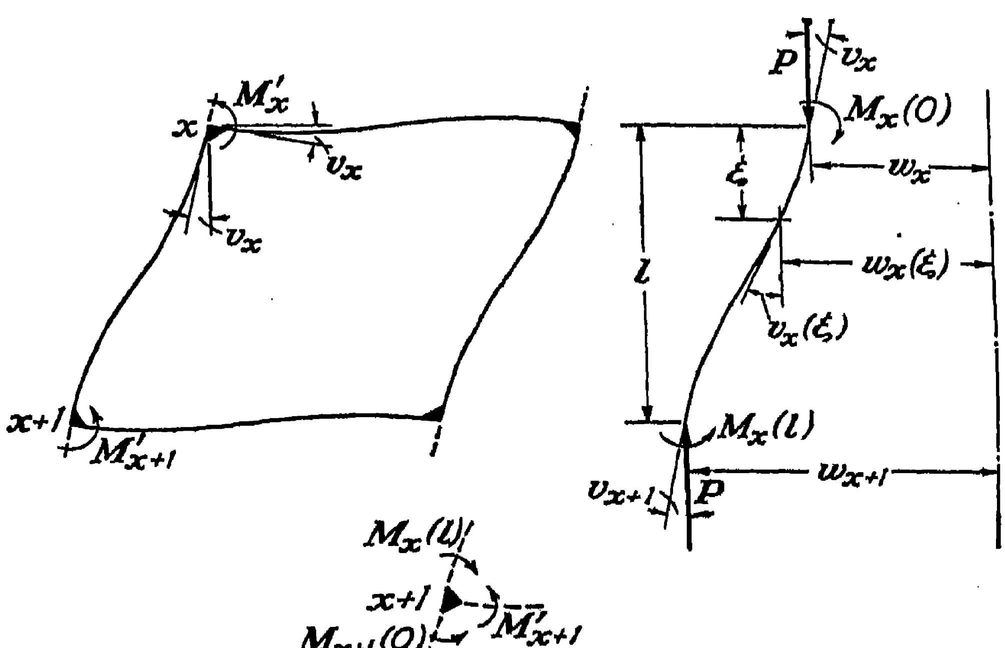

FIG. 4.2.—Buckling of a truss under axial load. Equilibrium between the forces and moments acting on the elements of the truss. Deformation of the elements.

this element, which is a section of the length l of the column with the flexural stiffness EI, is denoted by $w_x(\xi)$, where ξ is a coordinate running from $\xi = 0$ to $\xi = l$. The slope $dw_x/d\xi$ is denoted by $v_x(\xi)$ and the bending moment by $M_x(\xi)$. The element considered is loaded by end moments and by the axial force P. Therefore, $M_x(\xi) = M_x(0) + P\,[w_x(\xi) - w_x(0)]$ The differential equation for $w_x(\xi)$ is

$$EI\,\frac{d^2w_x}{d\xi^2} + M_x(0) + P\,[w_x(\xi) - w_x(0)] = 0$$

We obtain, by differentiation,

$$EI \frac{d^2 v_x}{d\xi^2} + Pv_x = 0 \tag{4.2}$$

The general solution of Eq. (4.2) is

$$v_x(\xi) = A_x \cos \alpha\xi + B_x \sin \alpha\xi \tag{4.3}$$

where

$$\alpha = \sqrt{(P/(EI)}$$

In order to express the constants A_x and B_x in terms of the slopes, we remember that $v_x(0) = v_x$ and $v_x(l) = v_{x+1}$ and obtain from (4.3)

$$v_x(\xi) = v_x \cos \alpha\xi + \frac{v_{x+1} - v_x \cos \alpha l}{\sin \alpha l} \sin x\xi \tag{4.4}$$

The bending moment $M_x(\xi)$ at an arbitrary point of the column is given by

$$\begin{aligned} M_x(\xi) &= -EI \frac{d^2 w_x(\xi)}{d\xi^2} = -EI \frac{dv_x(\xi)}{d\xi} \\ &= \alpha EI v_x \sin \alpha\xi - \frac{\alpha EI(v_{x+1} - v_x \cos \alpha l)}{\sin \alpha l} \cos \alpha\xi \end{aligned}$$

At the point $\xi = 0$, we have

$$M_x(0) = \frac{\alpha EI(v_x \cos \alpha l - v_{x+1})}{\sin \alpha l} \tag{4.5}$$

The moment $M_x(l)$, acting at the point $\xi = l$, is

$$M_x(l) = -\frac{\alpha EI(v_{x+1} \cos \alpha l - v_x)}{\sin \alpha l} \tag{4.6}$$

Having the expressions (4.5) and (4.6) for the bending moments acting on the column at the joints, we can set up the equilibrium condition for the joints. Consider, for instance, the joint $x + 1$ (Fig. 4.2). Since the moments acting about the joint must be in equilibrium, we have

$$M_{x+1}(0) + M'_{x+1} - M_x(l) = 0$$

Substituting the expressions (4.1), (4.5), and (4.6) for the moments, we find the following difference equation for the slope v_x:

$$v_{x+2} - 2\left(\cos \alpha l + 3\mu \frac{\sin \alpha l}{\alpha l}\right) v_{x+1} + v_x = 0 \tag{4.7}$$

where

$$\mu = \frac{E_1 I_1 l}{E I l_1}$$

In order to solve (4.7), it is convenient to try a solution of the form:

$$v_x = Ce^{\lambda x} \tag{4.8}$$

rather than one of the form $v_x = C\beta^x$. Substituting (4.8) in equation (4.7), we obtain

$$e^{\lambda(x+2)} - 2\left(\cos \alpha l + 3\mu \frac{\sin \alpha l}{\alpha l}\right)e^{\lambda(x+1)} + e^{\lambda x} = 0$$

or, dividing by $e^{\lambda(x+1)}$,

$$\cos \alpha l + 3\mu \frac{\sin \alpha l}{\alpha l} = \frac{e^{\lambda} + e^{-\lambda}}{2} = \cosh \lambda \tag{4.9}$$

Hence, the general solution of (4.7) can be written in the form:

$$v_x = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

or

$$v_x = C_1' \cosh \lambda x + C_2' \sinh \lambda x \tag{4.10}$$

where λ is determined by the relation (4.9).

The boundary conditions are $v_x = 0$ for $x = 0$ and $x = n$. From $v_0 = 0$ follows $C_1' = 0$. Hence, we have

$$v_n = C_2' \sinh \lambda n = 0$$

This equation cannot be satisfied by real values of λ, (except $\lambda = 0$), so that we must write $\lambda n = r\pi i$ or

$$\lambda = i\frac{r\pi}{n}$$

where r is an integer.

Then relation (4.9) becomes

$$\cos \alpha l + 3\mu \frac{\sin \alpha l}{\alpha l} = \cos \frac{r\pi}{n} \tag{4.11}$$

We obtain the lowest critical value of the buckling load P by substituting in Eq. (4.11) the smallest integer r for which v_x [cf. Eq. (4.10)] can be made to comply with the boundary conditions

$v_0 = v_n = 0$. For $r = 0$ we obtain as the only solution $v_x = 0$ for all x, which means that the column remains straight. For $r = 1$ we have from (4.10)

$$v_x = C \sin \frac{x\pi}{n} \tag{4.12}$$

It is seen that in this case the column has no vertical tangent except at the ends. Therefore, Eq. (4.12) represents a mode of buckling connected with horizontal deflection of the ends. If no deflection at the ends is allowed, we must put $r = 2$, and obtain

$$v_x = C \sin \frac{2x\pi}{n} \tag{4.13}$$

In this case the truss takes the shape shown in Fig. 4.1. We calculate this case in detail. We have from (4.9)

$$\cos \alpha l + 3\mu \frac{\sin \alpha l}{\alpha l} = \cos \frac{2\pi}{n} \tag{4.14}$$

Equation (4.14) determines the parameter α and the buckling load, which is $P = \alpha^2 EI$.

Let us first consider $n = 1$, *i.e.*, a truss consisting of two columns without a crossbeam. Then the buckling load is equal to the buckling load of a single column of the length l clamped to fixed bases at the ends, *i.e.*, $P = 4\pi^2 \frac{EI}{l^2}$. This gives $\alpha l = 2\pi$ which satisfies Eq. (4.14) for $n = 1$. In the case $n = 2$, there is one crossbeam at the mid-point of the truss. However, since the crossbeam is not deformed when the truss buckles, it does not influence the magnitude of the buckling load. Since now the column acts as a beam of length $2l$, $P = \pi^2 \frac{EI}{l^2}$ or $\alpha l = \pi$. This value of αl, in fact, satisfies Eq. (4.14) for $n = 2$.

If $n \geqq 3$, the crossbeams modify the buckling load. Equation (4.14) gives the value of $\alpha l = (Pl^2/EI)^{1/2}$ as function of μ and n. We write $P = P_E\, f(\mu,n)$ where $P_E = \pi^2 \frac{EI}{l^2}$ is the so-called *Euler's buckling load* of one section of a column. In Fig. 4.3 the ratio $P/P_E = \eta$ is plotted as function of the dimensionless ratio

$\mu = E_1 I_1 l / E I l_1$ for various values of n. For $\mu = 0$

$$\frac{Pl^2}{EI} = \left(\frac{2\pi}{n}\right)^2$$

or with $L = nl$, $PL^2/EI = 4\pi^2$, as expected, since when $\mu = 0$ the two columns buckle independently. For $\mu = \infty$, *i.e.*, when the bending stiffness of the crossbeams is very large, we find $Pl^2/EI = \pi^2$. In this case every section of the column buckles as if it were clamped between horizontal surfaces which could move freely in the horizontal direction. This is approximately correct for high buildings, for example, when the construction

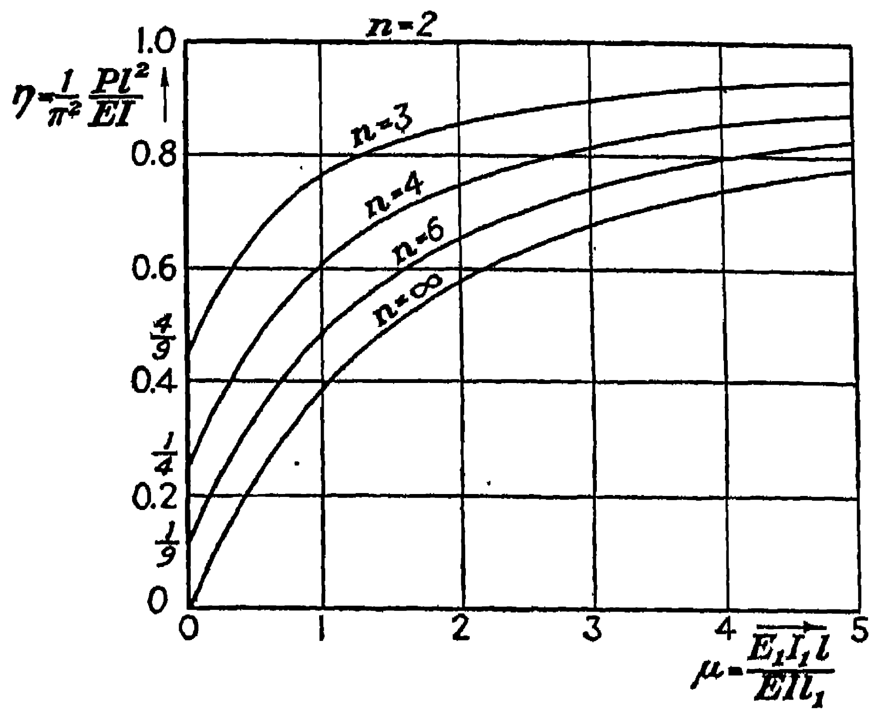

FIG. 4.3.—Buckling of a truss under axial load. Diagram for the buckling load.

of the floors is relatively more rigid than the columns. We notice that if $n \to \infty$ and l is finite, so that the structure becomes infinitely tall, P remains different from zero. This is due to the assumption that the two columns buckle identically, which excludes the bending proper of the structure as a whole.

5. Voltage Drop in a Chain of Electric Insulators.—We shall insert here an interesting problem of electrical engineering, which can be treated very conveniently by means of difference equations. The arrangement shown in Fig. 5.1 is called a *string insulator*. The $n - 1$ insulators of the chain are connected by metallic conductors. The first insulator is grounded by being attached to the tower; the last one is connected to the line

conductor, which carries an alternating current of frequency ω. Consider now one of the metallic conductors between two insulators. The rate of change of the charge of this conductor depends on its capacity with respect to all the other conductors and the ground. However, it is sufficient to take into account the capacity with respect to the two adjoining conductors and the ground. If we denote the voltage of the xth conductor by V_x,

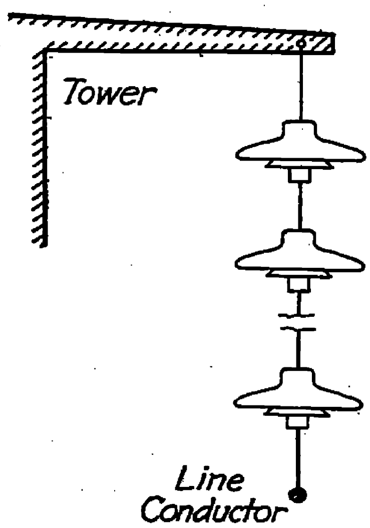

FIG. 5.1.—String insulator for an electric power-line.

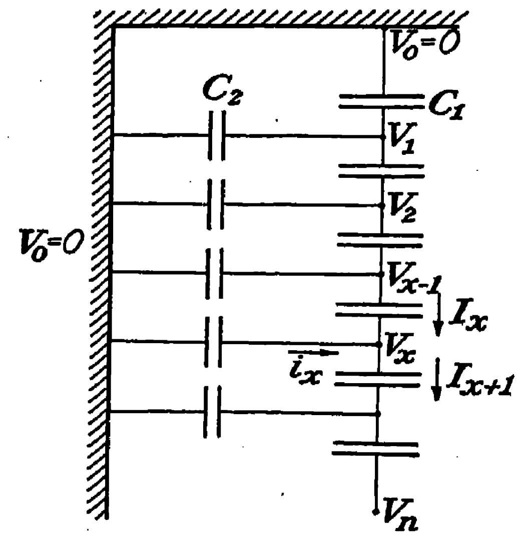

FIG. 5.2.—Equivalent network of a string insulator.

the current between the $(x-1)$th and the xth conductor is $I_x = i\omega C_1(V_{x-1} - V_x)$; between the xth and $(x+1)$th conductor $I_{x+1} = i\omega C_1(V_x - V_{x+1})$, and between the xth conductor and the ground $i_x = -i\omega C_2 V_x$, where C_1 is the capacity between two adjoining conductors and C_2 is the capacity of a conductor with respect to the ground. We assume that the conductors and insulators are all identical.

The network of capacities corresponding to the chain is schematically shown in Fig. 5.2. It is seen from the figure that $I_{x+1} = I_x + i_x$, and, therefore,

$$C_1(V_{x+1} - V_x) - C_1(V_x - V_{x-1}) - C_2 V_x = 0 \tag{5.1}$$

or

$$V_{x-1} - \left(2 + \frac{C_2}{C_1}\right)V_x + V_{x+1} = 0 \tag{5.2}$$

Equation (5.2) is a homogeneous difference equation of second order. We substitute $V_x = e^{\lambda x}$ in Eq. (5.2) and obtain

$$e^{\lambda(x-1)} - \left(2 + \frac{C_2}{C_1}\right)e^{\lambda x} + e^{\lambda(x+1)} = 0$$

or, dividing by $e^{\lambda x}$,

$$e^{-\lambda} - \left(2 + \frac{C_2}{C_1}\right) + e^{\lambda} = 0$$

It follows that

$$\cosh \lambda = 1 + \frac{C_2}{2C_1} \tag{5.3}$$

Equation (5.3) has two roots λ and $-\lambda$, and, therefore, the general solution of Eq. (5.2) is

$$V_x = A_1 e^{\lambda x} + A_2 e^{-\lambda x}$$

or

$$V_x = B_1 \cosh \lambda x + B_2 \sinh \lambda x \tag{5.4}$$

The boundary conditions are $V_0 = 0$ and $V_n = U$, where U is the voltage of the line conductor. Hence, $B_1 = 0$ and $B_2 \sinh \lambda n = U$. The distribution of voltage along the chain is, therefore, given by the equation

$$V_x = U\frac{\sinh \lambda x}{\sinh \lambda n} \tag{5.5}$$

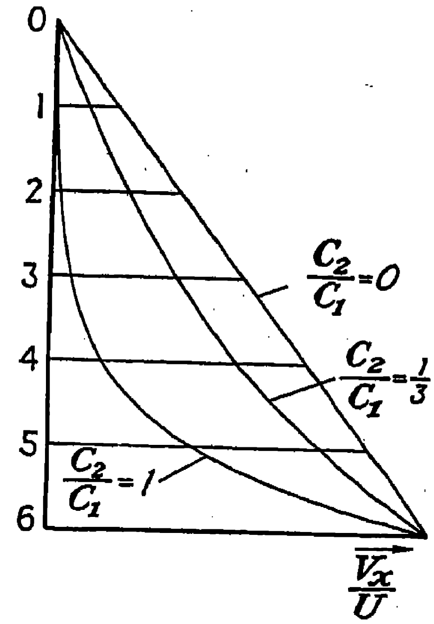

FIG. 5.3.—Distribution of potential in a chain of power-line insulators.

where λ is determined by Eq. (5.3). If the capacity with respect to the ground, C_2, is negligible in comparison with the capacity C_1 between adjoining conductors, $\lambda \rightarrow 0$ and $V_x = U\frac{x}{n}$ (Fig. 5.3). In this case the total drop of voltage is equally distributed between the insulators, the drop in each insulator being equal to U/n. However, if the ratio C_2/C_1 is not small and λn is large, the voltage drop per insulator decreases rapidly as we go farther away from the line conductor. In Fig. 5.3 the ratios V_x/U are plotted for $C_2/C_1 = 0$, $\frac{1}{3}$ and 1. Hence, the insulators near the line are much more heavily loaded than those adjoining the tower. It is also seen that beyond a certain limit the increase of the number of insulators is of little use.

6. Critical Speeds of a Multicylinder Engine.—It is generally assumed and fairly well confirmed by experience that the torsional

frequencies of the crankshaft of a multicylinder engine can be calculated with good approximation by replacing the moving system by a massless cylindrical shaft which carries a certain number of disks.

The moments of inertia of the disks are determined in such a way that the disks are dynamically equivalent to the masses of the cranks and to the masses of the reciprocating parts.

Let us consider a shaft of constant cross section carrying n identical disks spaced at equal intervals of the length l (Fig. 6.1). The moment of inertia of each disk is denoted by I.

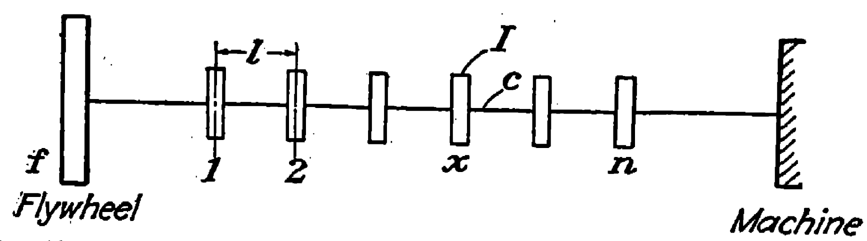

FIG. 6.1.—Distribution of the moments of inertia of the rotating parts of a multicylinder engine.

The torsional stiffness of each section of the shaft between two disks is determined by the constant c, such that if the relative angular displacement of two neighboring disks is equal to θ, the torque transmitted by the section is equal to $c\theta$.

Let us denote the angular displacement of the xth disk by θ_x. Then the equation of motion of this disk is given by

$$I\frac{d^2\theta_x}{dt^2} = c(\theta_{x-1} - \theta_x) - c(\theta_x - \theta_{x+1}) \tag{6.1}$$

Assuming harmonic motion with the amplitude Θ_x and with an angular frequency ω, we substitute $\theta_x = \Theta_x \sin \omega t$ and obtain

$$I\omega^2\Theta_x + c(\Theta_{x-1} + \Theta_{x+1} - 2\Theta_x) = 0$$

or with $\alpha^2 = I\omega^2/c$,

$$\Theta_{x-1} - (2 - \alpha^2)\Theta_x + \Theta_{x+1} = 0 \tag{6.2}$$

It is seen from the figure that this difference equation does not apply to the first and last disks, but is correct for $x = 2, 3, \cdots, n - 1$ and can be solved by the method used in section 5. We substitute in Eq. (6.2) $\Theta_x = e^{\lambda x}$. Then, we obtain

$$e^{-\lambda} - (2 - \alpha^2) + e^{\lambda} = 0$$

or

$$\cosh \lambda = 1 - \frac{\alpha^2}{2}$$

There are three cases:

a. If $|\alpha| < 2$, $|\cosh \lambda| < 1$, and λ is a pure imaginary number. Substituting $\lambda = \mu i$, we have

$$\cos \mu = 1 - \frac{\alpha^2}{2}$$

or

$$\alpha = 2 \sin \frac{\mu}{2} \tag{6.3}$$

The corresponding solution is

$$\Theta_x = A \cos \mu x + B \sin \mu x \tag{6.4}$$

b. If $|\alpha| > 2$, $1 - \frac{\alpha^2}{2}$ is negative and in absolute value larger than unity. In this case we put $\lambda = \mu + i\nu$, and obtain

$$\cosh \mu \cosh i\nu + \sinh \mu \sinh i\nu = 1 - \frac{\alpha^2}{2} \tag{6.5}$$

As $\cosh i\nu = \cos \nu$ and $\sinh i\nu = i \sin \nu$, we must have $\sin \nu = 0$. If we choose $\nu = 0$, we obtain $\cosh \mu = 1 - \frac{\alpha^2}{2}$, which is impossible. Therefore, we put $\nu = \pi$. Then

$$\cosh \mu \cos \pi = 1 - \frac{\alpha^2}{2}$$

and

$$\cosh \mu = \frac{\alpha^2}{2} - 1$$

or

$$\alpha = 2 \cosh \frac{\mu}{2} \tag{6.6}$$

Then $e^{\lambda x} = e^{x(\mu + i\pi)}$, or as $e^{i\pi} = -1$, $e^{\lambda x} = (-1)^x e^{\mu x}$. Hence, we write the general solution in the form

$$\Theta_x = A(-1)^x e^{\mu x} + B(-1)^x e^{-\mu x}$$

or

$$\Theta_x = C(-1)^x \cosh \mu x + D(-1)^x \sinh \mu x \tag{6.7}$$

c. The limiting case is $|\alpha| = 2$. Then Eq. (6.2) becomes

$$\Theta_{x-1} + 2\Theta_x + \Theta_{x+1} = 0 \tag{6.8}$$

The two independent solutions of this equation are $(-1)^x$ and $(-1)^x x$. The different types of solutions are shown in Fig. 6.2 for a shaft extending from $x = 0$ to $x = \infty$ (cf. section 7).

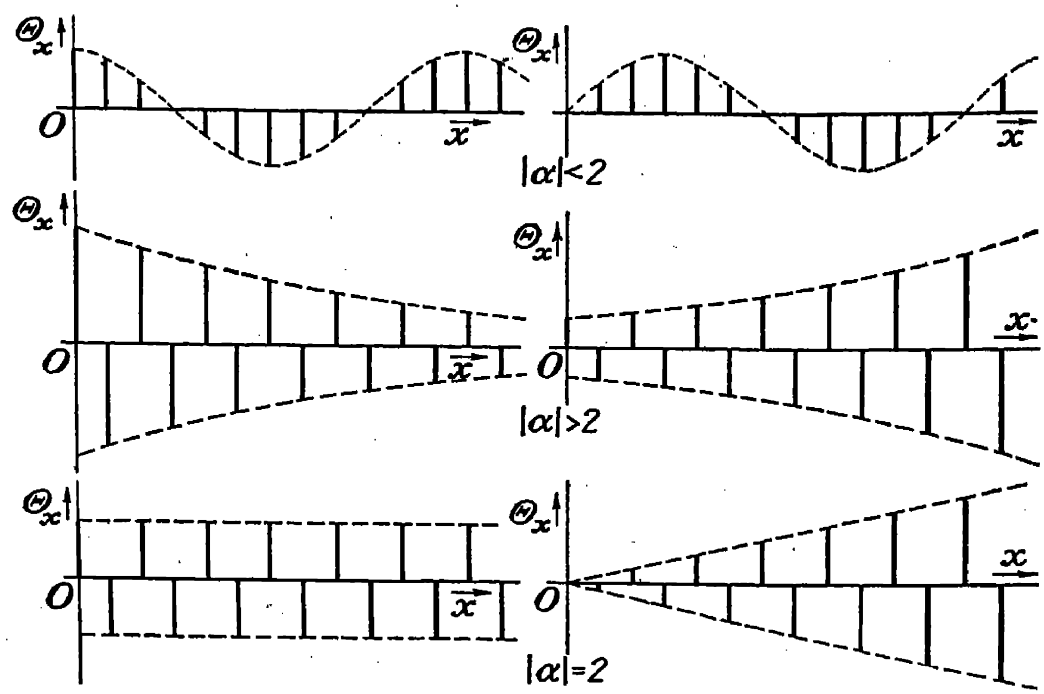

FIG. 6.2.—Types of forced oscillations in a system of elastically coupled disks.

The equation for the first disk becomes [cf. Eq. (6.1)]

$$I\frac{d^2\Theta_1}{dt^2} = c(\Theta_2 - \Theta_1) - k_0(\Theta_1 - \Theta_f) \tag{6.9}$$

where Θ_f is the angular deflection of the flywheel and k_0 is the elastic constant of the shaft between the first disk and the flywheel. For the flywheel we have the equation

$$I_0\frac{d^2\Theta_f}{dt^2} = k_0(\Theta_1 - \Theta_f) \tag{6.10}$$

If we put $d^2\Theta_1/dt^2 = -\Theta_1\omega^2$, $d^2\Theta_f/dt^2 = -\Theta_f\omega^2$, and eliminate Θ_f between (6.9) and (6.10), we obtain

$$-I\omega^2\Theta_1 = c(\Theta_2 - \Theta_1) - \frac{k_0}{1 - (k_0/I_0\omega^2)}\Theta_1 \tag{6.11}$$

We shall use the notation $K_0 = \dfrac{k_0}{1 - (k_0/I_0\omega^2)}$. (The constant K_0 is identical with the dynamic spring constant defined in Chapter IX, section 4.)

The significance of K_0 is the following: The effect of the flywheel on the motion of the first disk is the same as though the disk were elastically joined to a fixed base by a spring whose spring factor K_0 varied with the frequency. If ω is small, K_0 is small and negative. If ω passes through $\omega_0 = \sqrt{k_0/I_0}$, K_0 becomes $-\infty$ and then jumps to $+\infty$ and decreases with increasing value of ω. If ω is very large, $K_0 \cong k_0$. In this case the flywheel acts essentially as a fixed base. The factor K_0 is plotted against ω in Fig. 6.3.

From Eq. (6.11) it follows that

$$\Theta_2 - (1 - \alpha^2)\Theta_1 - \frac{K_0}{c}\Theta_1 = 0 \qquad (6.12)$$

The equation for the oscillation of the last disk is determined by the coupling of the crankshaft with the machine. Let us assume

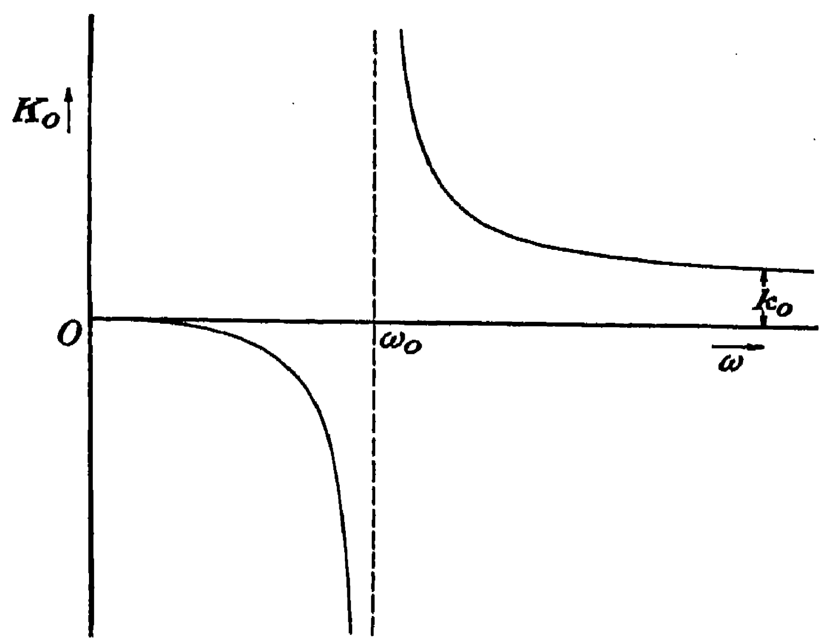

FIG. 6.3.—The dynamic spring constant K_0 as function of the frequency.

that the mechanical characteristics of the machine can also be replaced by a dynamic spring constant which we denote by $K_{n+1}(\omega)$. Then we obtain for the nth disk the equation

$$\Theta_{n-1} - (1 - \alpha^2)\Theta_n - \frac{K_{n+1}}{c}\Theta_n = 0 \qquad (6.13)$$

which is analogous to Eq. (6.12).

Equations (6.12) and (6.13) furnish two relations for the determination of the constants of integration occurring in the

solutions (6.4) and (6.7). The calculation can be simplified by the following procedure:

We define two quantities Θ_0 and Θ_{n+1} formally by substituting in the solution (6.4) [or (6.7)] $x = 0$ and $x = n + 1$. Using Eq. (6.4), we obtain

$$\Theta_0 = A$$

and

$$\Theta_{n+1} = A \cos \mu(n+1) + B \sin \mu(n+1) \tag{6.14}$$

Then we have

$$\Theta_0 - (2 - \alpha^2)\Theta_1 + \Theta_2 = 0$$

and

$$\Theta_{n-1} - (2 - \alpha^2)\Theta_n + \Theta_{n+1} = 0 \tag{6.15}$$

Comparing Eqs. (6.15) with Eqs. (6.12) and (6.13), which account for the end conditions, we obtain

$$\begin{aligned} \Theta_0 + \left(\frac{K_0}{c} - 1\right)\Theta_1 &= 0 \\ \Theta_{n+1} + \left(\frac{K_{n+1}}{c} - 1\right)\Theta_n &= 0 \end{aligned} \tag{6.16}$$

Equations (6.16) may be considered as the *boundary conditions* for $x = 0$ and $x = n + 1$.

Substituting for Θ_0, Θ_1, Θ_n, Θ_{n+1} their expressions from Eqs. (6.4) and (6.14), we obtain the following two equations for A and B:

$$A\left[1 + \left(\frac{K_0}{c} - 1\right)\cos \mu\right] + B\left(\frac{K_0}{c} - 1\right)\sin \mu = 0$$

$$A\left[\cos \mu(n+1) + \left(\frac{K_{n+1}}{c} - 1\right)\cos \mu n\right] + B\left[\sin \mu(n+1) + \left(\frac{K_{n+1}}{c} - 1\right)\sin \mu n\right] = 0$$

These two linear equations have solutions for A and B different from zero if

$$\sin \mu(n+1) + \left[\left(\frac{K_0}{c} + \frac{K_{n+1}}{c}\right) - 2\right]\sin \mu n + \left(\frac{K_0}{c} - 1\right)\left(\frac{K_{n+1}}{c} - 1\right)\sin \mu(n-1) = 0 \tag{6.17}$$

We remember that $\alpha = \omega\sqrt{\frac{I}{c}} = 2 \sin \frac{\mu}{2}$ and, on the other hand, that K_0 and K_{n+1} are given functions of ω. Hence, Eq. (6.17) is an equation determining ω, *i.e.*, it is the *frequency equation* of the oscillating system. It can be solved graphically by plotting the left side as function of ω. We have used only the solution (6.4) which assumes $\alpha < 2$. To be thorough we should also use the solution (6.7) and investigate whether there are frequencies for which $\alpha > 2$. The frequency equation for this case is similar to (6.17) and contains hyperbolic functions.

Let us consider the simple case $K_{n+1} = K_0 = 0$, *i.e.*, the shaft with free ends. Then Eq. (6.17) reads

$$\sin \mu(n + 1) - 2 \sin \mu n + \sin \mu(n - 1) = 0$$

or

$$\sin \mu n(\cos \mu - 1) = 0$$

If the factor $1 - \cos \mu = 2 \sin^2 \frac{\mu}{2} = 0$, then $\omega = 0$. All the other frequencies are obtained from $\sin \mu n = 0$, *i.e.*, $\mu = r\pi/n$, where $r = 1, 2, \cdots$ The corresponding values of α are $\alpha_r = 2 \sin \frac{r}{n}\frac{\pi}{2}$. The corresponding frequencies are

$$\omega_r = 2\sqrt{\frac{c}{I}} \sin \frac{r}{n}\frac{\pi}{2} \tag{6.18}$$

Substituting $r = n$ into (6.18), we obtain $\omega_n = 2\sqrt{c/I}$. However, if we calculate the values of Θ_x for this case, *i.e.*, the corresponding mode of oscillation, we find identically $\Theta_x = 0$ for all values of x. Hence, ω_n does not represent the frequency of an actual vibration. If we substitute $r = n + 1, n + 2, \ldots$ the values of $|\omega_r|$ are repeated. Hence, we have n frequencies including $\omega = 0$. Moreover, it is seen that $\omega_n = 2\sqrt{c/I}$ is an upper limit for the frequencies of the system, no matter how many disks there are. The *spectrum* of the frequencies of the system is shown in Fig. 6.4 for eight disks. The system has 8 degrees of freedom; however, one degree of freedom corresponds to the rotation of the whole system; this is formally verified by the fact that $\omega = 0$ satisfies the frequency equation. With an increase in

number of the degrees of freedom, the frequencies come closer together and especially crowd together near the limiting frequency.

It is interesting to investigate the behavior of the system when n approaches infinity. We can proceed to the limit in two different ways. If we keep the total length of the shaft constant and increase the number of disks, the frequencies of the torsional oscillations of the system will be identical in the limit with the oscillations of a shaft with uniform moment of inertia. In this case, the frequencies are proportional to the integral numbers and there is no upper limit for the frequencies. If we keep the

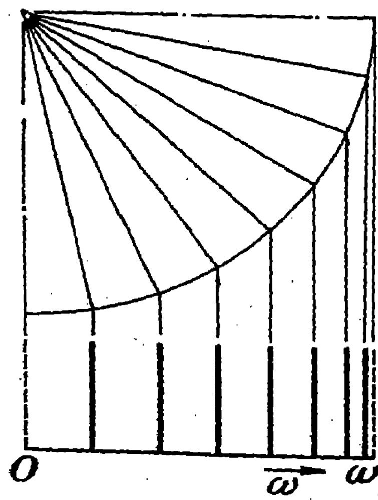

FIG. 6.4.—Frequency spectrum of an eight-cylinder in line engine.

distance between adjoining disks constant and increase the number of disks, the number of frequencies increases, but they all lie between zero and the upper limit $2\sqrt{c/I}$. We shall show in the next section that this fact greatly influences the type of forced oscillations occurring in such a system.

Before concluding this section let us consider the case in which one end of the shaft is free ($K_0 = 0$) and the other end is fixed a distance l from the last disk ($K_{n+1} = k$). Then from Eq. (6.17) we obtain

$$\omega_r = 2\sqrt{\frac{c}{I}} \sin \frac{2r - 1}{2n + 1}\frac{\pi}{2} \tag{6.19}$$

where $r = 1, 2, \ldots, n$.

The system considered in Chapter V, section 6, is a special case of the above arrangement for which $n = 3$. Therefore the frequencies are equal to $2\sqrt{c/I}$ multiplied by $\sin \pi/14$, $\sin 3\pi/14$,

and $\sin 5\pi/14$. These results will be found in agreement with the numerical values given on pages 179 and 180.

It is seen that if a system consists of identical units, the method of difference equations possesses the great advantage that the frequencies and the coefficients of normal modes can be expressed by elementary functions.

7. Waves in a Mechanical Chain.—Let us assume that the shaft extends indefinitely from $x = 0$ to $x = \infty$ and that the end at $x = 0$ performs angular oscillations described by $\theta_0 = \Theta \sin \omega t$. We assume that all disks oscillate with the frequency ω but with unknown amplitudes and phases. Hence, we write for the angular deflection of the xth disk

$$\theta_x = \Theta_x' \sin \omega t + \Theta_x'' \cos \omega t \tag{7.1}$$

Substituting the expression (7.1) in Eq. (6.1), we find that both Θ_x' and Θ_x'' must satisfy Eq. (6.2): $\Theta_{x-1} - (2 - \alpha^2)\Theta_x + \Theta_{x+1} = 0$. Let us first assume that $|\alpha| < 2$. Then Θ_x' and Θ_x'' are given by

$$\begin{aligned} \Theta_x' &= A' \cos \mu x + B' \sin \mu x \\ \Theta_x'' &= A'' \cos \mu x + B'' \sin \mu x \end{aligned} \tag{7.2}$$

where $\alpha = \omega\sqrt{I/c} = 2 \sin \mu/2$. The end conditions are $\Theta_0' = \Theta$ and $\Theta_0'' = 0$. Hence, $A' = \Theta$ and $A'' = 0$, while B' and B'' are undetermined, unless conditions at the other end of a system are given. Thus, the general solution of the problem is given by

$$\theta_x = \Theta \cos \mu x \sin \omega t + B' \sin \mu x \sin \omega t + B'' \sin \mu x \cos \omega t \tag{7.3}$$

Let us consider a special case, putting $B' = 0$ and $B'' = -\Theta$, then

$$\begin{aligned} \theta_x &= \Theta(\cos \mu x \sin \omega t - \sin \mu x \cos \omega t) \\ &= \Theta \sin (\omega t - \mu x) \end{aligned} \tag{7.4}$$

Equation (7.4) represents a *progressing wave*. If we compare two instants t and $t + \frac{\mu}{\omega}$, we see that $\theta_x(t) = \theta_{x+1}\left(t + \frac{\mu}{\omega}\right)$, *i.e.*, the $(x + 1)$th disk has the same angular deflection at the time $t + \frac{\mu}{\omega}$ that the xth disk had at the time t. Hence, the time in which a certain angular deflection progresses from disk to disk

is equal to μ/ω. If we denote by l the distance between adjoining disks, the *velocity of propagation* w is equal to

$$w = \frac{l\omega}{\mu} = 2l\sqrt{\frac{c}{I}}\frac{\sin\frac{\mu}{2}}{\mu} \tag{7.5}$$

The *wave length* is equal to $2\pi l/\mu$. For very long waves, *i.e.*, for the limiting case $\mu \to 0$, $\omega = l\sqrt{c/I}$. The wave velocity decreases if μ increases and reaches the value $w = \frac{2l}{\pi}\sqrt{\frac{c}{I}}$ for $\mu = \pi$. This value corresponds to the limiting case $|\alpha| = 2$. For this value of α, the wave length becomes $2l$, *i.e.*, the half-wave length is equal to the distance between two adjoining disks. Therefore, the adjoining disks are oscillating in opposite phase; it is seen that in this limiting case the notion of a progressing wave becomes confused with that of a standing wave. The limiting case can be interpreted either as a standing wave or as a progressing wave in which the wave reaches the next disk in a time equal to the half period.

Let us now consider $|\alpha| > 2$. According to Eq. (6.7), in this case the solution corresponding to Eq. (7.2) is given by

$$\begin{aligned} \Theta'_x &= A'(-1)^x e^{\lambda x} + B'(-1)^x e^{-\lambda x} \\ \Theta''_x &= A''(-1)^x e^{\lambda x} + B''(-1)^x e^{-\lambda x} \end{aligned} \tag{7.6}$$

where

$$\alpha = 2\cosh\frac{\lambda}{2}$$

If we assume that the angular deflection remains finite at infinity, we have $A' = A'' = 0$. From the end condition at $x = 0$, it follows that $B' = \Theta$ and $B'' = 0$. Therefore, we have

$$\theta_x = \Theta(-1)^x e^{-\lambda x}\sin\omega t \tag{7.7}$$

It is seen that no progressing wave is possible in the system. We obtain a standing wave with two adjoining disks oscillating in opposite phase. The amplitude of the oscillation decreases exponentially with x.

Hence, $\omega_1 = 2\sqrt{c/I}$ is the maximum frequency of torsional waves traversing the chain of disks considered. We may call

ω_1 the *cut-off frequency* of the system. We have seen in section 6 that the same value of ω is the upper limit for the free oscillation frequencies of the system. Therefore, it appears that no wave whose frequency is higher than the upper limit of the free oscillations can pass through the chain. The analogous electric device, consisting of a network of coils and capacitors, generally used in the technique of oscillating circuits, is called a *low-pass filter*.

We now modify the mechanical properties of the chain by inserting identical springs between each disk and a fixed base (Fig. 7.1). We assume that the restoring moment exerted by

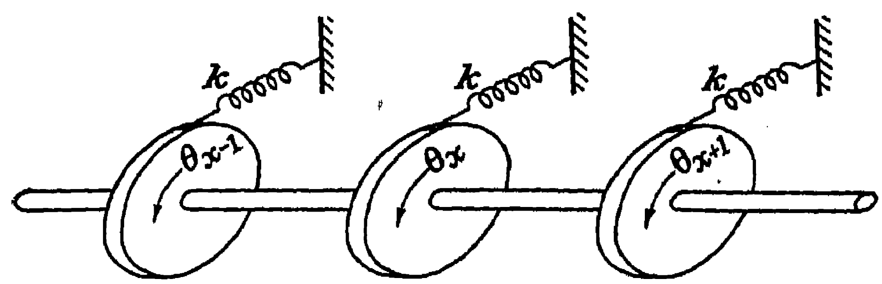

FIG. 7.1.—Mechanical model of a band-pass filter.

each spring is equal to $-k\theta_x$. Then the equation of motion for the xth disk is given by

$$c(\theta_{x-1} - \theta_x) + c(\theta_{x+1} - \theta_x) - k\theta_x = I\frac{d^2\theta_x}{dt^2} \tag{7.8}$$

Substituting $\theta_x = \Theta_x \sin \omega t$, we obtain

$$\Theta_{x-1} - 2\left(1 - \frac{I\omega^2}{2c} + \frac{k}{2c}\right)\Theta_x + \Theta_{x+1} = 0 \tag{7.9}$$

Let us write $\beta = 1 + \dfrac{k}{2c} - \dfrac{I\omega^2}{2c}$. Then Eq. (7.9) becomes

$$\Theta_{x-1} - 2\beta\Theta_x + \Theta_{x+1} = 0 \tag{7.10}$$

We have already discussed this equation for $\beta \leqq 1$ and have found two types of solutions: one for the case $-1 \leqq \beta \leqq 1$, and another for $-\infty < \beta < -1$. There remains the case $\beta > 1$ to be discussed. Putting $\Theta = e^{\lambda x}$, we obtain $\cosh \lambda = \beta$, which gives for $\beta > 0$ a real value of λ. The corresponding solution is

$$\Theta_x = Ae^{\lambda x} + Be^{-\lambda x} \tag{7.11}$$

The condition $\beta > 1$ means $k/c - I\omega^2/c > 0$ or $\omega^2 < k/I$, whereas $\beta < -1$ leads to $\omega^2 > \frac{k + 4c}{I}$. Hence, $\omega = \sqrt{k/I}$ represents a lower limit for solutions of the type

$$\Theta_x = A \cos \mu x + B \sin \mu x$$

We find that below this lower limit no progressing wave can travel through the chain. Now, if k is large in comparison with c, *i.e.*, the springs which connect the disks to the fixed base

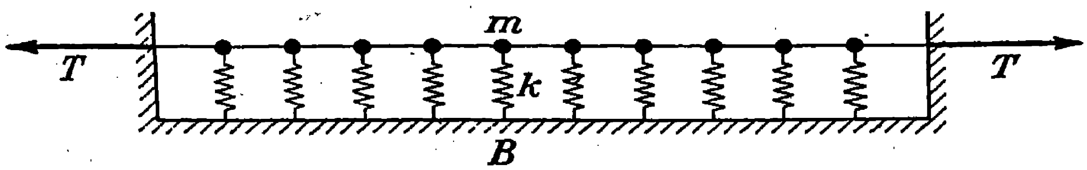

FIG. 7.2.—Mechanical model of a band-pass filter.

are much stronger than the springs between the disks, the two limits $\omega_1 = \sqrt{k/I}$ and $\omega_2 = \sqrt{\frac{k}{I}\left(1 + \frac{4c}{k}\right)}$ come close together. Hence, the behavior of such a mechanical system is similar to

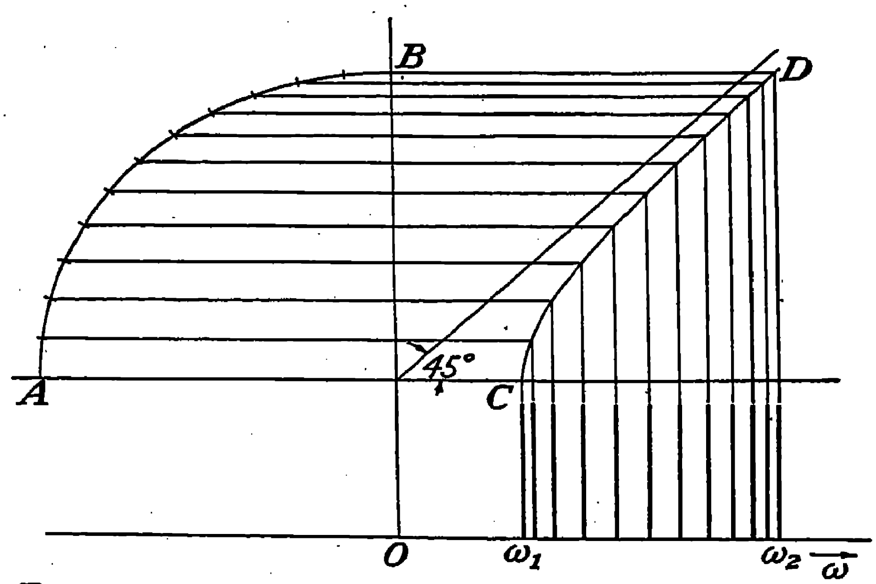

FIG. 7.3.—Frequency spectrum of the mechanical model shown in Fig. 7.2.

that of a selective filter that allows the passage of progressing waves only in a narrow band of frequencies. If the system is used as a *filter*, ω_1 and ω_2 are the *cut-off frequencies*.

The character of the oscillation in the range below the lower limit ω_1 is given by (7.11). If we exclude the case in which Θ is infinite at infinity, we obtain a standing wave; the amplitudes of the successive disks decrease exponentially with the distance from the end, and all disks oscillate in the same phase.

Figure 7.2 shows an arrangement that is mechanically similar to the system of disks shown in Fig. 7.1. Ten mass points are attached to a string held under tension and are bound to a fixed base B by springs. The 10 frequencies are all between the two limits ω_1 and ω_2, as shown in Fig. 7.3.

The frequencies are given by the equation $\omega_r^2 - \frac{k}{I} = \frac{4c}{I} \sin \frac{r\pi}{24}$, where $r = 1, 2 \ldots 10$. To obtain the ω_r's graphically, we draw a hyperbola, whose equation is $\omega^2 - y^2 = k/I$, where the ordinate y is measured from the line AC. The radius of the circle is equal to $2\sqrt{c/I}$.

8. Electric-wave Filters.—Let us consider the network shown in Fig. 8.1. Its arrangement is similar to a ladder. One side of the ladder, AB, is a conductor kept at zero potential; the other side, CD, consists of a series of identical coils and capacitors in series. The rungs represent another series of coils and capacitors.

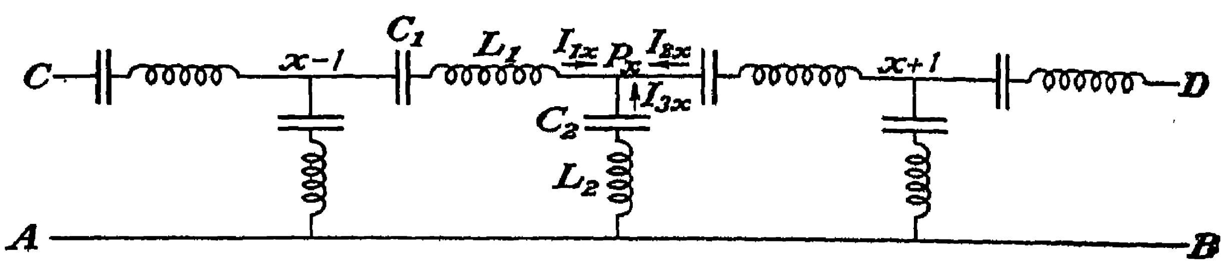

FIG. 8.1.—Network diagram of an electric wave filter.

We denote the voltage of the point P_x, where the xth rung of the ladder joins the side CD, by V_x, and the three currents meeting in P_x by I_{1x}, I_{2x}, I_{3x}, where

$$I_{1x} + I_{2x} + I_{3x} = 0 \tag{8.1}$$

The impedance Z_1 of the combination L_1 and C_1 is given by $Z_1 = L_1 i\omega + \frac{1}{C_1 i\omega}$; that of the combination C_2 and L_2 by

$$Z_2 = L_2 i\omega + \frac{1}{C_2 i\omega}$$

Then $I_{1x} = \frac{V_{x-1} - V_x}{Z_1}$, $I_{2x} = \frac{V_{x+1} - V_x}{Z_1}$, and $I_{3x} = -\frac{V_x}{Z_2}$

Substituting these expressions in Eq. (8.1), we obtain the following relation between the consecutive voltages V_{x-1}, V_x, V_{x+1}:

$$V_{x-1} - \left(2 + \frac{Z_1}{Z_2}\right)V_x + V_{x+1} = 0 \tag{8.2}$$

This equation is formally identical with (7.10). The ratio Z_1/Z_2 is real, because Z_1 and Z_2 are pure imaginary numbers. Substituting the values of Z_1 and Z_2, we have

$$\beta = 1 + \frac{Z_1}{2Z_2} = 1 + \frac{1}{2}\frac{L_1\omega^2 - (1/C_1)}{L_2\omega^2 - (1/C_2)} \tag{8.3}$$

The solutions of Eq. (7.10) have been discussed in sections 6 and 7. It was found that progressing waves can pass through the system only if $-1 < \beta < 1$. We distinguish between the following cases:

a. Assume that the network consists of capacitors or of coils only. In the first case $\beta = 1 + \frac{C_2}{2C_1}$; in the second case $\beta = 1 + \frac{1}{2}\frac{L_1}{L_2}$; in both cases $\beta > 1$; and, therefore, no wave can pass through the system. The device acts as a *wave trap*.

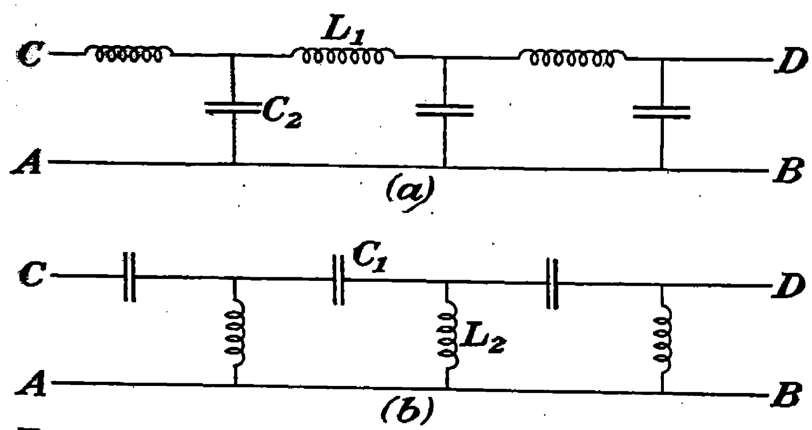

FIG. 8.2.—(*a*) Low-pass filter; (*b*) high-pass filter.

b. Assume that the line CD contains coils only and the rungs of the ladder merely capacitors, as shown in Fig. 8.2*a*. Then $L_2 = 0$, and $C_1 = \infty$; $\beta = 1 - \frac{1}{2}L_1C_2\omega^2$. The device lets through waves of the frequency ω if $\beta > -1$. Hence, the device acts as a *low-pass filter* with the cut-off frequency $\omega_1 = 2/\sqrt{L_1C_2}$.

c. Assume that the line BD contains capacitors only and the rungs are constructed of coils (Fig. 8.2*b*). Then we must substitute $L_1 = 0$ and $C_2 = \infty$ and obtain $\beta = 1 - \frac{1}{2C_1L_2\omega^2}$. We see that β is a large negative quantity for small values of ω, that it passes through $\beta = -1$ for $\omega_2^2 = 1/4C_1L_2$, and that it converges to 1 if $\omega \to \infty$. Therefore, the device cuts off the

waves of low frequency up to $\omega_2 = 1/2\sqrt{C_1L_2}$. The device is a *high-pass filter*.

d. If L_1 and L_2 are not zero, and C_1 and C_2 are finite, we obtain waves between two limiting frequencies ω_1 and ω_2, which correspond to $\beta = 1$ and $\beta = -1$. From Eq. (8.3) it follows that

$$\omega_1 = \frac{1}{\sqrt{L_1C_1}} \quad \text{and} \quad \omega_2 = \frac{1}{\sqrt{L_1C_1}}\sqrt{\frac{1 + \frac{4C_1}{C_2}}{1 + \frac{4L_2}{L_1}}}.$$

The device acts as a *band-pass filter*. If ω_1 and ω_2 are near together, only a narrow range of frequencies will pass through the filter. It is seen that this will be the case if C_1/C_2 and L_2/L_1, *i.e.*, C_1L_1 and C_2L_2 are only slightly different.

Figure 8.3 shows the parameter β as function of the frequency ω for the three cases *b*, *c*, and *d*.

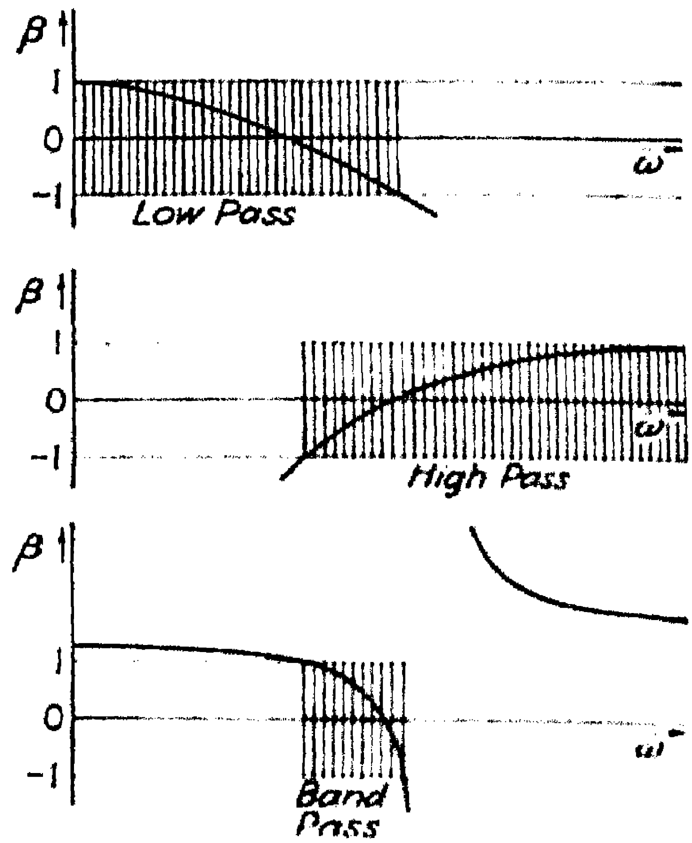

FIG. 8.3. Diagram for the frequency range of various types of filters.

Problems

1. The numbers 1, 3, 6, 10, . . . , etc., constitute an arithmetic progression of second order, *i.e.*, their differences of second order are constant. Find the value of the hundredth term of the progression by setting up and solving the difference equation.

2. Calculate the frequencies and normal modes for the mechanical systems described in (*a*) Prob. 10, Chapter V; (*b*) Prob. 13, Chapter V, by the method of difference equations.

3. Calculate the frequencies and the modes of vibration of a 10-story building. The mass of the building is equally distributed between the nine floors above the ground. The stories have equal rigidities with the exception of the first story, which is designed to have half the rigidity of the other stories.

4. Write the difference equation for the horizontal displacement of the truss joints in the buckling problem of section 4. Show that when the truss buckles, the joints lie on a sine curve.

5. Consider an infinitely extended string under a tension F. The string carries equidistant masses m, which are immersed in a viscous fluid. Discuss

the nature of the wave propagation when a harmonic transversal motion is imposed on one of the masses.

References

1. BLEICH, F., and E. MELAN: "Die gewöhnlichen und partiellen Differenzengleichungen der Baustatik," Verlag Julius Springer, Berlin, 1927.
2. FUNK, P.: "Die linearen Differenzengleichungen und ihre Anwendung in der Theorie der Baukonstruktionen," Verlag Julius Springer, Berlin, 1920.
3. BOOLE, G.: "A Treatise on the Calculus of Finite Differences," Macmillan & Company, Ltd., London, 1880.
4. SHEA, T. E.: "Transmission Networks and Wave Filters," D. Van Nostrand Company, Inc., New York, 1929.
5. GUILLEMIN, E. A.: "Communication Networks," vol. 2, John Wiley & Sons, Inc., New York, 1935.

WORDS AND PHRASES

"Words and Phrases" is the title of a well-known collection of legal terms, especially of the legal interpretation of expressions used in everyday life. The words and phrases collected here attempt to give the strictly mathematical definitions of some of the concepts and operations commonly used by physicists and engineers.

1. "God made the *integers;* all the rest is the work of man."—L. Kronecker.

2. *Negative integers* are introduced so that the equation $x + a = b$ always has a solution when a and b are positive integers. If $a > b$, x is a negative integer.

3. A *rational* number is a number that can be expressed as the quotient of two integers.

4. We say that a *real number a* is defined if there is a rule that decides whether an arbitrary rational number is smaller or greater than a. For example, the rule that any positive rational number is greater than a if its square is greater than 2 and any rational number is less than a if its square is smaller than 2 defines a positive number a called $\sqrt{2}$. If the number defined in this way is not a rational number, it is called an *irrational number*.

5. Every *irrational* number can be included in an arbitrarily small interval between two rational numbers; *e.g.*, $1.4 < \sqrt{2} < 1.5$, $1.41 < \sqrt{2} < 1.42$, $1.414 < \sqrt{2} < 1.415$, etc.

6. A *complex number* $a + bi$ is a number defined by a pair of real numbers a and b so that $a + bi$ formally obeys the rules of addition and multiplication, including the additional rule that $i^2 = -1$. Hence,

$$\begin{aligned}(a + bi) + (c + di) &= (a + b) + (c + d)i \\ (a + bi)(c + di) &= (ac - bd) + (ad + bc)i\end{aligned}$$

a is called the *real part*, bi the *imaginary part*, of the complex number $a + bi$.

7. A complex number whose real part is zero is called a *pure imaginary number*.

8. The *logarithm* of a number to a certain base is the exponent to which the base must be raised to produce the number. If the base is 10, the logarithm is known as the *Briggs's logarithm* or *common logarithm* of the number. If the base is the irrational number known as $e = 2.718281828$ . . . , the logarithm is known as the *Napierian logarithm* or *natural logarithm* of the number.

9. A *sequence of numbers* is a set of numbers arranged in a definite order. If the number of members of the set is greater than every integer N but the members are arranged in order according to a definite rule, the set forms an *infinite sequence.*

10. If for any given positive number ϵ, however small, an integer ν can be found such that $|a - a_n| < \epsilon$ whenever n is greater than ν, where a is a fixed number and a_n is a member of the sequence $a_1, a_2, \ldots, a_n, \ldots$, then a is the *limit of the sequence* as n tends to infinity; *i.e.*,

$$a = \lim_{n \to \infty} a_n$$

11. If for every positive integer N, however large, it is possible to choose ν such that $a_n > N$ whenever $n > \nu$, then a_n *tends to infinity.*

12. *Irrational numbers* can always be expressed as *limits of infinite sequences of rational numbers.*

13. An *infinite series* $a_0 + a_1 + \cdots + a_n + a_{n+1} + \cdots$ is said to be *convergent* if the sum of the first n terms $S_n = a_0 + a_1 + \cdots + a_n$ tends to a limit as n tends to infinity. A series that is not convergent is *divergent.* The necessary and sufficient condition for the convergence of a series is the following: For any given positive number ϵ, however small, it is possible to take n sufficiently large so that

$$|a_n + a_{n+1} + \cdots + a_{n+p}| < \epsilon$$

for every positive value of p.

14. If an infinite series is convergent, the limit of the sum of the first n terms as n tends to infinity is called the *sum of the infinite series:*

$$\sum_{k=0}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=0}^{n} a_k$$

15. A variable y is called a *function* of the independent variable x if its value or values are determined when the value of x is known. This is written in the form $y = y(x)$ or $y = f(x)$. If y has a value for $x = a$, it is said to be *defined* for $x = a$.

16. If for every value of x in an interval, y has only one value, y is a *single-valued* function of x in the interval; if y has more than one value for each value of x, it is a *multiple-valued* function of x.

17. If for any arbitrary number N, however large, $f(x)$ is greater than N whenever x is sufficiently near to a, then $f(x)$ is said to *tend to infinity* as x tends to a. This is written $f(x) \to \infty$ as $x \to a$.

18. The values of x for which the equation $f(x) = 0$ is true are called the *roots* of the equation. An equation $f(x) = 0$ that is true for arbitrary values of x is called an *identity.*

19. A *polynomial* in the variable x is the sum of a finite number of terms of the form a_nx^n, where a_n is a constant called the *coefficient* of x^n and n is a positive integer or zero. The *degree* of a polynomial is equal to the greatest exponent that occurs.

20. An equation of the form $f(x) = 0$, where $f(x)$ is a polynomial of the nth degree, is called an *algebraic equation* of the nth degree.

21. An equation $f(x) = 0$, where the expression $f(x)$ involves only a finite number of *algebraic operations* on x, *i.e.*, addition, subtraction, multiplication, division, and raising to rational exponents, can always be brought into the form $g(x) = 0$, where $g(x)$ is a polynomial. Hence, $f(x) = 0$ in this case is also called an *algebraic equation.*

22. A real number which can be a root of an algebraic equation with real coefficients is called an *algebraic number;* a real number that is not algebraic is *transcendental.*

23. An algebraic equation of nth degree can always be written in the form:

$$f(x) = a_n(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n) = 0$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation. A number α_k that occurs only once in the sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ is called a *simple root;* a number that occurs more than once is called a *multiple root;* if it occurs r times, it is called an *r-fold root.*

24. An infinite series of the form $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_1, a_2, \ldots, a_n, \ldots$ are a sequence of numbers independent of x, is called a *power series* in x.

25. If the terms of an infinite series are trigonometric functions of x multiplied by constant coefficients, the series is called an *infinite trigonometric series.*

26. If the terms of an infinite series are functions of a variable x and $S_n(x)$ denotes the sum of the first n terms, the series *converges* to the sum $S(x)$ for any given value of x if for every given ϵ a value of ν can be found such that $|S(x) - S_n(x)| < \epsilon$ whenever $n > \nu$.

27. If a series converges for every value of x in an interval, it *converges in the interval.*

28. If for every given ϵ it is possible to choose ν *independent of* x so that $|S(x) - S_n(x)| < \epsilon$ whenever $n > \nu$ for all values of x in an interval, the series *converges uniformly* to the sum $S(x)$ in the interval.

29. A variable y is a *function of several independent variables* $x_1, x_2, \ldots, x_n$ if its value or values are determined when $x_1, x_2, \ldots, x_n$ are known and $x_1, x_2, \ldots, x_n$ can vary arbitrarily in certain intervals. The function y is written in the form:

$$y = y(x_1, x_2, \cdots, x_n)$$

or

$$y = f(x_1, x_2, \cdots, x_n)$$

30. If $f(x_1, x_2, \ldots, x_n)$ is a polynomial of first degree in the variables $x_1, x_2, \ldots, x_n$, the equation

$$f(x_1, x_2, \cdots, x_n) = a_1x_1 + \cdots + a_nx_n + b = 0$$

is called a *linear equation* for $x_1, x_2, \ldots, x_n$. If $b = 0$, the equation is *homogeneous;* otherwise it is *nonhomogeneous.*

31. *The system of n homogeneous linear equations* for the n variables $x_1, x_2, \ldots, x_n$

$$\begin{array}{c} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = 0 \end{array}$$

has the unique solution $x_1 = x_2 = \cdots = x_n = 0$ if the *determinant*

$$\Delta = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

is different from zero.

32. The *determinant of the nth order*

$$\Delta = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

is a symbol for the sum of all products that can be made after the following rules:

a. Every term contains only one element from each row and from each column.

b. If in every product the factors are arranged in ascending order according to the columns from which they are drawn and the number of inversions in the sequence of the first subscripts (which designate the rows) is even, the term is given a positive sign in the sum. If the number of inversions is odd, the term is given a negative sign.

33. The array of $m \times n$ numbers

$$\begin{bmatrix} a_{11}a_{12} \cdot & \cdots & a_{1n} \\ a_{21}a_{22} \cdot & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1}a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called a *matrix* with m rows and n columns. If $m = n$, the matrix is called a *square matrix.* The coefficients of n linear equations for n unknowns constitute a *square matrix.*

34. If there is a number L such that for any given positive number ϵ, however small, a number δ can be found so that $|L - f(x)| < \epsilon$ whenever $0 < |x - x_0| < \delta$, it is said that $f(x)$ *tends to the limit* L as x tends to x_0; *i.e.*,

$$L = \lim_{x \to x_0} f(x)$$

35. If $\lim\limits_{x \to x_0} f(x) = f(x_0)$, then $f(x)$ is *continuous* at $x = x_0$.

36. If $f(x)$ is not continuous at $x = x_0$, but for any given ϵ, however small, a number δ can be found such that $|M - f(x)| < \epsilon$ whenever $x > x_0$ and $x - x_0 < \delta$, we say that $f(x)$ *tends to the limit M from the right* and denote M by $f(x_0 + 0)$. If $|N - f(x)| < \epsilon$ whenever $x < x_0$ and $x_0 - x < \delta$, we say that N is the *limit of $f(x)$ from the left* and denote N by $f(x_0 - 0)$. We say that the function $f(x)$ has a *finite discontinuity* at $x = x_0$.

37. The difference $f(x + \Delta x) - f(x)$ is called the *difference* Δf of the function $f(x)$ for the *increment* Δx of the independent variable x.

38. If the ratio $\Delta f/\Delta x$ has a limit as $\Delta x \to 0$, the function $f(x)$ is said to be a *differentiable function* of x, and the limit

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx} = f'(x)$$

is called the *derivative of $f(x)$ with respect to x.*

39. If $f'(x)$ is differentiable, the derivative of $f'(x)$ is called the *second derivative* of $f(x)$ and is denoted by $f''(x)$ or d^2f/dx^2. If $f''(x)$ is differentiable, its derivative is called the *third derivative* of $f(x)$, and so on.

40. If the function $f(x)$ is differentiable, its difference Δf is equal to $\Delta f = (A + \epsilon)\,\Delta x$, where A is a function of x, independent of Δx, and $\epsilon \to 0$ when $\Delta x \to 0$. Then $df = A\,\Delta x$ is called the *differential* of $f(x)$. The coefficient A is equal to the derivative of f, *viz.*, $f'(x)$, and, according to the definition of the differential, $dx = \Delta x$. Hence, the differential of $f(x)$ can be written in the form:

$$df = f'(x)\,dx$$

41. If $f(x_1,x_2)$ is a function of the independent variables x_1 and x_2, then $\Delta f = f(x_1 + \Delta x_1,\ x_2 + \Delta x_2) - f(x_1,x_2)$ is called the *difference* of the function $f(x_1,x_2)$ for the *increments* Δx_1 and Δx_2. If the difference Δf is expressible in the form $\Delta f = A_1\,\Delta x_1 + A_2\,\Delta x_2 + \epsilon\rho$, where $\rho = \sqrt{(\Delta x_1)^2 + (\Delta x_2)^2}$, A_1 and A_2 are independent of Δx_1 and Δx_2, and $\epsilon \to 0$ when $\rho \to 0$, *i.e.*, when Δx_1 and $\Delta x_2 \to 0$; then $f(x_1,x_2)$ is said to be a *differentiable function* of x_1 and x_2.

42. The *principal part of the difference* Δf, *viz.*, $A_1\,\Delta x_1 + A_2\,\Delta x_2$, is called the *differential* of f and is denoted by df; this is usually written in the form:

$$df = A_1\,dx_1 + A_2\,dx_2$$

since, according to the above definition, $\Delta x_1 = dx_1$ and $\Delta x_2 = dx_2$.

43. If $f(x_1,x_2)$ is differentiable, it has *partial derivatives*. The partial derivative with respect to x_1 is equal to the coefficient A_1; the partial derivative with respect to x_2 is equal to the coefficient A_2. They are denoted by $\partial f/\partial x_1$ and $\partial f/\partial x_2$, respectively.

44. An expression of the form $A_1\,dx_1 + A_2\,dx_2$, where A_1 and A_2 are differentiable functions of x_1 and x_2, is *the differential of a function $f(x_1,x_2)$* if the partial derivatives $\partial A_1/\partial x_2$ and $\partial A_2/\partial x_1$ are *equal* and are continuous functions of x_1 and x_2.

45. The words and phrases defined here for two independent variables can be extended to any *arbitrary finite number of variables*.

46. The function $y(x)$ defined by a relation of the form $F(x,y) = 0$ is said to be a function defined *implicitly* by this relation.

47. If z is a differentiable function of x and y, where y is a differentiable function of x, the equation

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

is again true, and the *derivative of* z considered as a function of x is

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}.$$

48. A *differential equation* is an equation in which at least one derivative occurs. If a partial derivative occurs, it is a *partial differential equation*. If there is only one independent variable, so that no partial derivatives occur, it is an *ordinary differential equation*.

49. A differential equation in which the highest derivative is the nth derivative is called a *differential equation of the nth order*.

50. The *indefinite integral* of a function $f(x)$ is a function $F(x)$ whose derivative is $f(x)$. It is denoted by $F(x) = \int f(x)\,dx$.

51. The *definite integral* of a continuous function $f(x)$ between the limits a and b is equal to the *limit of the sum*

$$S_n = (x_1 - a)f(a) + (x_2 - x_1)f(x_1) + \cdots + (b - x_{n-1})f(x_{n-1})$$

where $a < x_1 < x_2 < \ldots < x_{n-1} < b$, when the intervals $x_1 - a$, $x_2 - x_1$, $\cdots$, $b - x_{n-1}$ tend to zero and the number of subdivisions n tends to infinity. The definite integral is written in the form:

$$\lim_{n\to\infty} S_n = \int_a^b f(x)\,dx$$

52. If the function has a finite discontinuity (cf. **36**) at $x = x_0$ but is continuous for $a \leqq x < x_0$ and $x_0 < x \leqq b$, the definite integral

$$\int_a^b f(x)\,dx = \int_a^{x_0} f(x)\,dx + \int_{x_0}^b f(x)\,dx$$

where

$$\int_a^{x_0} f(x)\,dx = \lim_{\Delta x_1\to 0}\int_a^{x_0-\Delta x_1} f(x)\,dx \quad \text{and} \quad \int_{x_0}^b f(x)\,dx = \lim_{\Delta x_2\to 0}\int_{x_0+\Delta x_2}^b f(x)\,dx$$

(Δx_1 and Δx_2 are positive).

53. If $f(x)$ tends to infinity when $x \to b$ and $\int_a^{b-\Delta x} f(x)\,dx$ has a limit as $\Delta x \to 0$, then

$$\lim_{\Delta x \to 0} \int_a^{b-\Delta x} f(x)\,dx = \int_a^b f(x)\,dx$$

where Δx has been assumed to be positive. Such an integral is called an *improper integral.*

54. If $f(x)$ tends to infinity as $x \to \alpha$, where α is a value of x in the interval between a and b, and if the integrals $\int_a^{\alpha - \Delta x_1} f(x)\,dx$ and $\int_{\alpha + \Delta x_2}^{b} f(x)\,dx$ have limits as $\Delta x_1 \to 0$ and $\Delta x_2 \to 0$, where Δx_1 and Δx_2 are always positive, then the *definite integral* is defined by

$$\int_a^b f(x)\,dx = \lim_{\Delta x_1 \to 0} \int_a^{\alpha - \Delta x_1} f(x)\,dx + \lim_{\Delta x_2 \to 0} \int_{\alpha + \Delta x_2}^{b} f(x)\,dx$$

55. If the integrals involved in **54** have no limits, but their sum has a limit when $\Delta x_1 = \Delta x_2$, the limit of the sum is called the *principal value* of $\int_a^b f(x)\,dx.$

56. The integral $\int_a^\infty f(x)\,dx$ is equal to $\lim_{x \to \infty} \int_a^x f(x)\,dx$ *if this limit exists.* If the value of $|f(x)x^n|$, where $n > 1$, is smaller than a fixed number for all values of x, however large, *the integral* $\int_a^x f(x)\,dx$ *has a limit* for $x \to \infty$.

57. The *coordinates of a point* P are three numbers which determine the location of P in three-dimensional Euclidean space.

58. A *rectilinear* or *Cartesian coordinate system* is defined by its *origin* and *three directions.* The straight lines that are parallel to the three directions and pass through the origin are called the *coordinate axes.*

59. If x_1, x_2, and x_3 are the *rectilinear coordinates* of a point P, the point P is located at the end point of a succession of three straight-line segments, where the first segment begins at the origin and is parallel to the x_1-axis, the second to the x_2-axis, and the third to the x_3-axis, and their lengths are x_1, x_2, and x_3, respectively.

60. If the axes of a rectilinear coordinate system are *perpendicular* to each other, the coordinate system and the coordinates are called *rectangular Cartesian.*

61. The coordinates of points that lie on a given surface satisfy an equation of the form $F(x_1, x_2, x_3) = 0$, which is called the *equation of the surface.* The equation of a surface often can also be written in other forms, such as $x_3 = f(x_1, x_2)$.

62. A linear equation $A_1x_1 + A_2x_2 + A_3x_3 + A_4 = 0$, where A_1, A_2, A_3, and A_4 are constants, is the *equation of a plane.* The equation of the x_1x_2 plane is, for example, $x_3 = 0$.

63. The coordinates of the points that lie on a curve in the x_1x_2 plane satisfy an equation of the form $F(x_1,x_2) = 0$, which is called the *equation of the curve*. The *equation of a straight line*, for example, is of the form $A_1x_1 + A_2x_2 + A_3 = 0$. The equation of a curve can also be written in the form $x_2 = f(x_1)$ or in the *parametric form* $x_1 = g_1(t)$, $x_2 = g_2(t)$.

64. The *equation of a curve* in three-dimensional space is written in *parametric form*: $x_1 = g_1(t)$, $x_2 = g_2(t)$, $x_3 = g_3(t)$. The *equation of a straight line*, for example, is given by $x_1 = A_1 + B_1t$, $x_2 = A_2 + B_2t$, $x_3 = A_3 + B_3t$.

ANSWERS TO PROBLEMS

Chapter I

5.
$$\frac{1}{x^2} + \frac{1}{y^2} = 2 \log \frac{Cx}{y}$$

6.
$$\tan^{-1} \frac{y}{a} - \tan^{-1} \frac{\sqrt{ax - a^2}}{a} = C$$

or, in the equivalent form,
$$y - \sqrt{ax - a^2} = C\left(a^2 + y\sqrt{ax - a^2}\right)$$

7.
$$\log y = -\frac{x}{p} + C$$

8.
$$y = \frac{x}{2} + \frac{x\sqrt{3}}{2} \tan\left(\frac{\sqrt{3}}{2} \log x + C\right)$$

9.
$$y = e^{\frac{x}{\sqrt{2}}}\left(A \cos \frac{x}{\sqrt{2}} + B \sin \frac{x}{\sqrt{2}}\right) + e^{-\frac{x}{\sqrt{2}}}\left(C \cos \frac{x}{\sqrt{2}} + D \sin \frac{x}{\sqrt{2}}\right)$$

$$y = Ae^x + Be^{-x} + C \cos x + D \sin x$$

10.
$$y = Ce^{-\sin x} + \sin x - 1$$
$$y = (x + 1)^n(e^x + C)$$

11.
$$\frac{\alpha S}{Q} t = -\frac{1 + \beta\theta_f}{1 + 2\beta\theta_f} \log(1 - \eta) - \frac{\beta\theta_f}{1 + 2\beta\theta_f} \log\left(1 + \frac{\beta\theta_f}{1 + \beta\theta_f}\eta\right)$$

12.
$\frac{x}{l} = \frac{h_1}{h_1 + h_2}$ gives the location of $p_{\max}$.

$$P = 6\mu\, U \cot^2\alpha \left(\log \frac{h_2}{h_1} - 2\frac{h_2 - h_1}{h_2 + h_1}\right).$$

13. If the slab is pulled:

$$F = \left(1 + \frac{\alpha}{f}\right)\left[1 - \left(\frac{h_2}{h_1}\right)^{\frac{f}{\alpha}}\right] kh_2; \qquad \left(\frac{h_2}{h_1}\right)_{\min} = \left(\frac{\alpha + f}{\alpha}\right)^{-\frac{\alpha}{f}}.$$

If the slab is pushed:

$$F = \left(1 + \frac{\alpha}{f}\right)\left[\left(\frac{h_1}{h_2}\right)^{\frac{f}{\alpha}} - 1\right] kh_1; \qquad \left(\frac{h_2}{h_1}\right)_{\min} = \left(\frac{\alpha + 2f}{\alpha + f}\right)^{-\frac{\alpha}{f}}$$

$$\text{when } \frac{f}{\alpha} \to 0, \left(\frac{h_2}{h_1}\right)_{\min} \to \lim_{f/\alpha \to 0}\left(1 + \frac{f}{\alpha}\right)^{-\frac{\alpha}{f}} = \frac{1}{e}$$

14.
$$y = e^{-\frac{x}{2}}(Ae^{+\frac{x}{2}\sqrt{5}} + Be^{-\frac{x}{2}\sqrt{5}})$$
$$y = (A + Bx)e^{-x}$$

15. $$y = e^{-\frac{x}{2}}\left(A \cos \frac{x\sqrt{3}}{2} + B \sin \frac{x\sqrt{3}}{2}\right) - \frac{3}{13}\cos 2x + \frac{2}{13}\sin 2x$$

16. $$y = \cos 2x \cosh 5x + \sin 3x \cosh 4x - i(\sin 2x \sinh 5x + \cos 3x \sinh 4x)$$

17. $$\cosh^5 x = \tfrac{1}{16}\cosh 5x + \tfrac{5}{16}\cosh 3x + \tfrac{10}{16}\cosh x$$

Chapter II

1. $$y = AJ_0(kx) + BY_0(kx)$$

3. $$x = 1.667$$

4. $$y = AI_0(kx) + BK_0(kx) - \frac{a}{k^2}$$

5. $$y = \frac{1}{x^2}Z_{-\frac{2}{3}}\left(\frac{4i}{3}x^3\right)$$

This can be written with real coefficients in two equivalent forms:

$$y = \frac{1}{x^2}\left[Ai^{-\frac{2}{3}}J_{\frac{2}{3}}\left(\frac{4i}{3}x^3\right) + Bi^{\frac{2}{3}}J_{-\frac{2}{3}}\left(\frac{4i}{3}x^3\right)\right]$$

and

$$y = \frac{1}{x^2}\left[CI_{\frac{2}{3}}\left(\frac{4}{3}x^3\right) + DK_{\frac{2}{3}}\left(\frac{4}{3}x^3\right)\right]$$

Tables for $i^{-\frac{2}{3}}J_{\frac{2}{3}}(it)$ and $i^{\frac{2}{3}}J_{-\frac{2}{3}}(it)$ will be found in Jahnke-Emde, "Tables of Functions," 2d ed., page 285.

6. $$y = \frac{1}{x}[AJ_1(2x) + BY_1(2x)]$$
$$y = e^{-\frac{x}{2}}(Ae^{\frac{x}{\sqrt{2}}} + Be^{-\frac{x}{\sqrt{2}}})$$

7. $$y = A \cos(\log x) + B \sin(\log x) + \frac{4}{5}x^2 + \frac{x^3}{5}$$

9. $$a = \tfrac{1}{3}$$

10.
$n = 0,\ y = 1;\ n = 2,\ y = \frac{1}{2}(3x^2 - 1);$
$n = 4,\ y = \frac{1}{8}(35x^4 - 30x^2 + 3).$
$n = 1,\ y = x;\ n = 3,\ y = \frac{1}{2}(5x^3 - 3x);$
$n = 5,\ y = \frac{1}{8}(63x^5 - 70x^3 + 15x).$

Chapter III

1. *a*) Both M_1 and M_2 rise with the speed of $\frac{1}{2}$ ft./sec.
b) M_1 rises with $\frac{3}{4}$ ft./sec.; M_2 with $\frac{1}{4}$ ft./sec.
c) M_1 rises with $\frac{1}{3}$ ft./sec.; M_2 with $\frac{2}{3}$ ft./sec.

2. The amplitude is 0.182 in. The required counterweight is 2.44 lb. The remaining amplitude is $\frac{3}{107}l\left(1 - \frac{l}{\sqrt{l^2 + r^2}}\right) = 0.0036$ in.

3. The beam turns through the angle

$$\varphi = 2\sqrt{\frac{4r^2 - l^2}{4i^2 + 4r^2 - l^2}}\sin^{-1}\frac{l}{2\sqrt{i^2 + r^2}}$$

where $i^2 = I/m$; I is the moment of inertia of the beam with respect to the center of the circle, m is the mass of the dog. If $i = 0$, $\varphi = 2\sin^{-1}\frac{l}{2r}$, *i.e.*, the point B reaches the initial position of A.

4. $\theta = 0$ or $\cos\theta = g/l\Omega^2$ hence θ can be different from zero when $\Omega^2 > g/l$.

5.

$$(I + ml^2\sin^2\theta)\Omega = I\Omega_0$$

$$\frac{1}{2}(I + ml^2\sin^2\theta)\Omega^2 + \frac{1}{2}ml^2\left(\frac{d\theta}{dt}\right)^2 + mgl(1 - \cos\theta) = \frac{I}{2}\Omega_0^2 + \frac{m}{2}v_0^2$$

For small values of θ:

$$\frac{1}{2}ml^2\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}ml(g - l\Omega_0^2)\theta^2 = \text{const.}$$

If $\Omega_0^2 < g/l$ the mass oscillates around the vertical position $\theta = 0$.
If $\Omega_0^2 > g/l$, the mass oscillates around a mean value of θ, different from zero.

6. The gyroscopic pitching moment of the two-bladed propeller in foot-pounds, is $1700\cos^2\beta$, where β is the angle of rotation of the propeller blade measured from the vertical.

7. For $\theta = 30°$, the two speeds of precessions are 167 r.p.m. and 33.6 r.p.m. For $\theta = 90°$, 28.2 r.p.m. and infinite.

8. The path of the C.G. is an ellipse. There is one stable and one unstable equilibrium position.

10. $W/4$

11. The pendulum is in stable equilibrium position at $\theta = 0$ if $\Omega^2 < g/l$.

12. Unstable for $\Omega^2 > g/l$.

13. The pressure in pounds per square foot is $p = 200\tanh^2(t/14.4)$, where t is the time in seconds.

14.

$$\left[(m_1 + m_2)r^2\sin^2\theta + I_2\frac{r^2}{l^2}\cos^2\theta + I_3\right]\ddot{\theta} + \left(m_1 + m_2 - \frac{I_2}{l^2}\right)r^2\dot{\theta}^2\sin\theta\cos\theta = prS\sin\theta - Q$$

15.
$$m\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = -kx$$
$$m\ddot{x}l\cos\theta + ml^2\ddot{\theta} = -mgl\sin\theta$$

For small values of θ:

$$\left(\frac{mg}{k} + l\right)\ddot{\theta} = -g\theta$$

16. Two unstable equilibrium positions on the circle, whose diameter is $\overline{AB} = l$ when $mgl/2k > 1$. One stable and one unstable equilibrium positions on the vertical through the midpoint of $\overline{AB}$ when $mgl/2k < 1$.

Chapter IV

1. $\dfrac{M}{\beta}\log\left(1 + \dfrac{\beta v_0}{\alpha}\right)$

2. 4.53 sec.

3. 18,130 slug ft.^{2}; the radius of gyration is 5.5 ft.

4. 81 lb.

5. Exact value: $7.4164\sqrt{I/mgr}$; approximation: $7.26\sqrt{I/mgr}$.

6. The velocity varies between $\dfrac{4}{\sqrt{3}}\sqrt{gr}$ and $\dfrac{2}{\sqrt{3}}\sqrt{gr}$; the time of one revolution is $4\dfrac{r}{v_0}F\left(\dfrac{\sqrt{3}}{2}\right)$.

7. $\dfrac{1}{\sqrt{1+c^2}}\left[F\left(\dfrac{c}{\sqrt{1+c^2}}, \dfrac{\pi}{2}\right) - F\left(\dfrac{c}{\sqrt{1+c^2}}, \dfrac{\pi}{2} - \psi\right)\right]$

8. $\frac{4}{3}F(\frac{1}{3})$

9. $\sqrt{2}F\left(\dfrac{1}{\sqrt{2}}\right)$

10. $\dfrac{2}{\sqrt{3}}\left[F\left(\sqrt{\dfrac{2}{3}}, \dfrac{\pi}{2}\right) - F\left(\sqrt{\dfrac{2}{3}}, \dfrac{\pi}{4}\right)\right]$

11. $e = r$

12. The frequency of the motion is $\dfrac{1}{2\pi}\sqrt{\dfrac{fg}{l}}$, where the distance between the pulleys is $2l$.

13. Yes, when $\beta^2 < (k - m\omega^2)^2/\omega^2$; no, when $\beta^2 > (k - m\omega^2)^2/\omega^2$.

14.
$$x = \frac{F_0}{m[(\omega_0^2 - \omega^2)^2 + 4\omega^2\lambda^2]}\left[(\omega_0^2 - \omega^2)\left(\sin\omega t - \frac{\omega}{\mu}e^{-\lambda t}\sin\mu t\right) - 2\omega\lambda\cos\omega t + 2\omega\lambda e^{-\lambda t}\left(\cos\mu t + \frac{\lambda}{\mu}\sin\mu t\right)\right]$$

15. $h_0 - h = \dfrac{v_f^2}{g}\log\cosh\left(\dfrac{gt}{v_f}\right)$, where $v_f^2 = \dfrac{mg}{k}$.

16. $x = 5{,}000$ ft.

17. $x = 4.830$ ft.

18. Approximately equal to the half period of a mathematical pendulum whose length is equal to the radius of the earth, *i.e.*, 42.5 min., $V_{\max} = 890$ m.p.h.

19. Nodal point; all integral curves are tangent to the straight line $x + y = 0$.

20. Saddle point; the approximate equations of the two integral curves passing through the singular point are

$$(\sqrt{5} \pm 1)(x - 3) \mp 2(y - 2) = 0$$

21. The differential equation is

$$\frac{dh}{dx} = \alpha \frac{1 - \dfrac{c_f}{2\alpha}\dfrac{Q^2}{gh^3}}{1 - \dfrac{Q^2}{gh^3}}$$

Complete discussion of the solution can be found in H. Rouse, "Fluid Mechanics for Hydraulic Engineers," McGraw-Hill Book Company, Inc. New York, 1938, pp. 290–294.

Chapter V

1. $\omega^2 = (3 \pm \sqrt{5})g$; $\tan 2\theta = 2$ (cf. p. 185).

2.
$$x = Ae^{\lambda t} + Be^{-\lambda t} + C \sin(\omega t + \psi)$$
$$y = -3.302(Ae^{\lambda t} + Be^{-\lambda t}) + 0.3025C \sin(\omega t + \psi)$$

where $\lambda = 1.614\sqrt{g}$, $\omega = 2.123\sqrt{g}$; A, B, C and ψ are arbitrary.

3. $\omega_1 = 47.7$/sec.; $\omega_2 = 137$/sec.; $\theta = 90°$

4. Frequency of vertical translation $\omega_1 = 4.90\sqrt{g}$; frequency of rotation about the vertical axis $\omega_2 = 13.8\sqrt{g}$; frequencies of the coupled oscillations consisting of horizontal translation and rotation in the vertical plane parallel to the direction of translation $\omega_3 = 3.55\sqrt{g}$ and $\omega_4 = 13.5\sqrt{g}$ (g in feet per second per second).

5. (*a*) $k < 72.5$ lb./in.

(*b*) Vertical amplitude at pump side, 0.00710 in.; at motorside, 0.01332 in.; horizontal amplitude, 0.00779 in.

6. (*a*) $\omega_1 = 11.7$/sec., $\omega_2 = 15.7$/sec.

$$A_1^{(1)} : A_2^{(1)} = 1.00 : 2.57, \quad A_1^{(2)} : A_2^{(2)} = 1.00 : -0.392$$

(*b*)
$$q_1 = -0.180 \cos 28.6t \text{ in.}$$
$$q_2 = -0.442 \cos 28.6t \text{ in.}$$

7. (*a*) $\omega_1 = 11.9$/sec., $\omega_2 = 50.4$/sec., $\omega_3 = 84.6$/sec.

$$A_1^{(1)} : A_2^{(1)} : A_3^{(1)} = 1.00 : 1.140 : 0.967$$
$$A_1^{(2)} : A_2^{(2)} : A_3^{(2)} = 1.00 : -0.286 : -1.07$$
$$A_1^{(3)} : A_2^{(3)} : A_3^{(3)} = 1.00 : -1.42 : 2.96$$

(b) $$q_1 = 10^{-5}\omega^2 \sin \omega t\left[\frac{1.30}{125 - \omega^2} - \frac{0.525}{2590 - \omega^2} - \frac{1.00}{7160 - \omega^2}\right]$$

$$q_2 = 10^{-5}\omega^2 \sin \omega t\left[\frac{1.85}{125 - \omega^2} + \frac{0.187}{2590 - \omega^2} + \frac{1.59}{7160 - \omega^2}\right]$$

$$q_3 = 10^{-5}\omega^2 \sin \omega t\left[\frac{0.83}{125 - \omega^2} + \frac{0.416}{2590 - \omega^2} + \frac{1.94}{7160 - \omega^2}\right]$$

8. (a) $\dfrac{k_{23}}{k_{11} + k_{33}} = -\dfrac{a}{r}$.

(b) The resulting moment of X and R with respect to the center of gravity vanishes if the condition (a) is satisfied.

9. $T = 0.76$ sec.

10. $\omega_1 = 0.535\sqrt{c/I}$

$$q_1^{(1)}:q_2^{(1)}:q_3^{(1)}:q_4^{(1)}:q_5^{(1)} = 1.00:0.713:0.235:-0.337:-0.806$$

11. $\omega_5 = 1.89\sqrt{c/I}$, $\omega_4 = 1.57\sqrt{c/I}$

$$q_1^{(5)}:q_2^{(5)}:q_3^{(5)}:q_4^{(5)}:q_5^{(5)} = 1:-2.564:3.010:-2.147:0.350$$
$$q_1^{(4)}:q_2^{(4)}:q_3^{(4)}:q_4^{(4)}:q_5^{(4)} = 1:-1.45:-0.33:1.61:-0.41$$

12. Assuming $q_5 = 0$; $\omega = 0.288\sqrt{c/I}$;

$$q_1:q_2:q_3:q_4 = 1:0.916:0.756:0.534$$

The exact solution is $\omega = 0.350\sqrt{c/I}$;

$$q_1:q_2:q_3:q_4:q_5 = 1:0.877:0.647:0.336:-0.368$$

13. For rigid connection $\omega = 0.347\sqrt{c/I}$; for free wheeling $\omega = 0.765\sqrt{c/I}$, where $c = C/l$.

14. $\omega_3 = 3.36\sqrt{c/M}$, $q_1^{(3)}:q_2^{(3)}:q_3^{(3)} = 1:-3.646:2$

$\omega_2 = 2\sqrt{c/M}$, $q_1^{(2)}:q_2^{(2)}:q_3^{(2)} = 1:0:-1$

$\omega_1 = 0.84\sqrt{c/M}$, $q_1^{(1)}:q_2^{(1)}:q_3^{(1)} = 1:1.646:2$

where c is the shearing rigidity and M is the mass of the building.

15. $$q_1 = \mu x_0 \sin \omega t\left(\frac{0.648}{0.708 - \mu} + \frac{0.333}{4 - \mu} + \frac{0.018}{11.29 - \mu}\right)$$

$$q_2 = \mu x_0 \sin \omega t\left(\frac{1.064}{0.708 - \mu} - \frac{0.065}{11.29 - \mu}\right)$$

$$q_3 = \mu x_0 \sin \omega t\left(\frac{1.295}{0.708 - \mu} - \frac{0.333}{4 - \mu} + \frac{0.036}{11.29 - \mu}\right)$$

where $\mu = M\omega^2/c$.

16. $x_1 = 1.0$, $x_2 = 0.667$, $x_3 = 0.736$, $x_4 = 0.738$

17. The exact values of the roots are -3, -1, 2, and 4.

Chapter VI

1. The characteristic equation for the oscillations of the center of gravity of the shaft is $m^2\lambda^4 + 2\lambda^2 km \cos\theta + k^2 = 0$ (m = mass of the shaft) Hence λ has a positive real part except when $\cos\theta = 1$.

2. $$\frac{p\epsilon/\gamma}{\sqrt{\left(\dfrac{6400\pi\mu l}{d_2^2\gamma T}\right)^2 + \left[1 - \dfrac{4\pi^2}{gT^2}(2h_0 + 100l)\right]^2}}$$

4. 2492 lb. sec./in.

5. The system is stable if either $k_1 > mg/2l$ and $k_2 > mg/2l$ or $k_1 < mg/2l$, $k_2 < mg/2l$, and

$$H > \sqrt{\frac{Amgl}{2}}\left(\sqrt{1 - \frac{2k_1 l}{mg}} + \sqrt{1 - \frac{2k_2 l}{mg}}\right)$$

6. $\mu = \dfrac{2}{C\Omega}(k_{22}r^2 - k_{11}a^2 - k_{33}a^2)$ (C = moment of inertia of the propeller).

7. $A\ddot{\theta} = C\Omega\dot{\varphi} + Wgh\theta$; $I\ddot{\varphi} = -C\Omega\dot{\theta} + W_o s\varphi$. The stability criterion is $C\Omega > \sqrt{AW_o s} + \sqrt{IWgh}$

8. $C_1C_2 > 0$; $I\beta k/m > C_1C_2$

9. $x_1 = 0.81114$; $x_2 = 4.64663$; $x_{3,4} = 1.771115 \pm 1.939435i$

10. $x_1 = 1$, $x_{2,3} = 1 \pm i\sqrt{2}$; $x_{4,5} = 2 \pm i$

Chapter VII

1. $w_{max} = \dfrac{385}{10{,}368}\dfrac{p_0 l^4}{F}$ at $x = \dfrac{31}{72}l$

2. $M = -\dfrac{P\sqrt{2}}{\beta}e^{-\frac{\beta}{\sqrt{2}}x}\sin\dfrac{\beta}{\sqrt{2}}x$

3. For $x \leqq \xi$; $g(x, \xi) = \dfrac{Px^2}{EIl^3}\left[l^3\xi - \dfrac{1}{3}l^3x - \dfrac{\xi^2}{6}(3l - \xi)(3l - x)\right]$

$$x \geqq \xi;\ g(x, \xi) = \frac{P\xi^2}{EIl^3}\left[l^3x - \frac{1}{3}l^3\xi - \frac{x^2}{6}(3l - x)(3l - \xi)\right]$$

4. If $\xi < l/2$, the maximum moment occurs when the wheels are located at $x = \xi$ and $x = \xi + a$; it is equal to $P\xi\left(2 - \dfrac{2\xi}{l} - \dfrac{a}{l}\right)$. If $\xi > \dfrac{l}{2}$, the maximum moment occurs when the wheels are at $x = \xi - a$ and $x = \xi$; it is equal to $P(l - \xi)\left(2\xi - \dfrac{a}{l}\right)$.

5. $w(x) = \dfrac{8}{3E}\sqrt{\dfrac{\sigma^3 l^3 b}{6P}}\left[\left(1 - \dfrac{x}{l}\right)^{3/2} + \dfrac{3x}{2l} - 1\right]$; σ is the maximum bending stress; l is the length and b the width of the beam.

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for the hinged beam is

$$\text{ʒ}l - \cot \beta l = \frac{2k}{EI\beta^3}$$

$$= \frac{k}{EI\beta^3} (\tanh \beta l - \tan \beta l)$$

where $\beta^4 = \rho\omega^2/EI$. When $k \to 0$, $\coth \beta l = \cot \beta l$ in the case of the hinged beam and $\cosh \beta l \cos \beta l + 1 = 0$ in that of the clamped beam. the hinged beam and $\tanh \beta l = \tan \beta l$

is

$$\text{nh } \beta l) - \frac{C\omega^2}{EI\beta} (\tan \beta l + \tanh \beta l)$$

$$+ \frac{Cm\omega^4}{E^2I^2\beta^4} (1 - \operatorname{sech} \beta l \sec \beta l) = 0$$

aft is negligible ($\rho \to 0$), β is

$$\frac{Cm\omega^4 l^4}{12E^2I^2} = 0$$

critical speed is given by the smallest root of the equation:

$$\tanh \frac{\beta l}{2} + \tan \frac{\beta l}{2} + \frac{2EI\beta}{k} = 0$$

$^4 = \rho\omega^2/EI$.

he frequency equation is

$$1 + \operatorname{sech} \beta l \sec \beta l + \tfrac{1}{2}\beta l (\tanh \beta l - \tan \beta l) = 0$$

$= \rho\omega^2/EI$.

14. (a) $\theta = \theta_0 + (\theta_1 - \theta_0) \dfrac{\cosh \sqrt{\alpha/kt}(l - x)}{\cosh \sqrt{\alpha/kt}\, l}$;

$$H = \sqrt{kt\alpha}(\theta_1 - \theta_0) \tanh \sqrt{\frac{\alpha}{kt}} l$$

(b) $\theta = \theta_0 + (\theta_1 - \theta_0) \dfrac{I_0\left(2\sqrt{\dfrac{\alpha l(l - x)}{kt_0}}\right)}{I_0(2\sqrt{\alpha l^2/kt_0})}$;

$$H = \sqrt{kt\alpha}(\theta_1 - \theta_0) \frac{I_1(2l\sqrt{\alpha/kt})}{I_0(2l\sqrt{\alpha/kt})}$$

15. 25,700 B.t.u./sq. ft. hr.

16. $$\theta_1 = \frac{1}{e^{\beta l} - \gamma}[\theta_1'(e^{\beta x} - \gamma) - \theta_2'(e^{\beta x} - e^{\beta l})]$$

$$\theta_2 = \frac{1}{e^{\beta l} - \gamma}[\theta_1'\gamma(e^{\beta x} - 1) - \theta_2'(\gamma e^{\beta x} - e^{\beta l})]$$

where $\gamma = \frac{v_1c_1}{v_2c_2}$, $\beta = \alpha\left[\frac{1}{v_2c_2} - \frac{1}{v_1c_1}\right]$.

17. $\omega = \sqrt{\frac{EI}{\rho}}\frac{\pi^2}{l^2}\sqrt{1 - \frac{Pl^2}{EI\pi^2}}$; if $P \to EI\pi^2/l^2$, $\omega \to 0$. ρ = mass per unit length.

19. $\omega = 4.599\frac{1}{l^2}\sqrt{\frac{EI_0}{\rho A_0}}$; ρ_0 = density of the material, A_0 = area, I_0 = moment of inertia of the root section.

Chapter VIII

1. $a_n = 0$ if n is even; $a_n = 4/n\pi$ if n is odd.

2. $M_{\max} = \frac{4Wl}{\pi^3}\sum_{n=1,3,\ldots}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n\left(n^2 - \frac{P}{P_c}\right)}$; $P_c = \pi^2\frac{EI}{l^2}$. W is the total weight of the beam.

4. $$M_{\max} = \frac{2pa^2}{\pi}\int_0^{\infty}\frac{\zeta\sin\zeta}{\zeta^4 + \alpha^4}\,d\zeta$$

6. $$\begin{aligned} x = r(1 - \cos\theta) + \frac{r^2}{2l}\Bigg[&\left(\frac{1}{2} + \frac{3r^2}{32l^2} + \frac{5r^4}{128l^4} + \frac{175r^6}{8192l^6} + \cdots\right) \\ &- \left(\frac{1}{2} + \frac{r^2}{8l^2} + \frac{15r^4}{256l^4} + \frac{35r^6}{1024l^6} + \cdots\right)\cos 2\theta \\ &+ \left(\frac{r^2}{32l^2} + \frac{3r^4}{128l^4} + \frac{35r^6}{2048l^6} + \cdots\right)\cos 4\theta \\ &- \left(\frac{r^4}{256l^4} + \frac{r^6}{1024l^6} + \cdots\right)\cos 6\theta + \left(\frac{r^6}{8192l^6} + \cdots\right)\cos 8\theta + \cdots\Bigg] \end{aligned}$$

7. $M = \frac{2\gamma l^3}{\pi^4}\sum_{n=1}^{\infty}\frac{n}{n^4 + \frac{48l^4}{\pi^4 d^2 t^2}}\sin\frac{n\pi x}{l}$; γ = specific weight of water, l = height of the tank.

8. Both ends hinged:

$$M = \frac{\pi^2}{864}\gamma l^3\left(1 - \frac{2x}{3l}\right)\left(3.080\sin\frac{\pi x}{l} - 0.7100\sin\frac{2\pi x}{l} + 0.3492\sin\frac{3\pi x}{l}\right)$$

Clamped at the bottom:

$$M = -\frac{\pi^2}{864}\gamma l^3\left(1 - \frac{2x}{3l}\right)\left(0.159 \cos\frac{\pi x}{l} - 0.171 \cos\frac{3\pi x}{2l} + 0.073 \cos\frac{5\pi x}{2l}\right)$$

9. $\omega = \frac{k}{l^2}\sqrt{\frac{EI_0}{\rho A_0}}$; approximation of the deflection curve by two terms of a power series, $k = 4.654$; by three terms, $k = 4.599$ (cf. Prob. 19, Chap. VII)

Chapter IX

1. $I = \frac{E_0}{L\omega}\left(-\cos\omega t + \frac{1}{9}\cos 3\omega t\right)$

$I = E_0C\omega(\cos\omega t - \cos 3\omega t)$

2. $I = \frac{E}{Z}$ where $Z = \dfrac{(0.2128 + 92.73i) - (968.1 - 4875i)\gamma}{(7.6 + 0.456i) - (2.4 - 2440i)\gamma}$

Hence I is of the form $\frac{a + b\gamma}{c + d\gamma}$, where $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$, $d = d_1 + id_2$. Putting $I = I_1 + iI_2$,

$$c_1I_1 - c_2I_2 + (d_1I_1 - d_2I_2)\gamma = a_1 + b_1\gamma$$
$$c_2I_1 + c_1I_2 + (d_2I_1 + d_1I_2)\gamma = a_2 + b_2\gamma$$

Eliminating γ, we have

$$(c_1d_2 - c_2d_1)(I_1^2 + I_2^2) - (c_1b_2 - c_2b_1 + a_1d_2 - a_2d_1)I_1 + (c_2b_2 + c_1b_1 - a_1d_1 - a_2d_2)I_2 + a_1b_2 - a_2b_1 = 0$$

which is the equation of a circle.

3. $\theta = -\dfrac{M_0 \sin\omega t}{I\omega^2\left(1 - \frac{I\omega^2}{c}\right)\left(3 - \frac{I\omega^2}{c}\right)} - \dfrac{M_0 \sin 3\omega t}{36I\omega^2\left(1 - \frac{9I\omega^2}{c}\right)\left(3 - 9\frac{I\omega^2}{c}\right)}$

$\theta \to \infty$, if $\omega \to \sqrt{c/I}$, $\omega \to \sqrt{3c/I}$, $\omega \to \frac{1}{3}\sqrt{c/I}$,

$\omega \to \sqrt{\frac{c}{3I}}$

4. $K = \dfrac{Ip^2(Ip^2 + 2c)\,[Ip^2 + c)(Ip^2 + 2c) + cIp^2]}{Ip^2(Ip^2 + c)(Ip^2 + 3c) + cIp^2(Ip^2 + 2c)\ \ c^2(Ip^2 + c)}$

The frequency equation is

$$\omega^2(2c - I\omega^2)[(c - I\omega^2)(2c - I\omega^2) - cI\omega^2] = 0$$

5. The impedance of the network on the left,

$$\frac{1}{p}\,\frac{L_1L_2p^4 + \left(\frac{L_1 + L_2}{C_1} + \frac{L_1}{C_2}\right)p^2 + \frac{1}{C_1C_2}}{(L_1 + L_2)p^2 + \frac{1}{C_2}}$$

and the impedance of the network on the right,

$$\frac{1}{p}\frac{L_3L_4p^4 + \left(\frac{L_4}{C_3} + \frac{L_3}{C_4}\right)p^2 + \frac{1}{C_3C_4}}{(L_3 + L_4)p^2 + \frac{1}{C_3} + \frac{1}{C_4}}$$

are equal if the conditions given in the problem are satisfied.

6. (a) $I = 46.8$ amp.
   (b) $I = 194$ amp.

7. The impedance is

$$\frac{R\left(R + Lp + \frac{1}{Cp} + \frac{L}{CR}\right)}{R + Lp + \frac{1}{Cp} + R}$$

It is equal to R if $L/CR = R$.

Chapter X

1 and 2.

$$I = \frac{\sin\varphi\, e^{-\frac{R}{L}t} + \sin(\omega t - \varphi)}{\sqrt{L^2\omega^2 + R^2}}; \tan\varphi = \frac{\omega L}{R}$$

3. The maximum torque in the shaft is

$$M_{\max} = M_0\frac{I}{I_1 + I}\left(1 + \frac{\sin\frac{1}{2}\omega_0t_0}{\frac{1}{2}\omega_0t_0}\right)$$

where $\omega_0 = \sqrt{\frac{c}{I_1} + \frac{c}{I}}$ and c is the spring constant of the shaft. If the driving torque is suddenly applied

$$M_{\max} = 2M_0\frac{I}{I_1 + I}$$

4. $$I = \frac{E_0C}{t_0}\left(1 - e^{-\frac{t}{RC}}\right)1(t) - 2\frac{E_0C}{t_0}\left(1 - e^{-\frac{t - t_0}{RC}}\right)1(t - t_0) + \frac{E_0C}{t_0}\left(1 - e^{-\frac{t - 2t_0}{RC}}\right)1(t - 2t_0)$$

5. In the interval $nt_0 < t < (n + 1)t_0$

$$I = \frac{E_0C}{t_0}(-1)^n\left(1 - \frac{2e^{\frac{(n+1)t_0 - t}{RC}}}{1 + e^{\frac{t_0}{RC}}}\right) + \frac{E_0C}{t_0}e^{-\frac{t}{RC}}\frac{e^{\frac{t_0}{RC}} - 1}{e^{\frac{t_0}{RC}} + 1}$$

The last term represents the transient current.

6. In the interval $nt_0 < t < (n+1)t_0$

$$I = \frac{E_0}{Rt_0}[t1(t) - 2(t-t_0)1(t-t_0) + 2(t-2t_0)1(t-2t_0)\ \ldots\ ]$$

$$- \frac{E_0}{R}\frac{L}{t_0R}(-1)^n\left(1 - \frac{2e^{\frac{R}{L}[(n+1)t_0 - t]}}{e^{\frac{Rt_0}{L}} + 1}\right)$$

$$+ \frac{E_0}{R}\frac{L}{t_0R}e^{-\frac{Rt}{L}}\frac{1 - e^{\frac{Rt_0}{L}}}{1 + e^{\frac{Rt_0}{L}}}$$

The last term is the transient current.

7. $I = \dfrac{p_1B_1}{\omega}e^{p_1t} + \dfrac{p_2B_2}{\omega}e^{p_2t} + iB_3e^{i\omega t} - iB_4e^{-i\omega t}$

With the notations of section 14, Chap. X.

8. $$I = B_1\left(\cos\varphi + \frac{p_1}{\omega}\sin\varphi\right)e^{p_1t} + B_2\left(\cos\varphi + \frac{p_2}{\omega}\sin\varphi\right)e^{p_2t}$$
$$+ B_3e^{i(\omega t+\varphi)} + B_4e^{-i(\omega t+\varphi)}$$

With the notations of section 14, Chap. X.

9. For $0 < t < T$ the pressure on the cam is given operationally by

$$F = P + ak_1\frac{(p^2 + \omega_1^2)\omega^2}{(p^2 + \omega_0^2)(p^2 + \omega^2)}1(t)$$

where $\omega_0 = \sqrt{\dfrac{k_1 + k}{m}}$, $\omega_1 = \sqrt{\dfrac{k}{m}}$, $\omega = \dfrac{2\pi}{T}$.

$$F = P + ak_1\frac{\omega^2}{\omega_0^2}\frac{\omega_0^2 - \omega_1^2}{\omega_0^2 - \omega^2}(1 - \cos\omega_0t) + ak_1\frac{\omega_1^2 - \omega^2}{\omega_0^2 - \omega^2}(1 - \cos\omega t)$$

F is always positive if $\omega^2 < \omega_1^2 + \dfrac{k_1}{m}\dfrac{P}{P + 2ak_1}$, and the roller cannot jump off the cam. The displacement of the valve is

$$x = x_1 - \frac{F - P}{k_1}$$

10. The acceleration of the last car due to a unit braking force is operationally

$$-\frac{1}{m}\cdot\frac{(\beta p + k)^2}{(mp^2 + \beta p + k)(mp^2 + 3\beta p + 3k)}1(t)$$

For $\beta = 2\sqrt{km}$ this acceleration is

$$a = -\frac{7000}{m}\left[\frac{1}{3} + \frac{1}{2}\left(t\sqrt{\frac{k}{m}} - 1\right)e^{p_1t} - \frac{1}{12}\left(\frac{3}{\sqrt{6}} - 1\right)e^{p_3t}\right.$$
$$\left. + \frac{1}{12}\left(\frac{3}{\sqrt{6}} + 1\right)e^{p_4t}\right]$$

$$p_1 = -\sqrt{\frac{k}{m}},\ p_3 = -(3 - \sqrt{6})\sqrt{\frac{k}{m}},\ p_4 = -(3 + \sqrt{6})\sqrt{\frac{k}{m}}$$

For $\beta = \frac{3}{2}\sqrt{km}$,

$$a = -\frac{7000}{m}\left[\frac{1}{3} - \frac{1}{2}e^{\alpha_1 t}\left(\cos \alpha_2 t - \frac{3}{\sqrt{7}}\sin \alpha_2 t\right) - \frac{1}{12}\left(\frac{9}{\sqrt{33}} - 1\right)e^{p_3 t} + \frac{1}{12}\left(\frac{9}{\sqrt{33}} + 1\right)e^{p_4 t}\right]$$

$$\alpha_1 = -\frac{3}{4}\sqrt{\frac{k}{m}}, \qquad p_3 = -\frac{1}{4}(9 - \sqrt{33})\sqrt{\frac{k}{m}}$$

$$\alpha_2 = \frac{\sqrt{7}}{4}\sqrt{\frac{k}{m}}, \qquad p_4 = -\frac{1}{4}(9 + \sqrt{33})\sqrt{\frac{k}{m}}$$

11.

$$I = \frac{PE}{akL(p_1 - p_2)}\,(e^{p_1 t} - e^{p_2 t})$$

$$p_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC_0} + \frac{E^2}{kLa^2}}$$

$$p_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC_0} + \frac{E^2}{kLa^2}}$$

Chapter XI

1. 5050

2. (a) $\omega = 2\sqrt{\frac{c}{I}}\sin\frac{\mu}{2},\ \varphi_k^{(r)} = \text{const.}\cos\frac{2k-1}{2}\mu_r$

where $\sin 4\mu = 2\sin 5\mu$.

(b) $\omega = 2\sqrt{\frac{c}{I}}\sin\frac{\mu}{2},\ \varphi_k^{(r)} = \text{const.}\cos\frac{2k-1}{2}\mu_r$

where $\cos\frac{9\mu}{2} = \frac{c}{c+k}\cos\frac{7\mu}{2}$ or $\mu = 0$.

3. $\omega = 2\sqrt{\frac{c}{m}}\sin\frac{\mu}{2},\ \varphi_k^{(r)} = \text{const.}\cos\frac{19-2k}{2}\mu_r$

where $\cos\frac{17\mu}{2} = 2\cos\frac{19\mu}{2}$, m = mass of one floor, c = coefficient of shearing rigidity, and k = number of a floor above ground.

4. $w_{x+3} - \left(1 + 2\cos\alpha l + 6\mu\frac{\sin\alpha l}{\alpha l}\right)(w_{x+2} - w_{x+1}) - w_x = 0$. The general solution of this equation is $w_x = A + B\sin\lambda x + C\cos\lambda x$, where

$$\sin\frac{3\lambda}{2} = \left(1 + 2\cos\alpha l + 6\mu\frac{\sin\alpha l}{\alpha l}\right)\sin\frac{\lambda}{2}$$

or

$$\cos \lambda = \cos \alpha l + 3\mu \frac{\sin \alpha l}{\alpha l}$$

From the boundary conditions $w = 0$ for $x = 0$ and $x = n$, it follows that

$$\lambda = \frac{2r\pi}{n}; \qquad w_x = A\left(1 - \cos \frac{rx\pi}{n}\right) \text{ with } r = 2.$$

5. Progressing damped waves ($x \geqq 0$):

$$w_x = \text{const. } e^{-px} \sin (\omega t - qx)$$

where p and q are determined by

$$1 - \cos q \cosh p = \frac{lm\omega^2}{2F}$$

$$\sin q \sinh p = \frac{lm\omega\beta}{2F}$$

provided these equations can be satisfied by real values of p and q. l = distance between masses, β = damping factor and ω = angular frequency of the harmonic motion.

INDEX

A

B

C

I

N

Q

R